Big Ideas in Mathematics
for Future Elementary Teachers

Big Ideas in Geometry
and Measurement

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(Updated Summer 2017)
Dear Future Teacher,

We wrote this book to help you to see the structure that underlies elementary mathematics, to give you experiences really doing mathematics, and to show you how children think and learn. We fully intend this course to transform your relationship with math.

As teachers of future elementary teachers, we created or gathered the activities for this text, and then we tried them out with our own students and modified them based on their suggestions and insights. We know that some of the problems are tough – you will get stuck sometimes. Please don’t let that discourage you. There’s much value in wrestling with an idea.

All our best,

John, Jason, Eric, Amy, Carol & Jen
Hey! Read this. It will help you understand the book.

*The only way to learn mathematics is to do mathematics.*
*Paul Halmos*

This book was written to prepare future elementary teachers for the mathematical work of teaching. The focus of this module is geometry – and this domain encompasses many deep and wonderful mathematical ideas. This text is not intended to help you relearn your elementary mathematics; it is about teaching you to think like a mathematician and it is about helping you to think like a mathematics teacher. The National Council of Teachers of Mathematics (NCTM, 2000) writes:

Teachers need several different kinds of mathematical knowledge – knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level; knowledge about the challenges students are likely to encounter in learning these ideas; knowledge about how the ideas can be represented to teach them effectively; and knowledge about how students’ understanding can be assessed (p. 17).

We are going to work toward these goals. *(Read them again. This is a tall order. In which areas do you need the most work?)* Throughout this book, we will ask you to consider questions that may arise in your elementary classroom.

- *Is a square always a rectangle?*
- *What does this number called \( \pi \) represent?*
- *What does it mean to measure “area”?*
- *Can two rectangles with the same area have different perimeters?*
- *What is so special about right triangles?*
- *If I buy a 5-inch pizza and my friend buys a 10-inch pizza, does she get twice as much to eat?*
- *What is geometry about?*

As mathematicians we will also convey to you the beauty of our subject. We view mathematics as the study of patterns and structures. We want to show you how to reason like a mathematician – and we want you to show this to your students too. This *way of reasoning* is just as important as any content you teach. When you stand before your class, you are a representative of the mathematical community; we will help you to be a good one.
No one can do this thinking for you. Mathematics isn’t a subject you can memorize; it is about ways of thinking and knowing. You need to do examples, gather data, look for patterns, experiment, draw pictures, think, try again, make arguments, and think some more. The big ideas of geometry are not always easy – but they are fundamentally important for your students to understand and so they are fundamentally important for you to understand.

Each section of this book begins with a Class Activity. The activity is designed for small-group work in class. Some activities may take your class as little as 20 minutes to complete and discuss. Others may take you two or more class periods. The Read and Study, Connections to the Elementary Grades, and Homework sections are presented within the context of the activity ideas. No solutions are provided to activities or homework problems – you will have to solve them yourselves.

The mathematics content in this book prepares you to teach the Common Core State Standards for Mathematics for grades K - 8. These are the standards that you will likely follow when you are an elementary teacher, so we will highlight aspects of them throughout the text. In order for you to see how the mathematical work you are doing appears in the elementary grades, we have made explicit connections to Bridges in Mathematics from The Math Learning Center. This is the online elementary grades mathematics curriculum adopted by the Oshkosh Area School District. You will often be asked to go to the site bridges.mathlearningcenter.org to read or do problems. Your instructor will provide you with a code so that you can access these materials.
Big Ideas in Geometry and Measurement

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Shulman, 1985, p. 47

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Chapter One

Seeing the World Geometrically
Class Activity 1: Triangle Puzzle


What happened to the missing square?
Mathematical objects – geometric objects included – are not real objects. They are ideal objects. This might seem disturbing because this means that geometric objects – like triangles and circles – do not exist in the physical world. You can draw something that looks like a triangle or a circle, but it won’t have the precise and perfect properties that the ideal mathematical triangle or circle has. The sketch will simply call to mind the ideal object.

While we draw lots of pictures in geometry, we need to keep in mind that the pictures can be misleading. Take the Triangle Puzzle as an example. This puzzle is compelling because it really looks like the top and bottom figures are both triangles. They are not. Only by reasoning about how the idealized pieces fit together can we discover the truth.

The puzzle is governed by an underlying structure – the properties of the shapes involved determine how they will fit together. Mathematicians love to reveal hidden structure and to explain patterns. This is what mathematics is about. And this is what geometry, in particular, is about.

One definition of geometry is that it is the study of ideal shapes and their properties, of the patterns those shapes can form, and of the actions on those shapes that preserve their properties. We will be giving you other definitions of geometry as we go along. Watch for the variety of ways of thinking about geometry.

This book is designed so that you get to do geometry. We generally do not teach you techniques and then have you practice. Instead, we ask that you work on problems to help you construct important ideas. The problems can be difficult – which is why we hope that you will work on them in groups and then discuss them as a class. We made them that way on purpose. We believe that a problem is only a problem if you don’t know how to solve it. If you do know how to solve it, then it is just an exercise. We hope this book is full of problems for you and that you will ‘get stuck’ a lot. Try not to let being stuck discourage you. It is part of doing mathematics.

There is no magic technique for solving a mathematics problem – which is good, because otherwise mathematics wouldn’t be any fun. Basically, you just have to wrestle with the
problem. The process might take minutes, or hours, or even years. There are strategies that may prove helpful, and a purpose of this first section is to make you aware of some of the things people do to work on problems.

For now, we would like you to separate your thinking about problem solving into four categories:

1) **Understanding the problem.** What does it mean to solve this problem? Do you understand the conditions and information given in the statement of the problem? (For the Triangle Puzzle above, this means understanding that since area must be preserved, the pictures have to be misleading in some way. Solving this problem means showing exactly how the pictures are misleading.)

2) **Reflecting on your problem solving strategies.** What did you do to work on the problem? (Did you study the pieces to see how they fit together? Randomly or in some systematic manner? Did you keep track of anything? Did you draw pictures? Did you compute something?)

3) **Explaining the solution.** What is the answer to the question? (Exactly what is wrong with the pictures?)

4) **Justifying that you are both done and correct.** Why does your solution make sense? Can you prove that you are correct and that the problem is completely solved?

Before we get into this book any further, we might as well tell you that we’re bossy. Throughout the reading sections (which you must do – you owe it to the children in your future classrooms) we will ask questions and issue commands in italics. *Do the things we suggest in italics.* Don’t worry that it slows the reading down. Mathematicians read very, very, agonizingly slowly and carefully, with pencil in hand. We write on our books – all over them. We verify claims; we do the problems; we ask new questions and try to answer them.

So we challenge you to do four things this term. First and second, read every word of your text and work hard on each and every problem. Third, make a contribution to each discussion of a class activity. And finally, practice listening to and making sense of other students’ mathematical ideas. As a teacher, you will need to understand the mathematical thinking of others; use your class to practice that skill.
Connections to the Elementary Grades

Instructional programs from prekindergarten through grade 12 should enable all students to organize and consolidate their mathematical thinking through communication; to communicate their mathematical thinking coherently and clearly to peers, teachers, and others; to analyze and evaluate the mathematical thinking and strategies of others; and to use the language of mathematics to express mathematical ideas precisely.

NCTM Principles and Standards for School Mathematics, 2000

Learning via problem solving and communication of ideas are two major threads in elementary mathematics education. The National Council of Teachers of Mathematics (2000), in adopting the Principles and Standards for School Mathematics, advocated that all students of mathematics engage in problem solving and communication, both oral and written, at all grade levels. They write:

“Solving problems is not only a goal of learning mathematics but also a major means of doing so” (p. 52).

“Communication is an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding.... When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing” (p. 60).

Mathematics educators believe that problem solving and written communication are also essential components of your mathematical preparation to become elementary school teachers. We will be asking you to write about mathematics in this class. We will ask you to write interpretations of problems, descriptions of strategies, explanations of solutions, and justifications of solutions.

How do we write about mathematics? Isn’t mathematics all about numbers like 2 and π? And about symbols like ‖ and ⊥?

Well, no, it’s not.

Mathematicians use symbols like these to write statements and to solve some problems, and you can use symbols like ⊥ and ≡ when you write about geometry problems. But mathematics is much more about exploring patterns, making conjectures, explaining results and justifying solutions. These activities require us to write with words in complete sentences that use mathematical language and logic appropriately.
Symbols are a tool we will use. But for now, you should focus on writing with words – clearly, completely, correctly, and convincingly. As future teachers you must practice communicating in the language of mathematics. You will have a chance to practice right now in the homework.

**Homework**

*You always pass failure on the way to success.*

*Mickey Rooney (MQS)*

1) The Tangram puzzle is composed of seven shapes including one square, one parallelogram, two small isosceles right triangles, one medium-sized isosceles right triangle, and two large isosceles right triangles. In the diagram below, the seven pieces are arranged so that they fit together to form a square.

a) Trace the pieces, cut them out, and then identify each one. Look up the terms **isosceles**, **right triangle** and **parallelogram** in the glossary and learn those definitions.

b) Figure out how to rearrange all seven pieces to form a trapezoid. Notice that you first need to **understand the problem**. Look up the definition of a **trapezoid** if you need to do so.

![Tangram Diagram](image)

Reflect on your problem solving strategies and write a description of the strategies you used to work on the above Tangram puzzle.

d) **Explain the solution** by giving careful instructions, *using words only* (no pictures), for arranging the seven pieces to form a trapezoid.
2) Children in the early elementary grades can solve puzzles similar to the Tangram puzzle using pattern blocks (a set of flat blocks in six shapes: regular hexagon, isosceles trapezoid, two rhombi, square, and equilateral triangle). Look up the terms rhombus (the plural form of rhombus is rhombi) and hexagon in the glossary, then do the activity described below.

![Pattern Blocks Image]

a) Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 1 Teachers Guide (your professor should have a code for you to view this). Spend 10-15 minutes looking through Unit 5 Module 1. Then work through Pattern Block Puzzle 3. How many different ways are possible to fill this shape? See if you can find at least 5.

b) What might children learn about the relationships among the pattern blocks by working on these problems?
3) Here is a picture of all seven Tangram pieces rearranged to make a triangle.

You are going to begin to “**justify that we are correct.**” This involves arguing that the pieces really do fit together as shown. (Remember that just “looking like they fit” isn’t good enough.) You get to assume that the original pieces really are all perfect shapes and that they originally fit perfectly to form a square like this:

a) Argue that the vertex of the yellow triangle along with the vertices of the large orange and blue triangles really do meet to make 180 degrees (they form a straight angle) at the bottom edge of the puzzle.

b) Argue that the edge of the orange triangle fits perfectly with the edges of the square and yellow triangle and that the big blue triangle really fits perfectly along the edges of the parallelogram and yellow triangle.
Class Activity 2: Defining Moments

Where there is matter, there is geometry.
Johannes Kepler (1571-1630)

Mathematical definitions are important to mathematicians because they give us the exact criteria we need to classify objects. Use the definition of a polygon to decide whether each of the objects the sketch calls to mind are polygons. If an object does not meet the definition, explain exactly how it fails. Visit the glossary if you need to look up terms.

A polygon is a simple, closed curve in the plane composed only of a finite number of line segments.
Mathematicians care deeply about the words we use to talk about mathematics. We have many special words, like *isosceles*, that do not appear in everyday language. These words have precise definitions that provide powerful knowledge about the objects to which they refer. Even common words like *right* take on special meaning when they are used in mathematical talk.

We will say lots, and we mean *lots*, more about the terms we use in mathematics throughout the pages of this book. Every time you encounter a term you don’t know, look up the definition in the glossary and make certain you understand just how the word is used in mathematical talk. To help you do this, we will continue to underline and boldface mathematical terms the first time we use them. This will let you know that the word has a particular meaning in mathematics to which you need to pay attention.

Mathematical definitions are so important because:

1) Definitions provide precise criteria for describing and classifying these ideal objects;

2) Definitions describe relationships among objects; and

3) Definitions give us the power to make mathematical arguments.

Let’s talk a bit more about each of these.

In the *Class Activity* you had the opportunity to make sense of a definition and to use it to classify polygons. Did it surprise you that the definition relied on so many other terms? After all, the idea of a polygon doesn’t seem that complicated. However you will find that your students have many different ideas in mind about polygons. Some will think that solid shapes are polygons. Some might classify shapes with curved edges (like circles) as polygons. In order to be sure that we are all imagining the same ideal objects, we must all have the same definition.

That said, we have to start somewhere when writing our definitions, and that means that not all terms can be precisely defined. In particular, in geometry, we accept the following ideas as undefined: *point*, *line*, *plane*, and *space*. These terms correspond to the 0-dimensional, the 1-dimensional, the 2-dimensional, and the 3-dimensional objects to which the definitions of geometry apply.
Maybe it seems strange that such a fundamental object as a line is an undefined term, but even though we don’t define it, we can understand a line to be a collection of points (also undefined) that obeys a set of rules. We have intuitive ideas about what a point or line is, but we can best understand or talk about points or lines in terms of a model. Useful models of a line include the crease in a sheet of paper, the straight edge where the wall of a room meets the floor, a taut piece of thin string, or the picture below.

Of course each of these models is only a representation of a line. A “true” line has no width at all – only length – and it extends indefinitely in both directions. Like all other mathematical objects, a “true” line is an ideal object – it exists only in our minds.

It may also seem strange that we can’t define a line by saying that it is “straight.” The property of “straightness” is another intuitive idea that carries with it the notion of “shortest distance.” That is, we say that a line is “straight” if it is measuring the shortest distance between points (the “taut string” idea). These intuitive ideas of “straight” work well on flat surfaces (and in the world of Euclidean geometry), but are not as helpful on curved surfaces such as a sphere. What is the shortest distance between two points on a sphere? Find a ball (or an orange or a globe) and a string and have a look.

Two common models for a plane are a flat sheet of paper and the surface of a whiteboard (provided we remember that each is only a portion of the plane which actually extends infinitely in all directions).

A second way that we use definitions is to create relationships between objects. For example, we say that two lines are parallel if they lie in the same plane and do not intersect. This definition helps us understand the relationship between lines that are parallel provided we understand what a plane is and what it means for lines to intersect. Alternatively, we can say that two lines are parallel if they lie in the same plane and do not have any points in common. This second definition incorporates the idea of non-intersection without using a word that may not be known.

We use definitions in a third way when we make arguments. We will talk about this further in the next section, but for now, let’s look at a simple example: Suppose we want to argue that no triangle can also be a square. Since the definition of a triangle states that it is a polygon with exactly three sides and the definition of a square states that it is a polygon with exactly four congruent sides and four right angles, and since three does not ever equal four; we can conclude that it is not ever possible for a triangle to have four sides. So a triangle can never also be a square. Now this argument may seem trivial, but the point here is that we use definitions to make arguments.

*Use the definitions of “parallel” and “perpendicular” to argue that two lines that are parallel can never also be perpendicular.*
Connections to the Elementary Grades

In prekindergarten through grade 2 all students should recognize, name, build, draw, compare, and sort two- and three-dimensional shapes.

NCTM Principles and Standards for School Mathematics, 2000

In recent years most states (including Wisconsin) have adopted common standards for school mathematics. These standards, called the Common Core State Standards (CCSS), prescribe the mathematical content and practices that teachers should address at each grade level. As a future teacher, you will need to know and understand them. The practice standards describe expectations for students across all grade levels. Which of these standards have you experienced so far in this class?

<table>
<thead>
<tr>
<th>Common Core State Standards for Mathematical Practice</th>
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<tbody>
<tr>
<td>Children should ...</td>
</tr>
<tr>
<td>1. Make sense of problems and persevere in solving them.</td>
</tr>
<tr>
<td>2. Reason abstractly and quantitatively.</td>
</tr>
<tr>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
</tr>
<tr>
<td>4. Model with mathematics.</td>
</tr>
<tr>
<td>5. Use appropriate tools strategically.</td>
</tr>
<tr>
<td>6. Attend to precision.</td>
</tr>
<tr>
<td>7. Look for and make use of structure.</td>
</tr>
<tr>
<td>8. Look for and express regularity in repeated reasoning.</td>
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</table>


In this book, we will focus on the content standards related to geometry and measurement, and we will begin now with geometry for children in kindergarten and first grade. Take a minute read them.
In order to help children to distinguish between defining and non-defining attributes, you might ask children to sort shapes into categories. For example, you might ask that they identify all the triangles in the following group of shapes:
What conversations could you have with children regarding this activity?

In what ways might you use it to address the CCSS for kindergarten?

<table>
<thead>
<tr>
<th>CCSS Grade 1: Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reason with shapes and their attributes.</strong></td>
</tr>
<tr>
<td>1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.</td>
</tr>
<tr>
<td>2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.</td>
</tr>
<tr>
<td>3. Partition circles and rectangles into two and four equal shares, describe the shares using the words <em>halves</em>, <em>fourths</em>, and <em>quarters</em>, and use the phrases <em>half of</em>, <em>fourth of</em>, and <em>quarter of</em>. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.</td>
</tr>
</tbody>
</table>

(http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf)

Another idea is to ask children to make up the rule, sort the shapes, and then have other children figure out a rule that will give the same “sort.”

Children sometimes attend to attributes in solving sorting problems that we, as mathematicians, would not pay attention to. Thus activities like these provide opportunities to draw children’s attention to different things. For example, many children would say that this figure ▼ is “upside down” or that these are “different shapes” because they are oriented differently.

Mathematicians would say that the above figures are the same shape. They do not take orientation of two-dimensional shapes into account when deciding if those shapes are “the same.” Below we show a “student sort” from a first grade classroom. Can you figure out Devione’s rule? Is there more than one rule that could give the same sort?
These shapes fit My Rule

These shapes do not fit My Rule

The Common Core State Standards ask that you, as a teacher, also help children to talk about the position of objects, to reason about how objects are composed of other objects, and to recognize whether an object is two-dimensional (flat) or three-dimensional. What are some activities that you might do with children to accomplish these things?
Homework

The only place success comes before work is in the dictionary.
Vince Lombardi

1) Do all the italicized things in the Read and Study section.

2) Do all the italicized things in the Connections section.

3) There are several terms associated with lines that you need to understand and use with proper notation. Take a few minutes to study these.

Suppose we have the three points, $A$, $B$, and $C$. (Notice that mathematicians customarily use capital letters from the beginning of the alphabet to denote points. Sometimes we are talking about enough points that we make it all the way to $Z$, but we almost always start with $A$.)

The line $AB$ is the entire set of points extending forever in both directions. We commonly denote a line as $\overrightarrow{AB}$ and represent $\overrightarrow{AB}$ as shown below (note the arrows at each end indicating that the line continues):

![Line AB](image)

The ray $AB$ is the set of points including $A$ and all the points on the line $AB$ that are on the $B$ side of $A$. The point $A$ is called the vertex of the ray. We commonly denote a ray as $\overrightarrow{AB}$ and represent $\overrightarrow{AB}$ as shown below:

![Ray AB](image)

The line segment $AB$ is the set of points between $A$ and $B$, including both $A$ and $B$, which are called the endpoints of the line segment. We commonly denote a line segment as $\overline{AB}$ and represent $\overline{AB}$ as shown below:

![Line Segment AB](image)

4) Visit the glossary and learn the precise definitions for each of the following terms: square, parallelogram, rectangle, rhombus, and trapezoid. Make sure you can explain the definitions using good mathematical language. Sketch an example of each.
5) Which properties are sufficient to define a rectangle? That is, if a quadrilateral has a particular property, do you know for certain that the quadrilateral must be a rectangle? Explain why your answer is ‘yes’ or ‘no’ in each case.

a) If a quadrilateral has two sets of congruent sides, then it must be a rectangle.
b) If a quadrilateral has opposite angles congruent, then it must be a rectangle.
c) If a quadrilateral has diagonals that bisect each other, then it must be a rectangle.
d) If a quadrilateral has two right angles, then it must be a rectangle.
e) If a quadrilateral has congruent diagonals, then it must be a rectangle.
f) If a quadrilateral has perpendicular diagonals, then it must be a rectangle.
g) If a quadrilateral has two sets of parallel sides and one right angle, then it must be a rectangle.
h) If a quadrilateral has two sets of congruent sides and one right angle, then it must be a rectangle.

6) Study the following examples and form a definition of each of these terms: convex and concave, in your own words. Then look up the mathematical definitions in the glossary. Explain the mathematical definitions in your own words.

7) A third grade class we observed was learning about parallel lines. The teacher explained that parallel lines are lines in the plane that have no common points. Then she drew the picture below and asked the children whether the lines shown were parallel or not.

Several children argued that they were parallel. Why might they have said that, and what would you say to them as their teacher?
Class Activity 3: Get it Straight

Go back a little to leap further.

John Clarke

1) Here is an activity to help your upper elementary children make the conjecture that taken together, the angles of any triangle can form a straight angle.

Each group should cut out a large obtuse triangle, a large acute triangle and a large right triangle. For each triangle, label the vertex angles (in any order) #1, #2 and #3. Then, for each triangle, tear off the three corners and put them together so that the angles are adjacent. Do this, and discuss what children might learn.

Did this activity prove that the angles of a triangle always can form a straight angle? Why or why not?

2) Again, as in part 1), each group should create a large obtuse triangle, a large acute triangle and a large right triangle and, for each triangle, label the vertex angles #1, #2 and #3. But instead of tearing off the corners, this time have one person use a protractor to carefully measure each angle labeled #1, another person measure each angle labeled #2, and another person measure each angle labeled #3 (each without looking at the others’ measurements). Was the total angle measure for each triangle 180 degrees? Should it have always been? Explain any discrepancies.

(continued on the next page)
3) Our explorations in parts 1) and 2) might have convinced you that the sum of the vertex angles of any triangle is the same as a straight angle, but mathematicians do not consider either of those demonstrations to be a proof. Why not, do you think?

Now we are going to learn to mathematically prove that sum of the vertex angles of any triangle is the same as a straight angle.

Step 1. Start with any triangle:

Now we’ll create a line through one vertex that is parallel to the opposite side of the triangle and label all the angles so we can talk about them. We know we can always do this because it is an axiom of plane geometry that through a point not on a line there can be drawn one (and only one) line parallel to the given line.

By the way, it shouldn’t be obvious to you why we’ve decided to create this parallel line; it just turns out to give us a great way to begin our proof. In Step 2, we are going to prove that \( \angle 1 \) is congruent to \( \angle 5 \), and that \( \angle 2 \) is congruent to \( \angle 4 \). So for now, supposing this to be true, argue that angles 1, 2, and 3 would combine to form a straight angle.

(continued on the next page)
Step 2. This is the only missing piece of our proof. Look up the definitions of a transversal and of alternate interior angles. Can you see that $\angle 1$ and $\angle 5$ are alternate interior angles? What is the corresponding transversal?

Can you see that $\angle 2$ and $\angle 4$ are alternate interior angles? What is the corresponding transversal?

Now argue that if two parallel lines are cut by a transversal, then the alternate interior angles are congruent. You may assume that if two parallel lines are cut by a transversal, then the interior angles on the same side of the transversal form a straight angle. (Note: This is a big thing to assume. It is one of Euclid’s fundamental assumptions about geometry.)
Read and Study

I argue very well. Ask any of my remaining friends. I can win an argument on any topic, against any opponent. People know this and steer clear of me at parties. Often, as a sign of their great respect, they don’t even invite me.

Dave Barry

Mathematical thinking always involves reasoning and making arguments, and we have a whole vocabulary for describing that process. In this section, we highlight the terms we mathematicians use to describe facets of doing mathematics. These are important. Make sure you understand them.

1) An **axiom** is a statement that we agree to accept without proof. It is an assumption or starting point. (Note: Another word for *axiom* is *postulate*.)

2) **Inductive reasoning** is coming to a conclusion based on examples. For example, I observe that 3, 5 and 7 are all prime numbers. Now, based on these examples I might reason (incorrectly, by the way) that all odd numbers are prime. Or I might notice that the sun rose day before yesterday, it rose yesterday, it rose today. So I might conclude that the sun will rise tomorrow. This is inductive reasoning.

3) **Deductive reasoning** is coming to conclusion based on logic. For example, I will argue deductively that the sun will come up tomorrow: The earth is caught in the sun’s gravity so it won’t float away, and the earth is spinning. We are here on the earth and so when our part of the earth turns toward the sun, we say it “comes up.” As long as no catastrophe occurs to change these facts, the sun will rise tomorrow. We’ll give you another (more mathematical) example in a minute.

4) A **conjecture** is a hypothesis or a guess about what is true. For example, after some experience with circles, a student might conjecture that two **intersecting** circles always share two **distinct** points in common. Conjectures are often made based on inductive reasoning.

5) A **counterexample** is a specific example that shows a conjecture is false. For example, the two circles below are tangent (they share exactly one point in common) and so the above conjecture is shown to be false by the counterexample shown here:
6) **proof:** a mathematical proof consists of a deductive argument that establishes the truth of a claim. (Note: *By truth* we mean truth in the context of the mathematical world that is created by the axioms. Something is *true* if it is a logical implication of the axioms.)

7) **theorem:** a theorem is a mathematical statement that has been proved to be true. For example, it is a theorem that vertical angles are congruent. You proved this in the *Class Activity.* (Note: Another word for *theorem* is *proposition.*)

We are going to use what you did in the *Class Activity* to highlight the meaning of some of these terms.

First, you tore apart a variety of triangles and arranged them to see that each appeared to have a straight angle (180 degrees) of vertex angles. This was *inductive reasoning,* because you were testing examples of triangles to see what seemed true about their vertex angle measure.

At this point it would have been reasonable to *conjecture* that the sum of the vertex angles of a triangle is 180 degrees.

Then you were asked to make an argument using definitions, axioms, and logic that you were correct. In other words you gave a *proof* that the vertex angles of a triangle sum to 180 degrees, and now that conjecture is called a *theorem.*

One of the reasons that geometry class (remember tenth grade?) has traditionally focused on proof is that the axioms of geometry are easier to state than the axioms of arithmetic. But proof is part of *all* mathematics. Don’t worry, we are not planning to focus on two-column proofs or an axiomatic development of geometry, but we would be remiss if we didn’t at least state the original axioms (assumptions) of plane geometry.

The axioms were first made explicit by Euclid, a Greek mathematician who lived and worked at the *Academy* in Alexandria, Egypt. He is best known for writing a 13-volume book of mathematics called *The Elements* - the second most published book in the world. It has been used as a mathematics textbook for over 2000 years. In the first two volumes of this work, he developed all of the (then) known theorems about two-dimensional (plane) geometry starting with just five axioms:

**The Axioms of Euclidean Geometry:**

1. A unique straight line segment can be drawn from any point to any other point.
2. A line segment can be extended to produce a unique straight line.
3. A unique circle may be described with a given center and radius.
4. All right angles are equal to each other.

5a. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

5b. Through a point not on a line there can be drawn exactly one line parallel to the given line.

Upon assuming axioms 1-4, axioms 5a and 5b are equivalent. Axiom 5a is Euclid’s version, and 5b is a more modern interpretation known as Playfair’s Axiom. Look in this book’s appendix and you will find the axioms (postulates) and theorems (propositions) from Book 1 of Euclid’s Elements, written more than 2,000 years ago. Axiom 5a is his fifth postulate. You may find it interesting that until the late 1800s, many mathematicians thought Euclid’s fifth axiom was redundant – that it already had to be true if axioms 1-4 were assumed to be true – but it has since been proven that those mathematicians were wrong.

Notice that each of the axioms describes something that can be constructed with the exception of the fourth. Axiom 4 says that the Euclidean plane is, in some sense, uniform (no distortions). Namely, it says that wherever you construct a perpendicular lines on the plane (and so form four angles), those angles will all have the same measure.

It turns out that Euclid made some implicit assumptions that he should have stated as additional axioms in order to do geometry rigorously – and so modern mathematicians have extended his list of axioms. But you don’t need to worry about those technicalities in this course.

*Sketch a picture of axioms 1 – 3 and 5 to help you make sense of each. Then, describe how you can create an equilateral triangle by following Euclid’s axioms.*

We also want to note that Euclid often used the word “equal” when we would use the word “congruent.” Today’s mathematicians use “equal” when they want to compare two numbers. So we might say that ½ is equal to 0.5. We use the word “congruent” when we want to say that two objects (like two triangles or two segments) are the same size and shape. The basic idea
here is that two objects are congruent in the case where if one object was moved to lie on top of the other object, they would correspond exactly. We will do a more careful job of defining “congruent” later.

While proving theorems is not the focus of this course, we may occasionally ask that you try to prove a Euclidean theorem. When we do, you should turn to the Appendix (starting on p. 168) where Euclid’s postulates (axioms) and propositions are listed and find it. Then you are free to use (assume) any postulate and any proposition listed before the one you are trying to prove. For example, say you want to prove that in an isosceles triangle, the base angles are congruent. Go to the Appendix (really do it) and see if you can find that theorem. Then come right back here.

Let’s end this section with the big idea: Geometry in the plane arises from some intuitive ideal objects, their mathematical definitions, these five axioms, and a lot of deductive reasoning. In fact, a second possible definition of geometry is this: an axiomatic system about ideal objects called “points,” collections of points called “lines,” and the relationships between points and lines.

Connections to the Elementary Grades

Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics; to make and investigate mathematical conjectures; to develop and evaluate mathematical arguments and proofs; and to select and use various types of reasoning and methods of proof.


It is important that you provide children in your future classes with the opportunities to really do mathematics. They too need to use inductive reasoning, make conjectures, look for counterexamples, and make deductive arguments. In order to understand better what you should expect from children, read the following from the discussion of the Reasoning and Proof standard for the PreK – 2 grade band found in NCTM Principles and Standards for School Mathematics, 2000, pages 122 – 125. Notice their use of mathematical language.

What should reasoning and proof look like in prekindergarten through grade 2?

The ability to reason systematically and carefully develops when students are encouraged to make conjectures, are given time to search for evidence to prove or disprove them, and are expected to explain and justify their ideas. In the beginning, perception may be the predominant method of determining truth: nine markers spread far apart may be seen as “more” than eleven markers placed close together. Later, as students develop their
mathematical tools, they should use empirical approaches such as matching the collections, which leads to the use of more-abstract methods such as counting to compare the collections. Maturity, experiences, and increased mathematical knowledge together promote the development of reasoning throughout the early years.

Creating and describing patterns offer important opportunities for students to make conjectures and give reasons for their validity, as the following episode drawn from classroom experience demonstrates.

The student who created the pattern shown in figure 4.27 proudly announced to her teacher that she had made four patterns in one. “Look,” she said, “there’s triangle, triangle, circle, circle, square, square. That’s one pattern. Then there’s small, large, small, large, small, large. That’s the second pattern. Then there’s thin, thick, thin, thick, thin, thick. That’s the third pattern. The fourth pattern is blue, blue, red, red, yellow, yellow.”

Her friend studied the row of blocks and then said, “I think there are just two patterns. See, the shapes and colors are an AABBC pattern. The sizes are an ABABAB pattern. Thick and thin is an ABABAB pattern, too. So you really only have two different patterns.” The first student considered her friend’s argument and replied, “I guess you’re right—but so am I!”

Being able to explain one’s thinking by stating reasons is an important skill for formal reasoning that begins at this level.

Finding patterns on a hundred board allows students to link visual patterns with number patterns and to make and investigate conjectures. Teachers extend students’ thinking by probing beyond their initial observations. Students frequently describe the changes in numbers or the visual patterns as they move down columns or across rows. For example, asked to color every third number beginning with 3 (see fig. 4.28), different students are likely to see different patterns: “Some rows have three and some have four,” or “The pattern goes sideways to the left.” Some students, seeing the diagonals in the pattern, will no longer count by threes in order to complete the pattern. Teachers need to ask these students to
explain to their classmates how they know what to color without counting. Teachers also extend students’ mathematical reasoning by posing new questions and asking for arguments to support their answers. “You found patterns when counting by twos, threes, fours, fives, and tens on the hundred board. Do you think there will be patterns if you count by sixes, sevens, eights, or nines? What about counting by elevens or fifteens or by any numbers?” With calculators, students could extend their explorations of these and other numerical patterns beyond 100.

![Fig. 4.28. Patterns on a hundred board](p. 123)

Students’ reasoning about classification varies during the early years. For instance, when kindergarten students sort shapes, one student may pick up a big triangular shape and say, “This one is big,” and then put it with other large shapes. A friend may pick up another big triangular shape, trace its edges, and say, “Three sides—a triangle!” and then put it with other triangles. Both of these students are focusing on only one property, or attribute. By second grade, however, students are aware that shapes have multiple properties and should suggest ways of classifying that will include multiple properties.

By the end of second grade, students also should use properties to reason about numbers. For example, a teacher might ask, “Which number does not belong and why: 3, 12, 16, 30?” Confronted with this question, a student might argue that 3 does not belong because it is the only single-digit number or is the only odd number. Another student might say that 16 does not belong because “you do not say it when counting by threes.” A third student might have yet another idea and state that 30 is the only number “you say when counting by tens.”

Students must explain their chains of reasoning in order to see them clearly and use them more effectively; at the same time, teachers should model mathematical language that the
students may not yet have connected with their ideas. Consider the following episode, adapted from Andrews (1999, pp. 322–23):

One student reported to the teacher that he had discovered “that a triangle equals a square.” When the teacher asked him to explain, the student went to the block corner and took two half-unit (square) blocks, two half-unit triangular (triangle) blocks, and one unit (rectangle) block (shown in fig. 4.29). He said, “If these two [square half-units] are the same as this one unit and these two [triangular half-units] are the same as this one unit, then this square has to be the same as this triangle!”

Even though the student’s wording—that shapes were “equal”—was not correct, he was demonstrating powerful reasoning as he used the blocks to justify his idea. In situations such as this, teachers could point to the faces of the two smaller blocks and respond, “You discovered that the area of this square equals the area of this triangle because each of them is half the area of the same larger rectangle.”

What should be the teacher’s role in developing reasoning and proof in prekindergarten through grade 2?

Teachers should create learning environments that help students recognize that all mathematics can and should be understood and that they are expected to understand it. Classrooms at this level should be stocked with physical materials so that students have many opportunities to manipulate objects, identify how they are alike or different, and state generalizations about them. In this environment, students can discover and demonstrate
general mathematical truths using specific examples. Depending on the context in which events such as the one illustrated by figure 4.29 take place, teachers might focus on different aspects of students’ reasoning and continue conversations with different students in different ways. Rather than restate the student’s discovery in more-precise language, a teacher might pose several questions to determine whether the student was thinking about equal areas of the faces of the blocks, or about equal volumes. Often students’ responses to inquiries that focus their thinking help them phrase conclusions in more-precise terms and help the teacher decide which line of mathematical content to pursue.

Teachers should prompt students to make and investigate mathematical conjectures by asking questions that encourage them to build on what they already know. In the example of investigating patterns on a hundred board, for instance, teachers could challenge students to consider other ideas and make arguments to support their statements: “If we extended the hundred board by adding more rows until we had a thousand board, how would the skip-counting patterns look?” or “If we made charts with rows of six squares or rows of fifteen squares to count to a hundred, would there be patterns if we skip-counted by twos or fives or by any numbers?”

Through discussion, teachers help students understand the role of nonexamples as well as examples in informal proof, as demonstrated in a study of young students (Carpenter and Levi 1999, p. 8). The students seemed to understand that number sentences like $0 + 5869 = 5869$ were always true. The teacher asked them to state a rule. Ann said, “Anything with a zero can be the right answer.” Mike offered a counterexample: “No. Because if it was $100 + 100$ that’s 200.” Ann understood that this invalidated her rule, so she rephrased it, “I said, umm, if you have a zero in it, it can't be like 100, because you want just plain zero like $0 + 7 = 7$.”

The students in the study could form rules on the basis of examples. Many of them demonstrated the understanding that a single example was not enough and that counterexamples could be used to disprove a conjecture. However, most students experienced difficulty in giving justifications other than examples.

From the very beginning, students should have experiences that help them develop clear and precise thought processes. This development of reasoning is closely related to students’ language development and is dependent on their abilities to explain their reasoning rather than just » give the answer. As students learn language, they acquire basic logic words, including not, and, or, all, some, if...then, and because. Teachers should help students gain familiarity with the language of logic by using such words frequently. For example, a teacher could say, “You may choose an apple or a banana for your snack” or “If you hurry and put on your jacket, then you will have time to swing.” Later, students should use the words modeled for them to describe mathematical situations: “If six green pattern blocks cover a yellow hexagon, then three blues also will cover it, because two greens cover one blue.”
Sometimes students reach conclusions that may seem odd to adults, not because their reasoning is faulty, but because they have different underlying beliefs. Teachers can understand students’ thinking when they listen carefully to students’ explanations. For example, on hearing that he would be “Star of the Week” in half a week, Ben protested, “You can’t have half a week.” When asked why, Ben said, “Seven can’t go into equal parts.” Ben had the idea that to divide 7 by 2, there could be two groups of 3, with a remainder of 1, but at that point Ben believed that the number 1 could not be divided.

Teachers should encourage students to make conjectures and to justify their thinking empirically or with reasonable arguments. Most important, teachers need to foster ways of justifying that are within the reach of students, that do not rely on authority, and that gradually incorporate mathematical properties and relationships as the basis for the argument. When students make a discovery or determine a fact, rather than tell them whether it holds for all numbers or if it is correct, the teacher should help the students make that determination themselves. Teachers should ask such questions as “How do you know it is true?” and should also model ways that students can verify or disprove their conjectures. In this way, students gradually develop the abilities to determine whether an assertion is true, a generalization valid, or an answer correct and to do it on their own instead of depending on the authority of the teacher or the book.

_NCTM Principles and Standards for School Mathematics_

**Homework**

_Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions._

_Eric Temple Bell_

1) Do all the italicized things in the _Read and Study_ section. Write a description of the strategies you used in solving the problem of creating an equilateral triangle using only Euclid’s five axioms.

2) Which of the children from the _Connections_ reading are using deductive reasoning and which are using inductive reasoning? Explain.

3) According to the NCTM, what is the teacher’s role in promoting reasoning among children in the early elementary grades?

4) Explain why _vertical angles_ formed by intersecting lines are the same.
5) Have a look at Proposition 32 of the appendix. Do you see that it is the theorem you proved in the Class Activity? Which propositions that come before #32 did you use in the proof?

6) Decide if each of the following statements about Euclidean lines and angles is true or false by exploring examples and looking up definitions. If you decide that a statement is true, write a deductive argument based on axioms and definitions. If you decide the statement is false, give a counterexample or a deductive argument that it is not possible.

   a) Any two distinct lines will either intersect in exactly one point or they will be parallel.
   b) There exist two acute angles which are supplementary.
   c) Every two lines that are each parallel to a third line must be parallel to each other.
   d) Every two lines that are each perpendicular to a third line will be perpendicular to each other.
   e) Every two acute angles must be complementary.
   f) There exist two opposite sides in any trapezoid which are parallel.
   g) If one of two supplementary angles is acute, the other angle must be obtuse.

7) Go online to bridges.mathlearningcenter.org and find the *Bridges in Mathematics Grade 2 Teachers Guide* (your professor should have a code for you to view this). Spend 10-15 minutes looking through Unit 6 Module 1. Carefully read through the activity Guess My Shape, then make up your own riddle for your second grade students to solve.

8) We have a conjecture. Every rectangle is a parallelogram. Give an inductive argument that this conjecture is true. Now, since mathematicians are not satisfied until they have a deductive argument, give one of those.

9) We have another conjecture! Every rectangle is a square. Is this conjecture true or false? If it is true, give a deductive argument. If it is false, give a counterexample.
10) Here are some more of the Common Core State Standards related to reasoning about shapes. In what ways do HW problems 4) – 7) address these standards?

**CCSS Grade 3: Reason with shapes and their attributes.**

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.


11) Draw a **Venn diagram** showing the relationship between all the various quadrilaterals we have studied. How does this fit with the Common Core State Standards described below?

**CCSS Grade 5: Classify two-dimensional figures into categories based on their properties.**

1. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*

2. Classify two-dimensional figures in a hierarchy based on properties.

Class Activity 4: All the Angles

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.

Albert Einstein

1) Here is the description – from the Common Core State Standards for grade four – of how to think about measuring angles using degrees. Read it carefully and sketch a picture to help your group make sense of their explanation.

An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a “one-degree angle,” and can be used to measure angles.

An angle that turns through $n$ one-degree angles is said to have an angle measure of $n$ degrees.

2) Make sure everyone in your group can use their protractor to measure (in degrees) the angle indicated below:

(continued on the next page)
3) Find a formula for the sum of the measures of the **vertex angles** of an \( n \)-gon (a polygon with \( n \) sides). You may need to collect some data. In the end, make a mathematical argument that the formula you find will work for a convex polygon of any number of sides. (The formula that you developed will work for concave polygons as well, but the argument is trickier, so we are not asking you to justify it at this time.)
Read and Study

The knowledge of which geometry aims is the knowledge of the eternal.
Plato

A regular polygon is one in which all of the line segments are congruent and all of the vertex angles are also congruent. Sketch a regular triangle and a regular quadrilateral. Is the below rhombus a regular quadrilateral? Explain.

Polygons are often named for the number of sides they contain. In fact, the prefix “poly” means “many” and the root “gon” means “side.” So “polygon” means “many-sided” figure. In order to name polygons, you’ll need to know the following prefixes:

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Number</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>Five</td>
<td>5</td>
<td>“penta”</td>
</tr>
<tr>
<td>Six</td>
<td>6</td>
<td>“hexa”</td>
</tr>
<tr>
<td>Seven</td>
<td>7</td>
<td>“hepta”</td>
</tr>
<tr>
<td>Eight</td>
<td>8</td>
<td>“octa”</td>
</tr>
<tr>
<td>Nine</td>
<td>9</td>
<td>“nona”</td>
</tr>
<tr>
<td>Ten</td>
<td>10</td>
<td>“deca”</td>
</tr>
<tr>
<td>Twelve</td>
<td>12</td>
<td>“dodeca”</td>
</tr>
</tbody>
</table>

For example, five-sided polygons are called pentagons.

Here are a couple examples of convex pentagons.

Draw an example of a concave pentagon.

Mathematicians identify three types of angles in a polygon: the vertex angles, the central angles, and the exterior angles. Study the hexagon in the following diagram and then explain the difference between the three types of angles in your own words. The bold arcs mark the vertex angles; the thin solid arcs mark the central angles; and the dashed arcs mark the exterior angles. Point G is any interior point.
Think about the sum of the six central angles formed at point $G$. This sum will always be 360° regardless of where in the interior point $G$ is, regardless of how many sides the polygon has, and regardless of whether or not the polygon is regular. Why?

We also claim that the sum of the exterior angles of any polygon is 360°. We will ask you to make a mathematical argument to support this claim in the homework section.

The case of the sum of the vertex angles of a polygon is the interesting case. In the Class Activity you found that this sum does depend on the number of sides in the polygon. Does it depend on whether the polygon is regular? Explain.

**Connections to the Elementary Grades**

*Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.*

*CCSS, p. 6*

Angles are notoriously difficult ideal objects for children. On the one hand, they are often defined as a figure formed by two rays with a common endpoint (and, in fact, that is exactly how we have defined them). On the other hand, what is important in measuring an angle is its degree of turn.
When you talk about angles with children, we suggest that you always use your hand or arm to show the sweep of the angle in addition to showing the static picture.

Children in grade four learn to use a protractor to measure angles in degrees and to accurately estimate the measure of angles. Below you will find the CCSS related to angle measure in this grade. Read them carefully.

**CCSS Grade 4: Geometric measurement: understand concepts of angle and measure angles.**

1. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
   
   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a “one-degree angle,” and can be used to measure angles.

   b. An angle that turns through $n$ one-degree angles is said to have an angle measure of $n$ degrees.

2. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

3. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

**CCSS Grade 4: Draw and identify lines and angles, and classify shapes by properties of their lines and angles.**

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

Go online to bridges.mathlearningcenter.org and find the *Bridges in Mathematics Grade 4 Teachers Guide*. Spend 10-15 minutes looking through Unit 5 Module 1. *Print out and then work through the Measuring Pattern Block Angles activity.* (Note that in this activity students are not using protractors to measure the angles. You also will not need to use a protractor.)

### Homework

*Energy and persistence conquer all things.*

*Benjamin Franklin*

1) Do all the italicized things in the *Read and Study* section.

2) Do all the italicized things in the *Connections* section.

3) Children in your class may say that $\angle ABC$ is smaller than $\angle DEF$. Is this true? As their teacher, what would you say to these children?

4) Using the results of the class activity, find the measure of one vertex angle in an equilateral triangle, a square, a regular pentagon, a regular hexagon, a regular octagon, a regular decagon, and a regular dodecagon.

5) A student claims that the sum of the vertex angles of a hexagon is $6 \times 180$ because each triangle has 180 degrees of angles measure and she has shown that 6 triangles to make up the hexagon. What will you say as her teacher?

6) In the *Read and Study* section we claim that the sum of the exterior angles of *any* polygon is always $360^\circ$. Make a mathematical argument to support this claim.
**Class Activity 5: A Logical Interlude**

*Equations are just the boring part of mathematics. I attempt to see things in terms of geometry.*

*Stephen Hawking*

In the picture below, each card has a color on one side and a shape on the other side. Which card(s) would you have to turn over to be sure that the following statement is true?

**If a card is red on one side, then it has a square on the other side.**
**Read and Study**

*When introduced at the wrong time or place, good logic may be the worst enemy of good teaching.*

*George Polya*

*The American Mathematical Monthly, v. 100, no 3.*

Here are two theorems about parallel lines that can be proved from Euclid’s axioms.

1) If two lines are cut by a transversal and the alternate interior angles are congruent, then the lines are parallel.

2) If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

*Draw a sketch to make sure that you see what each of these has to say.*

Notice that they both have an ‘if-then’ statement form. Lots of mathematical theorems are like this. When a theorem is stated in ‘if-then’ form, whatever follows the ‘then’ is always true whenever the conditions stated in the ‘if’ part are met. You can think of an ‘if-then’ statement as a promise that is kept unless the ‘if’ part is true and the ‘then’ part is not.

We call statement 2) the *converse* of statement 1). These two theorems may sound the same to you, but they are *not* saying the same thing. To help you see this, let’s change the context. Here is another pair of statements in which the second is the converse of the first (and vice versa):

3) If I live in Chicago, then I live in Illinois.

4) If I live in Illinois, then I live in Chicago.

*Think about each of the statements. Which of these is true?*

Since one is true and the other false, these two statements cannot be saying the same thing. In other words, the converse of a statement is a logically different statement from the original statement.

Now *here* is a statement that is logically equivalent to the original statement made in 3) above. It is called the *contrapositive* of statement 3).

5) If I do not live in Illinois, then I do not live in Chicago.

This is essentially what you found when you worked on the *Class Activity. Write the contrapositive to statement 4) above.*
It is helpful to know that a statement and its contrapositive are equivalent because that means that you can prove a statement by proving its contrapositive.

**Connections to the Elementary Grades**

*The beginning of knowledge is the discovery of something we do not understand.*

Frank Herbert

Now that we have talked a bit about doing mathematics, we want to show you that the Common Core State Standards require that children also do mathematics. *Give an example of something you have done in class so far this term that meets each of these standards.*

**Common Core State Standards for Mathematical Practice**

Children should …

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Homework

*A multitude of words is no proof of a prudent mind.*

*Thales*

1) Do all the italicized things in the *Read and Study* section.

2) Do all the italicized things in the *Connections* section.

3) Decide if each of the following statements is true or false. If true, give a mathematical explanation. If false, give a counterexample.

   a) If a quadrilateral is a square, then it is a rectangle.
   b) If a quadrilateral has a pair of parallel sides, then it must have a pair of opposite sides that are congruent.
   c) If the diagonals of a quadrilateral are perpendicular to each other, then the quadrilateral is a rhombus.
   d) If a quadrilateral has one right angle, then all of its angles must be right angles.
   e) Write the converse of each of the statements in a) – d) above. Which are true?
   f) Write the contrapositive of each of the statements in a) – d) above. Which are true?

4) Go online to bridges.mathlearningcenter.org and find the *Bridges in Mathematics Grade 4 Teachers Guide*. Spend 10-15 minutes looking through Unit 5 Module 4. Then work through the Clock Angles & Shape Sketches from the homelink section. How is deductive reasoning used to solve these problems?
Class Activity 6: Enough is Enough

I learned very early the difference between knowing the name of something and knowing something.

Richard Feynman

Suppose you are given some information about a triangle ABC. In which of the following cases will the information be enough to allow you to determine the exact size and shape of the triangle? If you have enough information, draw a triangle guaranteed to be exactly the same size and shape as \( \triangle ABC \). If you do not have enough information, describe the problem you encounter in attempting to draw \( \triangle ABC \).

You will need to use a ruler to measure lengths in centimeters (cm) and a protractor to measure the angles in degrees.

a) \( \overline{AB} = 4 \text{ cm} \) and \( \overline{BC} = 5 \text{ cm} \)

b) \( \overline{AB} = 8 \text{ cm} \) and \( \overline{AC} = 6 \text{ cm} \) and \( \angle BAC = 45^\circ \)

c) \( \overline{AB} = 8 \text{ cm} \) and \( \overline{AC} = 7 \text{ cm} \) and \( \angle ABC = 45^\circ \)

d) \( \angle ABC = 75^\circ \), \( \angle BCA = 80^\circ \), and \( \angle CAB = 25^\circ \)

e) \( \overline{BC} = 7 \text{ cm} \), \( \overline{AC} = 8 \text{ cm} \), and \( \overline{AB} = 9 \text{ cm} \)

f) \( \overline{AB} = 9 \text{ cm} \), \( \overline{BC} = 3 \text{ cm} \), and \( \overline{AC} = 4 \text{ cm} \)

g) \( \overline{AB} = 7 \text{ cm} \), \( \angle ABC = 25^\circ \), and \( \angle BAC = 105^\circ \)

h) \( \overline{BC} = 11 \text{ cm} \), \( \angle ABC = 75^\circ \), and \( \angle BAC = 40^\circ \)
Now it is time to talk about the idea of congruence. You were working with this idea in the Class Activity.

For now we will use the following definition: two geometric objects are congruent if they can be moved so that they coincide (sit on top of one another and fit exactly). We will make this definition more precise later in the book.

The idea of congruence is related to the idea of equality, but it is not the same thing. Congruence is a relationship between objects whereas equality is a relationship between numbers.

We would say two line segments are congruent (coincide), and we would say that the measures of their lengths (numbers) are equal.

We would say that two angles are congruent (coincide), and we would say their measures in degrees (numbers) are equal.

We do not use the equals sign (=) for congruence. Instead we have a special symbol (\(\cong\)) to say that \(\overline{AB}\) is congruent to \(\overline{CD}\), \((\overline{AB} \cong \overline{CD})\). Be sure to use \((\cong)\) when you mean congruence and = when you mean equality.

In the Class Activity you investigated conditions that will ensure that two triangles are congruent. You found that having two pairs of congruent sides is not sufficient, but that having three pairs of congruent sides does guarantee the triangles coincide.

The case of angles is more complex. Having three pairs of congruent angles is not sufficient information, but if we have two pairs of congruent angles and one pair of congruent sides, we do get congruent triangles. And then there is the case where we have two pairs of congruent sides and one pair of congruent angles – sometimes we have congruent triangles and sometimes not – it makes a difference whether or not the angles in question are the angles between the two pairs of congruent sides.

Euclid compiled all of this information about when triangles are congruent into these four theorems. Notice that each of these theorems is in the “if-then” statement form. Read each one carefully – make certain you understand all the terms and can explain each one in your own words. Then sketch a picture to illustrate what each one is saying.
1) **Angle-Side-Angle Triangle Congruence (ASA):** If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

2) **Side-Angle-Side Triangle Congruence (SAS):** If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

3) **Side-Side-Side Triangle Congruence (SSS):** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

4) **Angle-Angle-Side Triangle Congruence (AAS):** If two angles and the side opposite one of them in one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent.

You might have noticed that Theorem 4) is redundant to Theorem 1), in light of a fact we’ve already established about triangles in a previous section. *To what fact are we referring?*

*We want you to take time to relate these theorems to the cases you investigated in the Triangle Exploration Activity. In fact, we are going to give you space here to revisit each set of conditions and decide which, if any, of the above four theorems apply to the given cases. Really do this.*

a) \( \overline{AB} = 4 \text{ cm} \) and \( \overline{BC} = 5 \text{ cm} \)

b) \( \overline{AB} = 8 \text{ cm} \) and \( \overline{AC} = 6 \text{ cm} \) and \( \angle BAC = 45^\circ \)

c) \( \overline{AB} = 8 \text{ cm} \) and \( \overline{AC} = 7 \text{ cm} \) and \( \angle ABC = 45^\circ \)

d) \( \angle ABC = 75^\circ, \angle BCA = 80^\circ, \) and \( \angle CAB = 25^\circ \)

e) \( \overline{BC} = 7 \text{ cm}, \overline{AC} = 8 \text{ cm}, \) and \( \overline{AB} = 9 \text{ cm} \)

f) \( \overline{AB} = 9 \text{ cm}, \overline{BC} = 3 \text{ cm}, \) and \( \overline{AC} = 4 \text{ cm} \)

g) \( \overline{AB} = 7 \text{ cm}, \angle ABC = 25^\circ, \) and \( \angle BAC = 105^\circ \)

h) \( \overline{BC} = 11 \text{ cm}, \angle ABC = 75^\circ, \) and \( \angle BAC = 40^\circ \)

Okay, now that you know the triangle congruence theorems, let’s take a look at some of the other nice theorems about triangles:
5) The sum of the angle measures in any triangle is 180 degrees. (You already proved this.)

6) In a triangle, angles opposite congruent sides are congruent.

7) In a triangle, sides opposite congruent angles are congruent.

Theorems 6) and 7) are often called the Isosceles Triangle Theorems – and they are quite useful. *Draw a sketch of each to be certain you understand what they say.*

Notice that 6) and 7) are not theorems about two different triangles being congruent. Both theorems talk about a single triangle in which either two sides of that triangle are congruent or two angles of that triangle are congruent.

Here is one more idea that is commonly used as part of triangle congruence proofs:

8) **Corresponding parts of congruent triangles are congruent.**

We will use theorem 8) almost every time we make an argument involving triangles from now on. Since, by our definition of congruent objects, two congruent triangles coincide, every pair of corresponding parts or measurements of an attribute must be identical. So, in two congruent triangles, the smallest angles (if there are smallest angles) are congruent, and the longest sides (if there are longest sides) are congruent, and so on. You may (fondly) recall the anagram for this theorem, CPCTC, from a high school geometry course.
Home computers are being called upon to perform many new functions, including the consumption of homework formerly eaten by the dog.

Doug Larson

1) Do all the italicized things in the Read and Study section.

2) Study each bold and underlined term used in this section. This means you should be able to explain the definition using good mathematical language and that you should be able make examples and non-examples of each term.

3) For each of Theorems 1), 2), 3), 5), 6), and 7) of Read and Study, find Euclid’s corresponding proposition in the Appendix. (Euclid did not state Theorems 4) or 8).)

4) At which step do you know enough to draw a triangle that is congruent to the one we are describing? Explain your answer.
   
   I. One of the sides is 8 cm long.
   II. One of the sides is 4 cm long.
   III. The angle between the sides mentioned above is 60 degrees.
   IV. The triangle has a 90 degree angle.

5) At which step do you know enough to draw a triangle that is congruent to the one we are describing? Explain your answer.

   I. One of the sides is 3 cm long and another is 7 cm long.
   II. The angle between the 7 cm side and the unknown side is 20 degrees.
   III. The unknown side is the longest side.
   IV. The triangle has an obtuse angle.

6) At which step do you know enough to draw a triangle that is congruent to the one we are describing? Explain your answer.

   I. One of the angles measures 140 degrees.
   II. Another of my angles measures 25 degrees.
   III. One of my sides measures 7 cm.
   IV. My longest side measures 7 cm.

7) Make a mathematical argument that a triangle can only have one obtuse angle.
8) Make a mathematical argument that the two acute angles of a right triangle are complementary.

9) Make a mathematical argument for Theorem 7), that in a single triangle, the angles that are opposite the congruent sides must be congruent.

10) Make a mathematical argument that two acute angles of an isosceles right triangle are each 45°.

11) Make a mathematical argument that each angle in an equilateral triangle is 60°.
Summary of Big Ideas from Chapter One

Hey! What’s the big idea?

Sylvester

- Geometry is a study of ideal objects – not real objects.

- Definitions allow us to name and categorize ideal objects, to create relationships between objects, and to make arguments about the properties of objects.

- One definition of geometry is that it is the study of ideal shapes and their properties, of the patterns those shapes can form, and of the actions on those shapes that preserve their properties.

- A second definition is that geometry is an axiomatic system about objects called “points,” collections of points called “lines,” and the relationships between points and lines.

- Mathematical thinking also involves deductive reasoning and making arguments – this means using logic as well as using definitions and axioms.

- Congruence is an important relationship between geometric objects. We say two objects are congruent if they coincide when placed on top of each other.

- In Euclidean geometry the sum of the angles in a triangle is always 180 degrees and you can explain why this is so.

- You will represent the mathematical community for your students. They will look to you to understand what we mathematicians do and how we think. Your students will try to discern the meanings that you, your curriculum materials, and other students give to ideas, strategies, and symbols through their participation in doing mathematics in your classroom. You must be aware that talking about the meanings of words, ideas, and symbols is an important part of your role as a teacher; and you must be transparent and careful in your use of words, symbols, and notation when you are teaching mathematics.
Chapter Two

Transformations, Tessellations, and Symmetries
Class Activity 7: Slides

For the things of this world cannot be made known without a knowledge of mathematics.

Roger Bacon

Think of a translation as a motion which “slides” the entire plane in one direction a particular distance. In order to translate an object we must know how far to slide it and we must know the direction to use. These two pieces of information are usually given to us in the form of a translation vector (also called a translation arrow).

1) On the square grid below you are given translation vector \( RS \) and several geometric objects on the plane. Show where each object ends up after the plane is translated by vector \( RS \).

2) Here is a definition to study: A translation \( AA' \) is a rigid motion of the plane that takes \( A \) to \( A' \), and for all other points \( P \) on the plane, \( P \) goes to \( P' \) where vector \( PP' \) and vector \( AA' \) have the same length and direction. Discuss, in your groups, how this definition fits with the above idea.

3) What relationships do you see between the original figures and their translated images? Between the objects, their images, and the translation vector? Make as many conjectures as you can about translations.

4) If we translate the plane using \( RS \) and then perform a second translation, say, \( ST \), what is the resulting rigid motion? Explain.
**Read and Study**

*Only the curious will learn and only the resolute overcome the obstacles to learning. The quest quotient has always excited me more than the intelligence quotient.*

_Eugene S. Wilson_

**Transformations** are a big category of motions we can apply to the points of a plane which cause objects in the plane to change their position, or their size, or even their shape. Some transformations, like **scaling**, only change the size of an object. Other transformations, like a **shearing** can change both the size and shape of an object. The object that is the result of a transformation applied to an object is called the **image** of the object under that transformation. For example, the rectangle below is enlarged 1½ times to produce the scaled image and is sheared horizontally to produce the sheared image.

In this and the next two sections, we will study three transformations that change the position of an object but do not change its size or its shape. These types of transformations are called **rigid motions**.

**Rigid motions** are the transformations of the plane for which the distance between points is preserved. In other words, if two points were a certain distance apart before the motion, then they are still that same distance apart after the motion. *(Why does the name “rigid motion” make sense?)* Using the idea of rigid motions, we can more precisely define congruence: two objects are **congruent** if there exists a series of rigid motions where one object is the image of the other.

The rigid motions are the **translation**, the **rotation**, and the **reflection**. Each of these transformations will move the plane in a unique way. The translation will slide the plane a particular distance in a particular direction. The rotation will turn the plane either clockwise or counterclockwise around a fixed center. The reflection will move the plane by flipping it across a line. It turns out that all rigid motions of the plane are combinations of just these moves.

As you discovered in the class activity, a translation vector is used to describe the distance and direction each point is moved in a translation. If the endpoints of the vector are given as coordinates on a square grid, we can describe the distance and direction each point is moved as so many units up or down and so many units right or left. *Use this language to describe the translation vector RS on the previous page.*
We use a standard notation to label the vertices of the image of an object under a translation. For example, if the original object is the rectangle $ABCD$, then its image is labeled $A'B'C'D'$. Here vertex $A'$ of the image corresponding to vertex $A$ of the original rectangle, vertex $B'$ to vertex $B$, etc. The points $A$ and $A'$ are called corresponding points. The sides $AB$ and $A'B'$ are called corresponding sides.

**Connections to the Elementary Grades**

*Instructional programs from prekindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations.*

*NCTM Principles and Standards for School Mathematics, 2000*

Students in the elementary grades can predict and describe the results of sliding, flipping, and turning two-dimensional shapes.

Various elementary curricula use different approaches. Some make use of manipulatives and others have students cut out shapes in order to physically perform the slide, flip, or turn the shapes.

**Homework**

*You may be disappointed if you fail, but you are doomed if you don’t try.*

*Beverly Sills*

1) Do all the italicized things in the Read and Study section.

2) Decide if each of the statements about translations is true or false. If true, give a mathematical explanation; if false, explain why or give a counterexample.

a) Corresponding sides of an object and its translated image are always parallel.

b) If $A'B'C'D'$ is the image of $ABCD$ under a translation, then the line segment joining vertex $A$ to vertex $A'$ is the translation vector.

c) If the parallel sides of a trapezoid are horizontal, then it is possible for the parallel sides of its image to be vertical after a translation.

d) If $A$ and $A'$ and $B$ and $B'$ are corresponding points under a translation, it is possible for the lines $AA'$ and $BB'$ to intersect.

e) Every point in the plane moves to a new position under a translation, i.e., there are no fixed points in a translation.
Below you will find a coordinate grid. Apply the following three translations to a triangle with vertices initially located at (0, 0), (-2, -3), and (3, -3). What is a shortcut way of doing part c)?

a) up 5, left 3       b) down 2, right 4       c) up 5, left 3 followed by down 2, right 4
Class Activity 8: Turn, Turn, Turn

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

S. Gudder

A rotation involves “turning” the plane about a fixed point. In order to specify a rotation, we need an angle (with direction, clockwise or counterclockwise) and the fixed point (called the center of the rotation). In each case your group should use tracing paper and a compass and protractor to figure out where each shape ends up after the given rotation. (There are questions on the next page too.)

1) Rotate the plane 60 degrees counterclockwise about point A.

![Diagram of a rotated triangle with point A marked]

2) Rotate the plane 140 degrees clockwise about point P.

![Diagram of a rotated triangle with point P marked]

(This activity is continued on the next page.)
3) Rotate the plane 90 degrees clockwise about a point Q in the exact center of the square.

4) Study the three rotations in this activity. What relationships do you see between the original figures and their rotated images? Between corresponding points and the point P? Make as many conjectures as you can about rotations.

5) Here is the definition. Study it to see how it fits with the idea of rotation. A rotation about a point P through an angle \( \theta \) is a transformation of the plane in which the image of P is P and, if the image of A is \( A' \), then \( PA \cong PA' \) and \( m \angle APA' = \theta \). Point P is called the center of the rotation.
We don’t see things as they are. We see things as we are.  
Anais Nin

A rotation is a “turn” about a given point called the center through a given angle of turn. (The turn can be made clockwise or counterclockwise. This is typically indicated in the problem.)

Formally, a rotation (about a point \( P \) through an angle \( \theta \)) is a transformation of the plane in which the image of \( P \) is \( P \) and, if the image of \( A \) is \( A' \), then \( PA \cong PA' \) and \( \angle APA' = \theta \). Point \( P \) is called the center of the rotation.

Have a look at the following illustration of the motion of turning triangle \( ABC \) clockwise 240° around point \( P \).

Notice how each vertex of the triangle moves along a circle whose center is \( P \). What part of the definition says that this must happen? How can we see the 240° angle in the picture above? How are the segments \( AP \) and \( A'P \) related? The segments \( BP \) and \( B'P \)? The segments \( CP \) and \( C'P \)?
Homework

*Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air.*

*John Quincy Adams*

1) Do all the italicized things in the *Read and Study* section.

2) Use the grid to rotate the plane 90 degrees counterclockwise about point P. Show the image of the figures after the rotation.
3) Decide if each of the statements about rotations is true or false. If true, give a mathematical explanation; if false, explain why or give a counterexample.

a) Corresponding sides of an object and its rotated image are always parallel.

b) If \( A B C D \) is the image of \( ABCD \) under a rotation, then the line segment joining vertex \( A \) to vertex \( A \) goes through the center of the rotation.

c) If the parallel sides of a trapezoid are horizontal, then it is possible for the parallel sides of its image to be vertical after a rotation.

d) If \( A \) and \( A \) and \( B \) and \( B \) are corresponding points under a rotation, it is possible for the lines \( AA \) and \( BB \) to intersect.

e) Every point in the plane moves to a new position under a rotation, i.e., there are no fixed points in a rotation.

f) The line segments joining corresponding vertices are congruent.
Class Activity 9: Reflecting on Reflection

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture.

Bertrand Russell

A reflection flips the entire plane about a given line resulting in its mirror image. Officially a reflection in a line \( l \) is a rigid motion of the plane in which the image of a point \( P \) on \( l \) is \( P \), and if \( A \) is a point not on \( l \) and if the image of \( A \) is \( A' \), then \( l \) is the perpendicular bisector of \( AA' \).

1) Study that definition. Make sure everyone in your group understands how it fits with the idea of a reflection. See if you can use the definition to help you to sketch the reflection of triangle \( ABC \) in line \( l \).
2) Carefully use the definition of a reflection to sketch the reflection of the trapezoid shown. First you will reflect it in line \( m \) and then you will reflect what you get in line \( n \) (that is parallel to line \( m \)).

What is the single rigid motion that would take the initial figure directly to the final figure? Explain.

3) What would happen if the lines intersected? Try it and then use your observations to make a conjecture.
There are several ways to help children picture the results of a reflection. Think about the reflection of the parallelogram in line shown below.

![Parallelogram and Reflection Line](image)

One way to see where the image should be located is to trace the parallelogram and the line of reflection on a sheet of thin paper and then physically flip the paper over and place it back on top of the original paper so that the two lines coincide. The copy of the parallelogram on the tracing paper is now positioned as the image of the reflection. *Use a sheet of paper and try this method.*

Another way to visualize a reflection image is to physically fold the original sheet of paper along the line of reflection. The original object and its image under reflection should now coincide, as in an “ink-blot” drawing. *Try this. Really do it.*

Recall that a **reflection** in a line \(l\) is a transformation of the plane in which the image of a point \(P\) on \(l\) is \(P\), and if \(A\) is a point not on \(l\) and if the image of \(A\) is \(A'\), then \(l\) is the perpendicular bisector of \(AA'\).

This definition of a reflection provides insight into how we can sketch the image of a reflection. Each point must travel along a line perpendicular to the line of reflection so that that line is the midpoint between of the line segment connecting corresponding points. *Use that method to sketch the image of the parallelogram above.*
There are many instances of reflection in physical phenomena. Common examples include the reflection of light, sound, and water waves. We are all familiar with the phenomena of light reflection – take a look in a mirror.

**Homework**

*We all have a few failures under our belt. It’s what makes us ready for the successes.*

Randy K. Milholland, Webcomic pioneer

1) Do all the italicized things in the Read and Study section.

2) On the square grid below, use line \( n \) as the **line of reflection** to reflect the given objects. Label each image appropriately.

3) What relationships do you see between original figures and their reflected images? Between the objects, their images and the given line of reflection? Make as many conjectures as you can about reflections.
4) Use the definition of a reflection to reflect the below object in line.

5) Decide if each of the statements about reflections is true or false. If true, give a mathematical explanation; if false, explain why or give a counterexample.

a) Corresponding sides of an object and its reflected image are always parallel.
b) If \( ABCD \) is the image of \( ABCD \) under a reflection, then the line segment joining vertex \( A \) to vertex \( A \) is perpendicular to the line of reflection.
c) If \( ABCD \) is the image of \( ABCD \) under a reflection, then the line segment joining vertex \( A \) to vertex \( A \) is bisected by the line of reflection.
d) If the parallel sides of a trapezoid are horizontal, then it is possible for the parallel sides of its image to be vertical after a reflection.
e) If \( A \) and \( A \) and \( B \) and \( B \) are corresponding points under a reflection, it is possible for the lines \( AA \) and \( BB \) to intersect.
f) Every point in the plane moves to a new position under a reflection, i.e., there are no fixed points in a reflection.
g) The line segments joining corresponding vertices are congruent to each other.
Class Activity 10: Zoom

One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.

Alfred Nobel

In the picture above, we started by drawing the smaller triangle on a computer screen, and then we zoomed in. The triangle got bigger and moved to the new position.

One point on our screen remained fixed when we did this zoom. Find that point.

What was the scale factor of the zoom? In as many ways as you can, find evidence for your answer.
We have different notions of “sameness” in geometry. In the strongest sense, if we say two objects are the same, we mean they are congruent. But we might also refer to two objects having the same shape even if they aren’t congruent. For instance, the two triangles in our Class Activity have the same shape, but they are not the same size. Observe that their corresponding angles are congruent, and that their corresponding sides are proportional. Make sure that you understand what this means. These two triangles aren’t congruent, but they are what we call “similar”.

Formally, we say that two objects in the plane are similar if one can be obtained from the other by composing a rigid motion (to change the object’s position if necessary) with a “dilation”. A dilation is a motion of the plane in which one point, P, remains fixed, and all other points are pushed radially outward from P or pulled radially inward toward P so that all distances have been multiplied by some scale factor.

It can be shown that two polygons are similar if and only if their corresponding vertex angles are congruent, and their corresponding sides are proportional. Euclid proved a theorem about similar triangles:

1) Angle-Angle Triangle Similarity Theorem (AA): If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

This theorem appeared in his Book VI (about similar geometric figures and proportional reasoning) rather than his Book I that we have studied previously.

**Homework**

1) Using your compass, draw a circle. Place a point C at the exact center of your circle, and a point P somewhere outside of the circle. Then beginning with that circle,

   a) draw the similar shape that is the result of a dilation of the plane with fixed point C and scale factor 2.
   b) instead, draw the similar shape that is the result of a dilation of the plane with fixed point P and scale factor 2.
   c) What commonalities and differences do you observe about the three shapes you have drawn?
2) Beginning with the triangle pictured below, draw the similar triangle that is the result of a dilation of the plane with fixed point P and scale factor 1/2.

![Triangle Diagram]

3) One night, a 6-foot tall man stood 10 feet from a lamppost. The light from the lamppost cast a 12 foot shadow of the man. How tall was the lamppost?

4) Suppose two objects are similar and the scale factor of the dilation is 1. What else can you say about the relationship between those two objects?

5) Look again at Euclid’s triangle similarity theorem (AA). Given the other things you have already learned about triangles, it is equivalent to saying: if all three angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar. Consider the following conjecture about quadrilaterals: if all four angles of one quadrilateral are congruent to the corresponding angles of another quadrilateral, then the quadrilaterals are similar. Is this conjecture true or false? Give an argument to support your answer.
Class Activity 11: Searching for Symmetry

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.

Aristotle

A symmetry of an object is a rigid motion of the plane in which the image coincides with the original object. There are two primary types of symmetry. Intuitively, an object has reflection symmetry if it can be cut by a line of reflection into two parts that are mirror images of each other. This butterfly has reflection symmetry and so does this arrow. Sketch the line of reflection in each case.

An object has rotation symmetry if it can be rotated around a center point through a certain angle and end up with the image coinciding with the original. We would say that the recycling sign below has 120, 240 and 360 degree rotational symmetry.

1) Find all the symmetries of the capital letters in the following typeface:

\[
\text{A B C D E F G H I J K L M N O P Q R S T U V W X Y Z}
\]

2) In each case, sketch a polygon with the given symmetries, or explain why such a polygon cannot exist.

   a) no lines of reflection symmetry, but 180° (and 360°) rotation symmetries
   b) 90° (and 360°) rotation symmetries and no other symmetries.
   c) 2 lines of reflection symmetry, 360° rotation symmetry, and no other symmetries
   d) 6 lines of reflection symmetry
   e) no lines of reflection symmetry, but 90°, 180°, 270° (and 360°) rotational symmetry
   f) any translation symmetry
Mathematics is the science which uses easy words for hard ideas.
E. Kasner and J. Newman

A figure, picture, or pattern is said to be symmetric if there is at least one rigid motion of the plane that leaves the figure unchanged. For example, this leaf is (pretty much) symmetric because there is a line of reflection symmetry.

Many objects in nature display this kind of bilateral symmetry. The letters in ATOYOTA also form a symmetric pattern: if you draw a vertical line through the center of the “Y” and then reflect the entire phrase across the line, the left side becomes the right side and vice versa. The picture doesn’t change.

The order of a rotation symmetry is determined by counting the number of turns the object can make and coincide with itself before returning to its original position. The angle measure of the smallest turn is determined by dividing $360^\circ$ by that number of turns. Why does this make sense?

For example, an equilateral triangle has “order 3 rotation symmetry.”

The turn of $120^\circ$ takes vertex A to vertex B; the turn of $240^\circ$ takes vertex A to vertex C; and the turn of $360^\circ$ takes vertex A back to vertex A. What are the rotation symmetries of the square? Of the regular hexagon? Of the regular octagon? Of the regular n-gon?
Only repeating infinite patterns have translation symmetry. They are the only type of object that can be slid and still fall back on themselves. Imagine that this pattern strip continues forever in both directions so if you slide it one pattern to the right (or two or three …) it looks just the same.

Connections to the Elementary Grades

I touch the future. I teach.
Christa McAuliffe

The Common Core State Standards for mathematics make informal ideas of symmetry a topic for grade 4. While they mention only reflection symmetry, children at this age (and even younger) are capable of exploring and recognizing “turn” symmetry as well. Read this standard.

CCSS Grade 4:

1. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Go on line to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 4 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 2. Carefully examine the Mosaic Game. Play the game one time and record your result. Explain how you know that your arrangement produces the most lines of symmetry. What might students learn about symmetry from completing this activity?

Play the game again, but this time, try to produce a figure with the largest order of rotational symmetry. Again, explain your reasoning.
Homework

By perseverance the snail reached the ark.
Charles Haddon Spurgeon

1) Do all the italicized things in the Read and Study and Connections sections.

2) Classify the symmetries for the following traffic signs – (consider the entire sign – not just the interior design).

3) Is it possible for an object to have rotation symmetries without having reflection symmetries? If it is, give an example of such an object. If it is not, give an argument to support that conclusion.

4) Is it possible for an object to have two reflection symmetries without having 180 degree rotation symmetry? If it is, give an example of such an object. If it is not, give an argument to support that conclusion.

5) Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 1 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 3 Sessions 1 and 2. What ideas about rotational symmetry are emphasized in these sections?
Class Activity 12: Tessellations

The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

Godfrey H. Hardy

There are many new definitions involved in this activity. Take a little time to study each of them.

A tiling is an arrangement of polygons that can be extended in all directions to cover the plane with no gaps and no overlaps.

A tessellation is a tiling in which all vertices meet only other vertices.

A regular tessellation is a tessellation that uses only one regular polygon.

Find all the possible regular tessellations, make a sketch of each one, and then make an argument that you have found them all.
Traceable copies of many regular polygons. All of them have been scaled so that they have the same side length.
A **tiling** is an arrangement of polygons that can be extended in all directions to cover the plane with no gaps and no overlaps. An example using a “T” shape polygon is shown below.

A **tessellation** is a tiling in which all vertices only meet other vertices. *Is the above tiling a tessellation? Explain.*

Some curriculum materials use the words tiling and tessellation interchangeably. We will not do so, but we want you to be aware of that fact.

Mathematically, we are interested in interpreting the definition of a tessellation. How can we know it will have no gaps and no overlaps? How can we determine that a given arrangement will extend indefinitely in all directions? First we will examine what happens at a single vertex point within a tessellation.

Take a close look at vertices A and B in the following arrangement composed of regular hexagons and equilateral triangles.

There are two hexagons and two triangles meeting at vertex A and six triangles meeting at vertex B. Since we know that the hexagons and the triangles are regular, we know that the vertex angles of the hexagon are 120° each. *(Have a look back at Class Activity 4, if you have*
forgotten how to think about this.) Likewise, we know that the vertex angles of the equilateral triangle are 60° each. This means that there are 2*120° + 2*60° = 360° at vertex A and there are 6*60° = 360° at vertex B. (Check this.) Now, since the sum of the angles at each vertex is exactly 360°, we have proved that there are no gaps or no overlaps in this arrangement.

Now we will note that if we made infinitely many copies of this arrangement, we could slide (translate) them around on the plane (e.g., slide one so that A goes to C) to cover the entire plane.

We will tell you that there are many tessellations of regular polygons that are not regular. For example, have a look back at the tessellation made of hexagons and triangles that we were just discussing above. Why is this tessellation not regular?

There are many websites that present tessellations- or tiling-type activities. In the Homework section you will be asked to explore several that blur the line between geometry and art.

Connections to the Elementary Grades

Learning is not compulsory... neither is survival.

W. Edwards Deming

Creating tessellations is an activity that can be adapted to every grade level – very young children can create patchwork quilts from construction paper using only rectangles or squares or isosceles right triangles. Older students can make more complex artwork using the mathematical concepts of congruency and transformations.

Online resources can allow students to create a variety of more complicated designs quickly and accurately. Spend 15 minutes at the website at http://www.mathcats.com/explore/tessellations/tesspeople.html to see an example of an interactive web-based exploration of tessellations used in the fifth grade Trailblazers curriculum in North Carolina.


**Homework**

*Creativity is allowing yourself to make mistakes. Art is knowing which ones to keep.*

*Scott Adams, ‘The Dilbert Principle’*

1) Do all the italicized things in the *Read and Study* and *Connections* sections. Make sure to spend some time at the website.

2) Any triangle (if you have enough copies of it) can be used to tessellate the plane. To explore this, fold a paper up so you can cut out 8 congruent *scalene* triangles all at once. Then have a look. Pay particular attention to the transformations you are using to move the triangle into new positions.

   a. Make a mathematical argument that your triangle would, in fact, tessellate the plane.
   b. Where in your proof did you use transformations of the plane?

3) True or false? Any quadrilateral (if you have enough copies of it) can be used to tessellate the plane. To explore this, fold a paper up so you can cut out 8 congruent quadrilaterals all at once. Then have a look. What about concave quadrilaterals? Make an argument to support your choice.

4) True or false? It’s possible to find a pentagon that can be used to tessellate the plane (if you have enough copies of it). Make an argument to support your choice.

5) Make a mathematical argument that the number of regular tessellations you found in the class activity is the exact number possible.

6) A polygon with more than six sides will not tile if it is convex. Explain why not. The following polygons have more than six sides, but they are concave. Sketch a portion of a tiling for each polygon.

   ![Diagram](image-url)
7) Describe some reflectional, rotational and translational symmetries for the tessellation below.
Summary of Big Ideas from Chapter Two

*If an idea’s worth having once, it’s worth having twice.*

Tom Stoppard

- There are three distinct ways to “move” the plane without changing the shape or size of objects: the translation, the rotation, and the reflection.

- A translation slides the plane a given distance in a given direction.

- A rotation turns the plane around a given point through a given amount of rotation (usually given in degrees).

- A reflection flips an object to its mirror image across a line of reflection.

- One of the goals of elementary school geometry instruction is that students learn to visualize and apply transformations.

- Two objects are similar if one can be obtained from the other by a rigid motion and a dilation.

- Symmetry is a phenomenon of the natural and artistic worlds that can be explained with the language of rigid motions.

- Mathematicians most often talk about two types of symmetry: reflection symmetry, in which an object is divided by a line of reflection into two parts that are mirror images of each other, and rotation symmetry, where an object is rotated around a center point through a certain angle and ends up occupying the same position in the plane.

- Creating tilings and tessellations is an activity that can be adapted to every grade level – very young children can create patchwork quilts from construction paper using only rectangles or squares or isosceles right triangles. Older students can make more complex artwork using the mathematical concepts of congruency and transformations.
Chapter Three
Measurement in the Plane
Class Activity 13: Measure for Measure

And there went out a champion out of the camp of the Philistines, named Goliath of Gath, whose height was six cubits and a span.

I Samuel 17:4

1) Using your cubit (length from elbow to fingertips) and hand span, determine how tall Goliath was by cutting a string that is as long as Goliath was tall. Compare your string length with the strings of others in your group. What are the difficulties that might arise from choosing and using units determined by each person’s own body? Why do you think we use the “foot” as a common unit of length?

2) On the next page, are four copies of the same line segment. Using each of the rulers provided, carefully measure the line segment.
The big idea of **measurement** is that of comparing an attribute of an object to an appropriate unit. For example, we might compare the length of our desk to a foot-long ruler or we might compare the area of sheet of paper to the area of a green triangle. So the key question regarding measurement is this: How many of the unit *fit* into the object?

What makes a unit appropriate? Well, first it must have a **dimension** that matches the attribute to be measured. For example, we measure length (or width or height) using one-dimensional units.

Here is an example of a one-dimensional unit: ______

We measure area using two-dimensional units like this: ...

We measure volume using three-dimensional units like this:

We will talk more about volume measurement in later sections.

Second, if you want to be able to communicate with others, it helps that the unit be a ‘standard’ one. A standard unit is one that the culture has agreed upon. Each person has a mental model of the unit so he or she can picture how big it is. For example, in our culture a foot is a standard unit for measuring length. In Europe (and most of the rest of the world) a meter is a standard unit for measuring length. If we want to talk among cultures, we need to be able to convert from one unit to another. *There are about 3.28 feet in a meter. If a room is 15 feet long, approximately how many meters is that?*

Third, it is useful that the unit be of reasonable size in relation to the attribute to be measured. It would be inconvenient to measure the length of football field using microns (a *really* small one-dimensional unit). In the metric system size is indicated by the prefix. For example, the prefix *kilo* means 1000 times. So a kilometer is 1000 meters. In the curriculum materials elementary students are expected to use the prefixes *milli* (one thousandth), *centi* (one hundredth), and *kilo*. You should memorize these and be able to make conversions. *Approximately how many centimeters are there in a foot?*

Because measurement always involves comparison, it is necessarily always an approximation. We often indicate our degree of certainty about a measurement based on the way we report it. For example, if I claim a desk is 1.23 meters long, this suggests that I am confident in the
accuracy to the hundredth of a meter (to the centimeter). When working with children you should explicitly discuss these facets of measurement and you should be sure to always report the unit with any actual measurement. Saying that a desk is 1.23 long means nothing. Saying that it is 1.23 meters long makes sense.

A common measurement we make for a plane object is to measure the distance around its boundary. We call this measurement the perimeter of the object. (When the object is a circle, we call this length the circumference.) The perimeter of a plane object is a one-dimensional measurement – so we use linear units like inches or centimeters. We calculate a perimeter by simply adding up the lengths of the curves that make up the object. We can measure the lengths of line segments with a ruler. We will find a formula for the circumference of a circle in a future activity.

**Connections to the Elementary Grades**

*Only the curious will learn and only the resolute overcome the obstacles to learning. The quest quotient has always excited me more than the intelligence quotient.*

Eugene S. Wilson

The Common Core State Standards for mathematics require that children begin to study standard measurement of length beginning in grade 2. Read the excerpt from the CCSS below.

**CCSS Grade 2: Measure and estimate lengths in standard units.**

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

3. Estimate lengths using units of inches, feet, centimeters, and meters.

4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.


Notice that children are to be learning both metric and English units. Come up with a benchmark (something to imagine that is the right length) for each of these units: inches, feet, centimeters, and meters.
The Common Core State Standards for measurement in grade 1 are shown below. *Compare them to the grade 2 standards. How do they differ?*

### CCSS Grade 1: Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.

2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*


Go on line to bridges.mathlearningcenter.org and find the *Bridges in Mathematics* Grade 2 Teachers Guide. Spend 10-15 minutes looking through Unit 4 Module 1 Session 1 Teacher Feet. *In what ways do the children need to analyze the process of measurement in this session?*

### Homework

"Many of life's failures are people who did not realize how close they were to success when they gave up.

*Thomas A. Edison*

1) Do all the italicized things in the *Read and Study* section.

2) Do the *Connections* problems.

3) What are good benchmarks for the millimeter, centimeter, decimeter, and kilometer? Find something that is about the length of each.

4) Use an appropriate English unit to estimate a) the length of your table, b) the distance from Chicago to Denver, and c) the width of a pencil.

5) Use an appropriate metric unit to estimate a) the length of your table, b) the distance from Chicago to Denver, and c) the width of a pencil."
6) Find the perimeters of the plane figures shown below by measuring them
   a) Using an appropriate English unit.
   b) Using an appropriate metric unit.
   c) Now convert your metric units to English units to check that your measurements in
      b) match your measurements in a).

7) Explain why it makes sense that the perimeter of a rectangle can be found by computing
   \(2l + 2w\) where \(l\) is its length and \(w\) is its width.

8) I want to run around the below lake. I plan to hug the shoreline. How far do I have to
   run? Think about various ways you could use a map to answer this question. How
   accurate is your estimate? How can you improve its accuracy?

9) Go on line to bridges.mathlearningcenter.org and find the \textit{Bridges in Mathematics}
   Grade 2 Teachers Guide. Spend 10-15 minutes looking through Unit 4 Module 1. Then,
   carefully examine the worksheet 4A Estimate & Measure Inches Record Sheet. What
   conversations could you have with your students about the big ideas of linear
   measurement based on this activity?
Go down deep enough into anything and you will find mathematics.
Dean Schlicter

Define **one triangle unit** to be the area of the green triangle pattern block. Using this unit, determine the area of this sheet of paper. Then use the triangle unit to estimate the area of each of the pattern block shapes pictures below.
Class Activity 14 B: Area Estimation

Here is a map of the great state of Wisconsin. Without looking anything up online, your group needs to make a reasonable estimate of its area in square miles. First discuss a couple of different ways of doing this using the map. Then go ahead and carefully compute your estimate.

http://www.nationsonline.org/one_world/map/USA/wisconsin_map.htm

Would you guarantee your estimate to the nearest 10 square miles? The nearest 100 square miles? The nearest 1000 square miles? Something else? How did you decide?
Read and Study

The man ignorant of mathematics will be increasingly limited in his grasp of the main forces of civilization.

John Kemeny

Mathematicians define area as the quantity of two-dimensional space enclosed by a closed plane figure. We commonly measure this area in terms of square units such as square inches or square centimeters. So to find the area of a figure you need to find the number of square units it would take to cover the figure. We want you to literally do this now.

Here is a unit of area (the square centimeter):

Trace it and then see how many of those units (including parts of units) it takes to cover the shape below:

If the shape is a representation of a lake, and a centimeter of length corresponds to 3 miles, then what is the area of the actual lake? Explain.

We often measure the area of irregularly shaped objects just by counting (and estimating) the number of square units within the boundary of the object. Sometimes we superimpose a grid to help with that estimate. Estimate the area of this object now assuming that each small square is a unit of area.

When we measure the area of an object in square units, notice that we are taking advantage of the fact that squares tessellate the plane – there are no gaps between the squares. We have already discovered that there are other shapes that tessellate the plane as well; two of them
are, like the square, also regular polygons. Which ones? Why do you think people chose to use square units instead of some other shape that tessellates the plane?

**Connections to the Elementary Grades**

*Teaching creates all other professions.*

*Author Unknown*

Children need to have experiences finding areas by covering figures with square units like you did in the Read and Study section. If you show them formulas too early, they will simply try to remember those formulas and they may not focus on the idea of area. Besides, formulas work only with a limited number of shapes, and so estimating areas using the definition is a useful skill in its own right.

We suggest that children in grades 2 and 3 spend many weeks estimating areas using square stickers or square grids to cover figures. Some of those figures should have curved boundaries and some should have special polygon shapes.

Children will quickly begin to propose shortcuts on their own that will lead naturally to the area formulas. For example, if children compute areas of the below rectangle shapes using stickers or a grid, some of them will notice that a shortcut for finding areas of rectangle shapes is to simply multiply the length of the rectangle by its width. Then you can talk with them about why this is so. Practice that now by doing the below tasks.

*First use a grid to estimate how many area units it takes to fill each rectangle shape. Then explain why it makes sense that the area of a rectangle can be found by multiplying its length by its width.*

Here is a unit of length: ______

Here is a unit of area: [unit of area]

[Rectangle shapes]

[Rectangles of various sizes]
The Common Core State Standards describe the following standards for children in grade 3. Read them to see that the things children should learn are many of the things we have described in this section.

CCSS Grade 3: Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

1. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   a. A square with side length 1 unit, called “a unit square,” is said to have “one square unit” of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by \( n \) unit squares is said to have an area of \( n \) square units.
   c. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

2. Relate area to the operations of multiplication and addition.
   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b. Multiply side lengths to find areas of rectangles with whole number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths \( a \) and \( b + c \) is the sum of \( a \times b \) and \( a \times c \). Use area models to represent the distributive property in mathematical reasoning.
   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.


We have not talked explicitly about 2. c. and 2. d. above. Sketch pictures to help you to make sense of what they mean by those.
The more I practice, the luckier I get.

Jerry Barber

1) Do all the italicized things in the Read and Study section.

2) Do all the italicized things in the Connections section.

3) Use a grid to estimate carefully the area of the below circular figure in square centimeters.

4) A student comes to you and asks, "Why do we use square centimeters to measure the area of the circle? A circle is round and not square." Explain to her why we still use square centimeters to measure the area of a circle.

5) We would like to have the above lake classified as one of the Great Lakes. As part of the application to the Department of the Interior, we have to report its area. Estimate the area of the lake in square miles.

Scale
1 cm = 3 miles
6) Here is a floor plan for the first level of a house.
   a) What is its area in square feet including the garage? Assume that 1 cm represents 6 feet of length.
   b) How good is your estimate? Are you confident to the nearest square foot?
   c) What Common Core State Standard is met by this problem? Explain.

7) Go on line to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 3 Teachers Guide. Spend 10-15 minutes looking through Unit 6 Module 3.
   a. How are area and perimeter introduced?
   b. Print out and then carefully work through the worksheet Bayard Owl’s Borrowed Tables. What conversations could you have with your students about the big ideas of perimeter and area based on this activity?

8) An NBA basketball court measures 50 feet by 96 feet. Use this information below to determine how many acres it covers.
   1 foot = 12 inches
   1 yard = 3 feet
   1 mile = 1760 yards
   1 acre = 4840 square yards
   1 square mile = 640 acres
Class Activity 15: Finding Formulas

*One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers.*

*Heinrich Hertz*

We commonly compute areas of some special polygon shapes with the use of formulas. These formulas can be *explained* – they are based on geometric definitions and theorems – and we want you to understand why they make sense. That is the goal of this activity.

1) **Rectangle:** Use the idea of area as the number of square units it takes to cover an object to explain why it makes sense that the area of a rectangle is simply the product of its length and width.

![Rectangle Diagram](image1)

2) **Parallelogram:** Show how to cut and rearrange a parallelogram to make a rectangle with the same area. Argue that the resulting figure is in fact a rectangle. Use the formula for the area of a rectangle to find a formula for the area $A$ of a parallelogram using the base $b$ and the height $h$ of the parallelogram.

![Parallelogram Diagram](image2)

3) **Triangle:** Show how to rearrange two copies of the same triangle to make a parallelogram. Argue that the resulting figure is in fact a parallelogram. Use the formula for the area of a parallelogram to find a formula for the area $A$ of a triangle using the base $b$ and the height $h$ of the triangle.

![Triangle Diagram](image3)
The essence of mathematics is its freedom.
Georg Cantor

Many of your students (and many of their parents) will think that formulas tell the whole story about area. In fact, some people mistakenly define area as “length times width.” Every closed two-dimensional shape has area, but as we have seen in an earlier section, only a very few of these shapes have formulas we can use to calculate the area.

The area of some geometric objects is more easily determined through the use of area formulas. In the Class Activity you developed several useful and well-known formulas that are readily found in the elementary school curriculum.

The most fundamental area formula is for the area of a rectangle: length \( \times \) width. All of the other formulas for area are built on that. As you have seen, these formulas are really theorems that have been proven to work. And, these theorems are only true when we use the unit square as our unit. Explain carefully why that previous statement is so important. What happens if we do not use squares as our unit? You should have your upper elementary students do activities like those in the Class Activity so that they can see this for themselves.

In the Bridges in Mathematics curriculum for grade 4, students discover the formulas for the area and perimeter of rectangles. Go on line to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 4 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 3 Session 3. In what ways does this activity help students to develop formulas for the perimeter and area of a rectangle?

Connections to the Elementary Grades

Knowing is not enough; we must apply. Willing is not enough; we must do.
Johann Wolfgang von Goethe

Once children understand the ideas of perimeter and area, they can solve some practical problems. We will pose some now that specifically address the standards discussed below.
Do these problems and then read to see which of the standards we’ve addressed with them.

a) If a rectangular room has a length of 5 feet and an area of 60 square feet, what is its width?
b) If a rectangle has a length of 10 feet and a perimeter of 36 feet, what is its width?
c) Is it true that rectangles with bigger perimeters always have bigger areas too? Explain.
d) Is it true that rectangles with bigger areas always have bigger perimeters too? Explain.
1) Do the problems and then do the italicized statement in the Connections section.

2) Practice explaining why each of the formulas from the Class Activity makes sense.

3) One of your students is confused about area calculations. Adam wonders why, if you take a rectangle and multiply the length of each side by 2, the area of the new rectangle isn't twice as big as the area of the old rectangle. Draw some pictures to help you see what is going on here. What will you say to him?

4) Explain why it makes sense that the area of a trapezoid is always $\frac{1}{2}h(b_1+b_2)$ where $b_1$ and $b_2$ are the lengths of the parallel bases and $h$ is the height.

You can do this by partitioning the trapezoid region into pieces, or by cutting it apart and rearranging it, or by adding on structure. You may use what you know about finding areas of rectangles, parallelograms, and triangles. See if you can find more than one way to do this.

5) Find the area of the trapezoid shown below in as many different ways as you can. Assume that each grid square represents one unit of area.
6) Suppose a rectangle has a perimeter of 36 units. What are all the possible whole number dimensions of the rectangle? Make a graph of width vs. area. Which width gives the greatest area?

7) Suppose you have 100 meters of flexible fencing to mark a pasture out on the plains. How would you set it up (what shape) to enclose the most grazing area for your cattle? What dimensions would you use if the pasture had to be a rectangle?

8) The triangle below is constructed on a 1 cm grid. Find the area of the triangle using at least three different methods.

9) Assume that the triangle area above represents part of a sign that needs to be painted. The scale of the drawing is that 1 cm represents 10 feet. The instructions on the paint can say that 1 gallon of paint will cover 100 square feet. How many gallons of paint will you need to buy in order to paint the triangle?
Class Activity 16: The Round Up

Do not disturb my circles!
Archimedes’ final words

1) A circle is defined as the set of all points in the plane that are equidistant from a given point called the center. Study this definition. If the word “all” was missing, how would that change things? What if the words “in the plane” were missing?

2) How many times does the diameter of a circle fit into its circumference? Gather some data to see.

3) Explore the idea of the area formula for a circular region by rearranging it into a parallelogram-shaped figure. Find the area of the “parallellogram” in terms of the radius of the circle. Why is this just the idea of the argument?
Read and Study

It is not once nor twice, but times without number that the same ideas make their appearance in the world.

Aristotle

Circles are as fundamental to Euclidean geometry as are points and lines. Recall that Euclid’s third axiom assures us that we can always make a circle of any size (radius) we want. Of course, the circles we draw are still only approximations of a true mathematical circle, just as the line segments we make are just approximations of a true line segment.

A circle is the set of all points in the plane that are equidistant from a given point, called the center (O in the diagram below). The diagram shows some of the other important terms associated with a circle. Be certain you understand each term and can explain its mathematical definition, which you will find in the glossary.
It is an amazing fact that, for any size circle, the ratio of the circumference to the diameter is the same number. This was known in all early civilizations. We call that ratio “pi.”

So π (pi) is the symbol for the number of times the diameter of a circle fits into its circumference. Read that again; it is important.

This number π is an irrational number. That means it has a decimal representation that neither ends nor repeats. This was proved in 1761 by a mathematician named Johann Heinrich Lambert. Even when we use the π key on a calculator, we are using an approximate value. Elementary students commonly use either 3.14 or \(\frac{22}{7}\) as an approximate value for π when carrying out calculations involving circles. Always be sure to make the point that this is just an approximation.

**Connections to the Elementary Grades**

At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love.

Bertrand Russell

In the *Bridges in Mathematics* curriculum for grade 4, students are introduced to basic circle terminology. Go on line to bridges.mathlearningcenter.org and find the *Bridges in Mathematics* Grade 4 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 1 Session 5. How are the parts of a circle introduced to the students?

Within the lesson, the students are asked to write their own definition of a circle. Issa and Suzi gave the following definitions. Are their definitions correct? How as a teacher do you respond to each of these students?

Issa’s definition: A circle is a shape that’s round and has 360°.

Suzi’s definition: A circle is a shape that has the same width all the way around.
Homework

Eureka! I've got it!

Archimedes

1) Do all the italicized things in the Read and Study section.

2) Solve the problems in the Connections section.

3) Study each bold and underlined term used in this section. This means you should be able to explain the definition using good mathematical language and that you should be able sketch examples and non-examples of each term.

4) How does the definition of $\pi$ lead to the formula $C = 2 \pi r$ (where $C$ is the circumference of a circle and $r$ is its radius)?

5) If a circle has a measured radius of 5 inches, then, using the formula for finding the area of a circle, we would say that the area of the circle is approximately 78.5 square inches. Give at least two distinct reasons why the calculation is approximate.

6) Explain the relationship between a secant and a chord of a circle, between a radius and a diameter, and between a secant and a tangent of a circle.

7) Here is another set of pictures designed to give an intuitive argument that the area of a circular region is $\pi r^2$. Imagine that the circle shown is made of circular strings sitting one inside the next. Then you take a scissors, snip a radius, and flatten the strings to make the triangular shape. What is the area of the triangle in terms of $r$ (the radius of the circle)?

8) If I double the diameter of a circular region, what happens to its circumference?

9) If I double the diameter of a circular region, what happens to its area?
10) If an 8-inch (diameter) pizza costs $5, how much should a 16-inch pizza cost? Justify your result.

11) Have a look at the circles below.

   a) Carefully measure the perimeter of the square that is inscribed inside the first circle.
   b) Carefully measure the perimeter of the regular pentagon inscribed inside the second circle.
   c) Carefully measure the perimeter of the regular hexagon inscribed inside the third circle.
   d) True or False? As the number of sides of the inscribed polygon grows, so does the perimeter of the polygon.
   e) True or False? If we inscribe a polygon with infinity many sides, then the perimeter of that polygon will be infinitely long. Explain your thinking.
   f) How could you use these measurements to get an estimate for π? Explain.

More than 2000 years ago Archimedes found approximate bounds for π using inscribed and circumscribed (outside) polygons with, get this, 120 sides. We still use Archimedes bounds today ($\frac{223}{71} < \pi < \frac{22}{7}$). The average of these two values is roughly 3.1419 which is correct to three decimal places. Not bad for 350 BC.
Class Activity 17: Playing Pythagoras

*I have had my results for a long time, but I do not know yet how I am to arrive at them.*

*Carl Friedrich Gauss*

The Pythagorean Theorem is thought to be almost 4000 years old. The Babylonians, the Egyptians, and the Chinese all knew it. That is, they knew that the sum of the area of the squares on the legs of a right triangle equals the area of the square on the hypotenuse, and they used this fact numerically in construction and commerce and surveying.

1) Carefully measure the areas of the squares in the below example to see if this theorem seems true.

![Diagram of squares and right triangle]

2) What happens if the triangle isn’t a right triangle? Is the sum of the squares on the “legs” (shorter sides) of an obtuse triangle more or less than the square on the longest side? What happens in an acute triangle? Draw some diagrams to see.

(This activity is continued on the next page.)
The first proof of the theorem is attributed to Pythagoras of Samos (it’s in Greece) around 600 B.C. Since then, hundreds of different proofs have been created. You are going to explore one of them.

a) The proof begins with any right triangle. Carefully draw your own and label the length of the hypotenuse $c$, the length of the longer leg $b$, and the length of the shorter leg $a$.

![Diagram of a right triangle]

b) Now make four congruent copies of your triangle, cut them out, and arrange them into a quadrilateral as shown below.

![Diagram of quadrilateral formed by four copies of a right triangle]

c) Justify that the boundary of the outer quadrilateral is a square.

d) Justify that the inner quadrilateral is a square.

e) Now give an algebraic proof that $c^2 = a^2 + b^2$ by using the fact that the five polygons form the large figure (so the area formulas for the five must sum to the area formula of the large square).

f) What things go wrong if the triangles are not right triangles?
Read and Study

I’m very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I’m teeming with a lot of news--
With many cheerful facts about the square of the hypotenuse.
Gilbert & Sullivan, “The Pirates of Penzance”

The Pythagorean Theorem is one of the most well-known and most important theorems of all of elementary mathematics. It is named after the Greek mathematician, Pythagoras, and Euclid included it as the fitting end to volume one of The Elements.

In Euclid’s words the theorem says this:

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Euclid intended the word “square” to mean the physical square drawn on the hypotenuse or leg of the right triangle. So his theorem says that the area of the yellow square (in the figure above) is equal to the sum of the areas of the red and blue squares when ABC is a right triangle.

Today we more commonly use an algebraic statement:

If a right triangle has legs of lengths a and b and a hypotenuse of length c, then \( c^2 = a^2 + b^2 \).

Notice that here, a, b, and c are numbers representing the lengths of the various sides of the triangle. So now the word “square” carries the algebraic meaning denoted by the exponent two.

There are many (at one count, at least 367) proofs of the Pythagorean Theorem. You recreated one of the proofs in the Class Activity. You will be asked to explore two others as part of the Homework for this section.
The Pythagorean Theorem is an example of a theorem whose converse is also a theorem. *State the converse of the Pythagorean Theorem. If you have to look up “converse,” visit the glossary and do so.*

The converse of the Pythagorean Theorem gives us a way to discover whether or not a triangle is a right triangle even when we know nothing about the angle measures of the triangle.

For example, suppose we know that the sides of a triangle are exactly 3, 4, and 5 inches long. Is this a right triangle? Let’s substitute the values 3 for $a$, 4 for $b$, and 5 for $c$ (*How do we know that the hypotenuse must be the side of length 5?*) and check out the Pythagorean relationship $c^2 = a^2 + b^2$. Does $5^2 = 3^2 + 4^2$? If the answer is “yes,” then the triangle is a right triangle.

### Connections to the Elementary Grades

*Instructional programs from prekindergarten through grade 12 should enable all students to create and use representations to organize, record, and communicate mathematical ideas; to select, apply, and translate among mathematical representations to solve problems; and to use representations to model and interpret physical, social, and mathematical phenomena.*

*NCTM Principles and Standards for School Mathematics, 2000*

Having two (or more) ways to interpret or represent a mathematical idea is an important characteristic of mathematics for teaching. The NCTM *Standards* recognize this and call for all elementary students to “select, apply, and translate among mathematical representations to solve problems.” And so, as teachers, it is also important that we understand a mathematical concept from more than one point of view in order to assist our students to use different representations of that concept to solve problems.

The Pythagorean Theorem is one such idea that can be understood from many viewpoints: geometric, numeric, and algebraic. Some of our students will more easily grasp the algebraic approach while others will prefer the more concrete geometric or numeric representation. As teachers, we must master all representations in order to carry out our responsibilities to support each student’s learning.
Homework

The difference between a successful person and others is not a lack of strength, not a lack of knowledge, but rather a lack in will.

Vince Lombardi

1) Do all the italicized things in the Read and Study section.

2) James Abram Garfield (1831-1881), the country’s twentieth president, created this proof of the Pythagorean Theorem in 1876, while he was a member of the House of Representatives. Find Garfield’s proof by using the diagram below by finding formulas for the area of the trapezoid in two different ways. How does his argument fail if the triangles are not right triangles?

![Diagram of trapezoid with labels a, b, c, and c']

3) When the lengths of the sides of a right triangle are all integers the three numbers (a, b, c) are known as a Pythagorean Triple. Explain why (3, 4, 5) is a Pythagorean Triple. Which of the following triples of numbers are Pythagorean Triples?

a) (4, 5, 6)  
   b) (4, 6, 8)  
   c) (6, 8, 10)

4) Suppose a triangle has sides of length 8, 15, and 17. Is it a right triangle?

5) What is the exact height of an equilateral triangle if all sides are of length 10? Of length 3? Of length 4? Of length s?

6) A closet is 3 feet deep 4 feet wide and 12 feet high. Find the distance from one corner at the floor to the diagonally opposite corner at the ceiling. You might want to have a look at a box to help you see what to do here.
Which of the triangles pictured above are similar to each other? Justify your answer in as many ways as you can. (You may assume that the dots are equally spaced on the grid in both the horizontal and vertical directions, and that the vertices of the triangles lie exactly on the dots as they appear to.)
We are going to use this section to discuss the idea of “coordinate geometry.”

In the 1700’s René Descartes (pronounced Day-cart) had the idea that he could solve some geometric problems more easily by translating them into algebraic problems. His idea was to place a structure (a grid) on top of the plane and to give names (like (-3, -1)) to the points.

The **coordinate plane** features two perpendicular axes, the horizontal *x*-axis and the vertical *y*-axis, that intersect at a point called the origin. We label each point on the plane with an ordered pair of coordinates (*x*, *y*). The *x*-coordinate tells us how far the point is from the origin (0, 0) in the horizontal direction and the *y*-coordinate gives the distance from the origin in the vertical direction. For example, the point (-3,-1) is located 3 units left and 1 unit down from the origin.

Using the Pythagorean Theorem we can find the distance between any two points on the coordinate plane. For example, let’s find the distance between points *A* and *D* in the picture above. The line segment *AD* is the hypotenuse of a right triangle (*sketch it on the picture above*) with a horizontal leg of length 5 = (2 – (-3)) and a vertical leg of length 2 = ((-1) – (-3)). So the
square of the distance between $A$ and $D$ is $5^2 + 2^2 = 25 + 4 = 29$ and the distance between $A$ and $D$ is $\sqrt{29}$. Find the distance between $(2, -3)$ and $(4, 0)$.

There are two facts about lines on the coordinate plane that are useful to recall. One is that every line has a slope, which is a measure of its inclination with the $x$-axis. **Slope** is the amount you need to move in the $y$-direction to stay on the line for a one unit change in the $x$-direction. So think about this. **What does a slope of 3 mean? Sketch a line with that slope. What does a slope of $-\frac{1}{3}$ mean? Sketch a line with that slope on the axis above.**

We can calculate the slope ($m$) of a line by using the coordinates of two points that lie on the line with the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $(x_1, y_1)$ and $(x_2, y_2)$ are the coordinates of the two points.

**How does the formula relate to the definition of slope as stated above? Explain.**

**Compute the slope of the line containing the points $(4, 0)$ and $(-2, 5)$.**

If two lines are parallel, then they will make the same angle with the $x$-axis and so will have the same slope – and vice versa, if two lines have the same slope, then they are parallel. **Think about how you could make an argument for this fact.** This turns out to be a very useful observation. If we need to show that two lines are parallel, we can simply calculate their slopes and show that they are equal.

What if two lines are perpendicular? How are their slopes related? It turns out that the slopes of perpendicular lines also have a numerical relationship. The product of the slopes of perpendicular lines is always -1. **Make an argument for this fact.**

**What is the slope of the line that is perpendicular to the line containing the points $(4, 0)$ and $(-2, 5)$?**
In the Class Activity, did you try using the Pythagorean Theorem together with a triangle congruence theorem? Did you try using slopes of lines to compare angles and apply Euclid’s triangle similarity theorem from our section on similarity? If not, you should go back and do those things now!

**Connections to the Elementary Grades**

*Often, when I am reading a good book, I stop and thank my teacher. That is, I used to, until she got an unlisted number.*

*Author Unknown*

Coordinate geometry is a topic to be introduced in grade 5 according to the Common Core State Standards for mathematics. Read the description below to be sure that it fits with what we have discussed in this section.

**CCSS Grade 5: Graph points on the coordinate plane to solve real-world and mathematical problems.**

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).

2. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.


Go on line to bridges.mathlearningcenter.org and find the *Bridges in Mathematics* Grade 5 Teachers Guide. Spend 10-15 minutes looking through Unit 6 Module 1. Pay particular attention to how the coordinate plane is utilized in the sessions. *Print out Rita’s Robot and do the activity.*
1) Do all the italicized things in the Read and Study and Connections sections.

2) A **geoboard** is a board containing a lattice of points, that is, the points are arranged in a geometric pattern. The most common arrangement is to have points evenly spaced in horizontal and vertical columns, forming a square grid design like the dot paper you used in the Class Activity or the coordinate grid discussed in the Read and Study section. Children can form polygons on a geoboard by placing rubber bands around the pegs.

A **geoboard polygon** (or a dot-paper polygon) is like any other polygon in that it is a simple closed curve made up of line segments. However, we require that the vertices of a geoboard polygon coincide with points on the geoboard.

[Image of a geoboard]


Which regular polygons can be made on the geoboard? If it is not possible to make a particular regular polygon, explain why it is not possible.

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3) Find the perimeters and areas of the figures on the geoboard square grids below. You might find the Pythagorean Theorem helpful.

4) There are 14 different segment lengths that are possible to make on a 5 by 5 geoboard. Find them all.
Summary of Big Ideas from Chapter Three

Although this may seem a paradox, all exact science is dominated by the idea of approximation.

Bertrand Russell

- Measurement is the comparison of an attribute of an object to a unit. To measure means to see how many of the unit fit into the object you are measuring.

- Length is a one-dimensional measurement; area is two-dimensional; and volume is three-dimensional.

- All measurement is approximate.

- The choice of a unit is a fundamental part of the measuring process.

- Area is the number of square units it takes to cover an object. It is not defined as “length times width,” and, in fact, that formula works only in a few limited cases.

- Formulas for the calculation of area can be explained by the geometry of the objects being measured. As a teacher, you will need to help your students to see where formulas come from and why they make sense.

- \( \pi \) is defined as the number of times the diameter of a circle fits into its circumference. It is an irrational number.

- Sometimes it is useful to place a structure (a grid) on top of the plane and to give names to the points. We call this coordinate geometry.
Chapter Four

The Third Dimension
Class Activity 19: Strictly Platonic (Solids)

If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent.

Sir Issac Newton

Here is a definition for you to study: A polyhedron is a finite set of polygon-shapes joined pairwise along the edges of the polygons to enclose a finite region of space within one chamber. The polygon-shaped surfaces are called faces. It is required that the surface be simple (not more than one chamber is enclosed) and closed (you can’t get in from the outside without tearing it). The segments where the polygons meet are called edges. The points where edges intersect are called vertices.

1) Use the definition to decide which of the below images of 3-dimensional objects represent polyhedra (“polyhedra” is the plural form of polyhedron)

A polyhedron is regular if all of the faces are the same congruent, regular polygon and all of the vertices have exactly the same number of polygons.

2) Use the pieces of a Frameworks™ set (manufactured by Polydron) to build all of the regular polyhedra. Be systematic so you can give an argument that you have found them all. (Polyhedra are named for the number of faces they contain; for example, a polyhedron with ten faces would be called a decahedron.)

3) Can you build a polyhedron that uses only one type of congruent regular polygon but is not regular? Explain.

4) See if you can find shortcut ways of counting the numbers of vertices or edges of regular polyhedra.
**Read and Study**

*Everything should be made as simple as possible, but not one bit simpler.*

*Albert Einstein*

In the *Class Activity* you were asked to build models of the regular polyhedra. These special objects, also called the Platonic solids, have been known since before the days of the Greek mathematics. Approximations of the regular polyhedra even occur in nature. In particular, the tetrahedron, cube, and octahedron shapes all appear as crystal structures. We also find polyhedral shapes among living things, such as the Circogonia icosahedra shown below, a species of Radiolaria, which is shaped like a regular icosahedron.

![Image](http://en.wikipedia.org/wiki/Platonic_solids)

Many viruses also have the shape of a regular icosahedron. Viral structures are built of repeated identical protein subunits and apparently the icosahedron is the easiest shape to assemble using these subunits.

**Nets** are two-dimensional figures that can be folded into three-dimensional objects. Below are nets for the regular tetrahedron and for the cube. *Imagine how each folds up to make the 3-dimensional object.*
Before you read further, go to this site and build yourself one of each of the regular polyhedra. You will need them for the homework. http://www.mathsisfun.com/platonic_solids.html

Connections to the Elementary Grades

Students in grades 3–5 should examine the properties of two-and three-dimensional shapes and the relationships among shapes. They should be encouraged to reason about these properties by using spatial relationships.

NCTM Principles and Standards, 2000

The following excerpt from the NCTM Standards for Grades 3 – 5 Geometry (p. 168) describes the importance of visualization and spatial reasoning as tools elementary students can use to understand the properties of geometric objects and the relationship between these properties and the shapes. Read these paragraphs and study the examples. Then build the building they describe.

Use visualization, spatial reasoning, and geometric modeling to solve problems

Students in grades 3–5 should examine the properties of two-and three-dimensional shapes and the relationships among shapes. They should be encouraged to reason about these properties by using spatial relationships. For instance, they might reason about the area of a triangle by visualizing its relationship to a corresponding rectangle or other corresponding parallelogram. In addition to studying physical models of these geometric shapes, they should also develop and use mental images. Students at this age are ready to mentally manipulate shapes, and they can benefit from experiences that challenge them and that can also be verified physically. For example, “Draw a star in the upper right-hand corner of a piece of paper. If you flip the paper horizontally and then turn it 180°, where will the star be?”

Much of the work students do with three-dimensional shapes involves visualization. By representing three-dimensional shapes in two dimensions and constructing three-dimensional shapes from two-dimensional representations, students learn about the characteristics of shapes. For example, in order to determine if the two-dimensional shape in figure 5.15 is a net that can be folded into a cube, students need to pay attention to the number, shape, and relative positions of its faces.
Students should become experienced in using a variety of representations for three-dimensional shapes, for example, making a freehand drawing of a cylinder or cone or constructing a building out of cubes from a set of views (i.e., front, top, and side) like those shown in figure 5.16.

Make a building out of ten cubes by looking at the three pictures of it below.

Fig. 5.15. A task relating a two-dimensional shape to a three-dimensional shape

Fig. 5.16. Views of a three-dimensional object (Adapted from Battista and Clements 1995, p. 61)
Homework

Getting ahead in a difficult profession requires avid faith in yourself. That is why some people with mediocre talent, but with great inner drive, go much further than people with vastly superior talent.

Sophia Loren

1) Do all the italicized things in the Read and Study and Connections sections.

2) Write out the details of a mathematical argument that there are exactly five regular polyhedra.

3) The following picture is often given as an example of a regular icosahedron. Examine the picture carefully and determine why this picture is claiming to be something that it is not.

4) Is it possible to build a polyhedron where all of the faces are congruent regular polygons but the polyhedron is not regular? Explain.

5) Use the five models you built to find a formula that relates numbers of vertices, edges, and faces in regular polyhedra.

6) Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 1 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 2 Session 2 Mystery Bag Sorting. Make up your own mystery for your students to solve similar to those given in the lesson.
Class Activity 20: Pyramids and Prisms

*It is better to know some of the questions than all of the answers.*

*James Thurber*

There are two special categories of polyhedra we will explore in this activity: pyramids and prisms.

A **pyramid** is a polyhedron in which all but one of the faces are triangles that share a common vertex (called the **apex**). The remaining face may be any polygon and is called the base. A pyramid is named for the shape of its base. For example, a square pyramid is one whose base is a square. (Notice that the base of a pyramid need not have the shape of a *regular* polygon. It could look like the figure below, for example.)

![Pyramid](image)

1) You have already built a pyramid with an equilateral triangle base (the tetrahedron). In your group, sketch a pyramid with a square base and another with a hexagonal base.

2) Use your pictures to help you find formulas for the number of faces, the number of vertices, and the number of edges in a pyramid whose base is an $n$-gon. Then prove that your formulas will work for all pyramids.

A **prism** is a polyhedron in which two of the faces (called the bases) are congruent and lie on parallel (non-intersecting) planes and the remaining faces are parallelograms. The prism is also named after its base. If the parallelogram faces are rectangular, the prism is a **right** prism. If the parallelogram faces are non-rectangular, the prism is an **oblique** prism.

3) Which (if any) of the regular polyhedra are prisms? Explain.

4) Sketch an oblique prism.

5) Sketch a prism with a triangular base and one with a hexagonal base. Then use your pictures to find formulas for the number of faces, the number of vertices, and the number of edges in a prism whose bases are congruent $n$-gons. Prove that your formula works in the case of all prisms.
Read and Study

The mediocre teacher tells. The good teacher explains. The superior teacher demonstrates. The great teacher inspires.

William Arthur Ward

The most famous pyramids are the Egyptian pyramids. These huge stone structures are among the largest man-made constructions. In Ancient Egypt, a pyramid was referred to as the "place of ascendance." The Great Pyramid of Giza is the largest in Egypt and one of the largest in the world with a base that is over 13 acres in area. It is one of the Seven Wonders of the World, and the only one of the seven to survive into modern times.

The Mesopotamians also built pyramids, called ziggurats. In ancient times these were brightly painted. Since they were constructed of mud-brick, little remains of them. The Biblical Tower of Babel is believed to have been a Babylonian ziggurat.

A number of Mesoamerican cultures also built pyramid-shaped structures. Mesoamerican pyramids were usually stepped, with temples on top, more similar to the Mesopotamian ziggurat than the Egyptian pyramid. The largest pyramid by volume is the Great Pyramid of Cholula, in the Mexican state of Puebla. This pyramid is considered the largest monument ever constructed anywhere in the world, and is still being excavated.

Modern architects also use the pyramid shape for building. An example is the Louvre Pyramid in Paris, France, in the court of the Louvre Museum. Designed by the American architect I. M. Pei and completed in 1989, it is a 20.6 meter (about 70 foot) glass structure which acts as an entrance to the museum.

The most common example of a prism in the “real world” is its occurrence in optics, where a prism is a transparent optical element with flat, polished surfaces that refract light. The exact angles between the surfaces depend on the application. The traditional geometrical shape is that of a triangular prism with a triangular base and rectangular sides, and in colloquial use "prism" usually refers to this type. Prisms are typically made out of glass, but can be made from any material that is transparent to the wavelengths for which they are designed. A prism can be used to break light up into its constituent spectral colors (the colors of the rainbow). They can also be used to reflect light, or to split light into components with different polarizations.
Connections to the Elementary Grades

It is the supreme art of the teacher to awaken joy in creative expression and knowledge.

Albert Einstein

Elementary students begin their study of 3-dimensional shapes in the first and second grades. Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 1 Teachers Guide. Spend 10-15 minutes looking through Unit 5 Module 2 Sessions 4 and 5. What ideas about prisms and pyramids are emphasized in these sections?

Now look at the Homelink section from Session 5. Examine questions 4 and 5 carefully. Determine several reasons why your chosen figure does not belong. In question 5, determine at least two different figures on which to place the “X”.

Homework

I attribute my success to this: I never gave or took any excuse.

Florence Nightingale

1) Do the problem in the Connections section.

2) Make a mathematical argument for the formulas you found for the number of faces, the number of vertices, and the number of edges in a pyramid whose base is an n-gon.

3) Make a mathematical argument for the formulas you found for the number of faces, the number of vertices and the number of edges in a prism whose bases are congruent n-gons.

4) Prove that the formula you found relating the vertices, edges, and faces of a regular polyhedra also holds for the n-gonal pyramid and for the n-gonal prism.

5) The following are descriptions of pyramids and prisms. Identify the prism or pyramid and sketch a net.

   a) A polyhedron with 5 faces and 9 edges.
   b) A polyhedron with two hexagonal faces and the remaining faces are rectangles.
   c) A polyhedron with 12 edges and 8 vertices.
   d) A polyhedron with 7 faces and 7 vertices.
Class Activity 21: Surface Area

*Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting.*

*Ernst Mach*

1) Use 14 interlocking unit cubes to build the 3-dimensional figure pictured.

   a) What is the **surface area** of this object?

   b) What is its **volume**?

   c) Is this a polyhedron? Explain.

2) What are all the possible values for the surface area of figures made with 14 interlocked cubes? Explain.
Read and Study

*Number rules the universe.*

*Pythagoras*

Surface area is the measure of the boundary of a three-dimensional object in the same way that perimeter is the measure of the boundary of a two-dimensional object. And just like we measure *perimeter* by adding up the *lengths* of each section of the boundary of the object, we measure *surface area* by adding up the *areas* of each face of the boundary of the object.

For example, suppose we have a rectangular prism that is 3 cm long by 5 cm wide by 8 cm tall. *Take a minute to sketch that figure.*

This means that we have two faces that are rectangles that are 3 cm by 5 cm (and so have an area of 15 square cm or $15 \text{ cm}^2$), two faces that are rectangles that are 5 cm by 8 cm (and so have an area of 40 square cm each), and two faces that are 3 cm by 8 cm (and so have an area of 24 square cm each). Then the surface area of the prism is $15 + 15 + 40 + 40 + 24 + 24 = 158$ square cm. *(Check our work.)* Notice that we use square units to measure surface area since it is a measure of two-dimensional (flat) space.

There is no need to develop completely new formulas for surface area. We can use whichever area formulas are appropriate given the shapes that make up the faces of the object.
Connections to the Elementary Grades

The important thing is not to stop questioning. Curiosity has its own reason for existing.

Albert Einstein

Elementary students can use interlocking cubes (such as the ones we used in the Class Activity) to create “buildings” and then use those buildings to build an understanding of surface area through drawing the buildings from various view-points. For example the building below can be viewed from the top, front, and side.

Build your own building out of 5 cubes. Then sketch the figure and its corresponding top, front, and side views.

Here are the top, front, and side views of another building. Sketch a possible building with these views.

How do these activities relate to the surface area of the building?

Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 5 Teachers Guide. Spend 10-15 minutes looking through Unit 1 Module 2 Session 4. The end of the lesson lists several questions for you to ask your students. Think up two more questions you could ask your students about the activity.
Homework

*Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air.*

*John Quincy Adams*

1) Do all the italicized things in the Read and Study and Connections sections.

2) Suppose you are asked to build an object using interlocking unit cubes that has a surface area of 36 square units.
   a) How many unit cubes at a minimum would you need?
   b) What is the maximum number of unit cubes you could use?
   c) For what three-dimensional shape will the volume be greatest for a fixed surface area? Make a conjecture.

3) Below is a net for a tetrahedron. If each equilateral triangle has sides of length 5 cm, what is the surface area of the tetrahedron? (Those little extra parts are tabs that help you to tape it together – don’t include those.)

4) What amount of paper (area) would you need to make a net for the cube with an edge length of 7 cm? (Ignore any paper needed for tabs.)

5) Sketch a net for a rectangular prism that is 4 cm by 5 cm by 7 cm. What is the surface area of that prism?
Class Activity 22: Nothing But Net

*Pure mathematics is, in its way, the poetry of logical ideas.*  
*Albert Einstein*

1) Figure out how to build a paper model of a right **cylinder** with a radius of 3 cm and a height of 10 cm. Use your model to help you find a formula for the surface area of a cylinder.

2) Build a paper model of a pyramid that has a 6 cm by 6 cm square base and a height of 4 cm. What is its surface area?
**Class Activity 23: Building Blocks**

*Measure what is measurable, and make measurable what is not so.*  
*Galileo*

**Volume** is a measure of the amount of space enclosed by a three-dimensional object. One way to define volume is as the number of cubes (cubic units) that it takes to fill the enclosed space. Using this definition, we can measure volume by carefully estimating the number of cubes that fit within the object.

Some objects have formulas that will help us to compute volume. What is a formula for the volume of a rectangular prism with length $l$, width $w$, and height $h$? *Why does it make sense?*

Next, you are going to build three objects out of small wooden cubes (with half-inch edges) and large wooden cubes (with one-inch edges). Follow the steps below.

**Step A:** Build something out of 10 small wooden cubes. We’ll call this figure, “Object A.”

**Step B:** Using 10 large wooden cubes, build a larger version of Object A. We’ll call this figure, “Object B.”

**Step C:** Using as many small cubes as necessary, reproduce Object B. (It should be the same size and shape as Object B, but build out of small cubes instead of large ones.) We’ll call this figure, “Object C.”

1) How much taller is Object B than Object A?

2) How much bigger is the surface area of Object B compared to Object A?

(This problem continues on the next page.)
3) How much bigger is the volume of Object B compared to Object A?

4) What is the relevance of Object C to answering these questions?

5) What if I gave you a larger block that is 3 times the length of the small cube; how would your answers to 1-3 change? What about 4 times the length?

6) What if I gave you a smaller block that is ½ times the length of the small cube; how would your answers to 1-3 change?
**Read and Study**

*Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost.*

*W. S. Anglin*

In the *Class Activity* you talked about why it made sense to calculate the volume of a rectangular prism by multiplying the length $l$ of the prism by the width $w$ of the prism by the height $h$ of the prism ($V = l \times w \times h$). The idea here is that we can think of a prism as **layers of the base** stacked one upon the next. So volume is the area of the base multiplied by the height ($V = \text{Area of Base} \times h$). *Have a look at the picture below to see what we mean.*

![Cube](image)

Will that same idea work for a cylinder? Is its volume the area of its base multiplied by its height? *Make a sketch of a cylinder and use it to help you to explain your thinking.*

**Connections to the Elementary Grades:**

*The cure for boredom is curiosity. There is no cure for curiosity.*

*Dorothy Parker*

Children often begin their explorations of volume by determining the volume of various rectangular boxes. Go online to bridges.mathlearningcenter.org and find the *Bridges in Mathematics* Grade 5 Teachers Guide. Spend 10-15 minutes looking through Unit 1 Module 1 Sessions 4 and 5. *How does this activity help students to understand volume?*
Homework

*When the world says, "Give up,"*  
*Hope whispers, "Try it one more time."*  
*Author Unknown*

1) Do all the italicized things in the *Read and Study* section.

2) Do all those problems in the *Connections* section.

3) When you double the length of all the sides of a cube, what happens to its volume? Why does this happen? What happens when you triple the length?

4) Below we have sketched a net for a cube.

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a) Build a paper model of a cube that is twice as long in each linear dimension.
b) Build a paper model of a cube that has twice the surface area of the cube suggested by the net.
c) Build a paper model of a cube that has twice the volume of the cube suggested by the net.
Class Activity 24: Volume Discount

*The laws of nature are but the mathematical thoughts of God.*

Euclid

1) Reason with unit cubes to create volume formulas for the following objects:

   a) a rectangular prism

   b) a triangular prism

   c) a cylinder

2) Use rice -- not formulas -- to answer the next four questions:

   a) How does the volume of the square pyramid compare to the volume of the square prism? Use this information to create a volume formula for a square pyramid.

   (This activity continues on the next page.)
b) How does the volume of the triangular pyramid compare to the volume of the triangular prism? Use this information to create a volume formula for a triangular pyramid.

c) How does the volume of the cone compare with the volume of the cylinder? Use this information to create a volume formula for a cone.

d) How does the volume of the cone compare with the volume of the sphere? Use this information to create a volume formula for a sphere.
Read and Study

*If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things.*

Rene Descartes

In the *Class Activity* you observed that the volume of a prism is about three times the volume of a pyramid with the same height and base, and that the volume of a cylinder is three times the volume of a cone with the same height and radius. It turns out that for the ideal objects, the factor of three is exactly correct. These nice relationships make formulas for volume relatively straightforward if we build on formulas we already know.

In an earlier *Class Activity* you explained why it makes sense that the volume of a rectangular prism is $V = (l \times w \times h)$. Now, since you have seen that this volume is three times the volume of the pyramid with the same height and rectangular base, the volume of the pyramid should be given by the formula $V = \frac{1}{3}(l \times w \times h)$.

We can generalize these two formulas so that they apply to *all* prisms and *all* pyramids (and to *all* cylinders and *all* cones). Notice in the volume formula for the rectangular prism that $l \times w$ is just the area of the rectangular base. If the base has a different shape, we just need to use the appropriate area formula to find the area of the base and then multiply by the height to find the volume of the prism: $V = (\text{Area of Base}) \times h$ for all prisms and cylinders.

![Rectangular prism](image1.png)

![Hexagonal prism](image2.png)

![Pentagonal prism](image3.png)

Figure from G.S. Rehill's Interactive Maths Series.
Likewise, we can use \( V = \frac{1}{3} (\text{Area of base} \times h) \) for all pyramids and cones.

The sphere is another three-dimensional shape that has a well-known volume formula, \( V = \frac{4}{3} \pi r^3 \), where \( r \) is the radius of the sphere. This formula comes from calculus – so we don’t have the machinery to prove it works – however you (and your students) can observe that the formula seems plausible using the water experiment. Here’s the idea.

Since a cylinder has volume \( \pi r^2 \times h \), and the cylinder you used in the Class Activity has a height of \( 2r \), this means that the cylinder you used to pour water has a volume of \( 2\pi r^3 \). \textit{Take a minute to carefully think this through.}

Now, since a cone has one third the volume of a cylinder with the same base, the cone you used must have a volume of \( \frac{1}{3} \times (2\pi r^3) \).

So if it takes two cones to fill a sphere it the water pouring experiment, then it makes sense to conjecture that the volume of a sphere must be given by the formula, \( V = \frac{4}{3} \pi r^3 \). \textit{Make sure that you understand what we’re saying here.}

The corresponding formula for the surface area of a sphere is \( SA = 4 \pi r^2 \).

Of course, there are many three-dimensional objects for which we do not have formulas to calculate their volume. For all of these, we can use the “capacity” definition that was discussed in the Class Activity. When we measure volume using capacity we commonly use units like the cup, the quart, the gallon, the liter, etc. When we find volume using a formula we commonly use units like cubic inches, cubic yards, and cubic centimeters. Volume is a three-dimensional measure so the units used will all be cubic units.
Children in grade three should have experiences measuring volumes using water and weighing physical objects. Here are the relevant Common Core State Standards. Read them.


In grade 5, students should learn to do many of the things we have talked about in the last two sections. They should think of volume measurement as both the number of cubic units required to fill a 3-dimensional object, and as the capacity of the object. They should understand how to think about the volume of a prism and make sense of some volume formulas.

You will find the relevant Common Core State Standards for grade 5 below. Read them. What do they mean when they say that students should recognize volume as “additive?”
CCSS Grade 5: Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

1. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
   a. A cube with side length 1 unit, called a “unit cube,” is said to have “one cubic unit” of volume, and can be used to measure volume.
   b. A solid figure which can be packed without gaps or overlaps using $n$ unit cubes is said to have a volume of $n$ cubic units.

2. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

3. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.
   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole number edge lengths in the context of solving real world and mathematical problems.
   c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.
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Homework

To climb steep hills requires a slow pace at first.
William Shakespeare

1) Do all the italicized things in the Read and Study section.

2) Do the italicized things in the Connections section.

3) Suppose you have a right circular cylinder with height $h$ and radius $r$ and an oblique circular cylinder with height $h$ and radius $r$. Do these two cylinders have the same volume? Compare a straight stack of pennies to a “slanted” stack to see. (Really do it.) In each case, how is height measured?

4) The closet in my living room has an odd shape because my apartment is on the top floor of a house with a slanted roof. The closet is 6 feet tall in front but only 4 feet tall in back. It is 3 feet deep and 12 feet wide.
   a) Build a scaled down paper model of my closet. Really do this. It will help you with the rest of this problem.
   b) How many cubic feet of storage does it hold?
   c) I want to paint the inside walls and ceiling of my closet. How many square feet will I need to paint?

5) How many cubic feet of water does a semi-cylindrical (half a cylinder) trough hold that is 10 feet long by 1 foot deep? How many cubic inches is that?

6) I used 1500 cubic inches of helium to fill my balloon. Assuming my balloon is a sphere, to the nearest tenth of an inch, what is its diameter? What is its surface area?

7) A movie theater sells popcorn in a box for $2.75 and popcorn in a cone for $2.00. The dimensions of the box and the cone are given. Which is the better buy? Explain.

8) Go online to bridges.mathlearningcenter.org and find the Bridges in Mathematics Grade 5 Teachers Guide. Spend 10-15 minutes looking through Unit 6 Module 3 Session 4. How does this activity further students’ understanding of volume?
Class Activity 25: Volume Challenge

*If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.*

*John Louis von Neumann*

1) Your group should work together to build paper models of each of the following objects in such a way that the volume of each is 60 cm$^3$. (Other than that, you may use any dimensions you like.)

   a) Cylinder
   b) Rectangular prism that is not a cube
   c) Pyramid with a square base
   d) Prism with an equilateral triangle base

2) Find the surface area of each of the objects you built in 1) above.
Read and Study and Connections

*It's not that I'm so smart; it's just that I stay with problems longer.*

*Albert Einstein*

By the time they reach upper elementary school, students can solve many practical problems in geometry. However these students are not ready to simply apply formulas to solve problems; instead they need to use models in order to make sense of problems. As their teacher, your job will be to make sure that children in your classes build and handle appropriate models.

Notice that the Common Core State Standards for grade 6 explicitly require that students use hands-on models to explore ideas. Students are asked to cut triangles and other shapes apart, to draw pictures, to pack spaces with unit cubes, to use the coordinate plane, and to build and use nets. *Read these standards. Have we done of these things in this book so far this term?*

**CCSS Grade 6: Solve real-world and mathematical problems involving area, surface area, and volume.**

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Homework

*The elevator to success is not running; you must climb the stairs.*

*Zig Ziglar*

1) Do all the italicized things in the *Read and Study* and *Connections* sections.

2) A triangle on a coordinate plane has vertices at (2, 0), (7, 0) and (7, 8).
   a) Sketch the polygon on a coordinate plane.
   b) What is the length of the hypotenuse?
   c) What is the area of the polygon?
   d) What *Common Core State Standard* is addressed by this problem?

3) A replica of the Great Pyramid stands 2 feet tall and is 3.15 feet long on a side (it has a square base).
   a) Approximately how much volume does this replica take up? Use the model you built in the *Class Activity* to help you to think about this problem.
   b) What is the surface area of the replica?
   c) Suppose the scale of the replica to the real thing is in is 1 foot to 240 feet. What is the volume of Great Pyramid? What is its surface area?
   d) Which of the *Common Core State Standards* are met by this series of problems?
Summary of Big Ideas from Chapter Four

*Man’s mind, once stretched by a new idea, never regains its original dimensions.*

*Oliver Wendell Holmes*

- There are exactly five regular polyhedra – and children can understand why this is so.

- Surface area is a measure of the sum of areas of the faces of a three-dimension object. It is the number of square units it takes to cover the faces of the object. A unit of surface area is flat like this:

- Volume is a measure of the number of cubic units it takes to fill a three dimensional object. A unit of volume is three-dimensional and looks like this:

- Volume can also be measured by the amount of liquid (or sand) it takes to fill a three dimensional object.

- The volume of a prism or cylinder can be found by computing the area of the base and multiplying that by its height (the number of layers of the base). This idea works for both right and oblique objects.

- It takes the volume of three pyramids to fill a prism with the same base and height, and it takes the volume of three cones to fill a cylinder with the same base and height.

- Children need to have many years of experiences building and using a variety of models.
APPENDICES
Euclid’s Postulates and Propositions

Euclid's Elements
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Book I

POSTULATES

Let the following be postulated:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any center and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS

1. Things which are equal to the same thing are also equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

4. Things which coincide with one another are equal to one another.

5. The whole is greater than the part.
BOOK I  PROPOSITIONS

Proposition 1.
On a given finite straight line to construct an equilateral triangle.

Proposition 2.
To place at a given point (as an extremity) a straight line equal to a given straight line.

Proposition 3.
Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Proposition 4.
If two triangles have the two sides equal to two sides respectively, and have angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Proposition 5.
In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Proposition 6.
If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Proposition 7.
Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

Proposition 8.
If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Proposition 9.
To bisect a given rectilineal angle.

Proposition 10.
To bisect a given finite straight line.

Proposition 11.
To draw a straight line at right angles to a given straight line from a given point on it.
Proposition 12.
To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Proposition 13.
If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

Proposition 14.
If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

Proposition 15.
If two straight lines cut one another, they make the vertical angles equal to one another.

Proposition 16.
In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Proposition 17.
In a triangle two angles taken together in any manner are less than two right angles.

Proposition 18.
In any triangle the greater side subtends the greater angle.

Proposition 19.
In any triangle the greater angle is subtended by the greater side.

Proposition 20.
In any triangle two sides taken together in any manner are greater than the remaining one.

Proposition 21.
If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

Proposition 22.
Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I.20]

Proposition 23.
On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.
Proposition 24.

If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

Proposition 25.

If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater that the other.

Proposition 26.

If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.

Proposition 27.

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

Proposition 28.

If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

Proposition 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

Proposition 30.

Straight lines parallel to the same straight line are also parallel to one another.

Proposition 31.

Through a given point to draw a straight line parallel to a given straight line.

Proposition 32.

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Proposition 33.

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.
Proposition 34.  
In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

Proposition 35.  
Parallelograms which are on the same base and in the same parallels are equal to one another.

Proposition 36.  
Parallelograms which are on equal bases and in the same parallels are equal to one another.

Proposition 37.  
Triangles which are on the same base and in the same parallels are equal to one another.

Proposition 38.  
Triangles which are on equal bases and in the same parallels are equal to one another.

Proposition 39.  
Equal triangles which are on the same base and on the same side are also in the same parallels.

Proposition 40.  
Equal triangles which are on equal bases and on the same side are also in the same parallels.

Proposition 41.  
If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

Proposition 42.  
To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Proposition 43.  
In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Proposition 44.  
To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.
Proposition 45.
To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

Proposition 46.
On a given straight line to describe a square.

Proposition 47.
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Proposition 48.
If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.
Glossary

"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean - neither more nor less."
"The question is," said Alice, "whether you can make words mean so many different things."
"The question is," said Humpty Dumpty, "which is to be master - that's all."
Lewis Carroll, Through the Looking Glass

Acute angle – an angle that measures less than 90 degrees

Acute triangle – a triangle with three acute angles

Adjacent angles – two non-overlapping angles that share a vertex and a common ray

Alternate exterior angles – two non-adjacent angles formed by a transversal of a pair of lines

that lie outside the lines and on opposite sides of the transversal

Alternate interior angles – two non-adjacent angles formed by a transversal of a pair of lines

that lie between the lines and on opposite sides of the transversal

Angle – the figure formed by two rays with a common endpoint

Angle bisector – the line through the vertex of an angle that divides the angle into two

congruent angles

Apex (of a pyramid) – the common point of the non-base faces of a pyramid

Apex (of a cone) – the common point of the line segments that create a cone

Arc – the set of points on a circle between two given points of the circle (There are actually two

arcs between any two given points; the shorter one is called the minor arc and the

longer one is called the major arc.)

Area – the quantity of two-dimensional space enclosed by a closed plane figure

Attribute – a property of a geometric object that can be measured (such as length) or

categorized (such as color)

Axiom – a statement that we agree to accept as true without proof

Axiomatic system – a set of undefined terms, definitions, axioms, and theorems that create a

mathematical structure

Axis (of a cone) – the line joining the apex to the center of the (circle) base

Axis of symmetry – a line in space around which a three-dimensional object is rotated
**Base angles** (of an isosceles triangle) – the angles that are opposite the congruent sides of an isosceles triangle

**Bilateral symmetry** – an object has bilateral symmetry when it has exactly one line of reflectional symmetry

**Bisect** – to divide a geometric object such as a line segment or an angle into two congruent pieces

**Boundary** – the set of points that separate the inside of a closed planar object from the outside

**Center** (of a circle) – the point that is equidistant from all points on the circle

**Central angle** – an angle whose vertex is a center of a geometric object

**Chord** – a line segment whose endpoints are distinct points on a given circle

**Circle** – the set of all points in the plane that are the same distance from a given point, called the center

**Circumcenter** – the point of intersection of the three perpendicular bisectors of a triangle

**Circumference** – the circumference of a circle is its perimeter

**Circumscribed circle** – the circle that contains all the vertices of a polygon

**Closed curve** – a curve that starts and stops at the same point

**Coincide** – two objects are said to coincide if they correspond exactly (are identical)

**Collinear points** – points that lie on the same line

**Compass** – an instrument used to construct a circle

**Complementary angles** – two angles whose measures sum to 90 degrees

**Concave polygon** – a polygon for which at least one diagonal lies outside the polygon

**Concurrent lines** – three or more lines that intersect in the same point

**Cone** (circular) - a three-dimensional geometric object consisting of all line segments joining a single point (called the apex) to every point of a circle (called the base)

**Congruent objects** – two geometric objects that coincide when superimposed

**Conjecture** – a guess or a hypothesis

**Contrapositive** (of “If A, then B.”) – “If not B, then not A,” where A and B are statements

**Converse** (of “If A, then B.”) – “If B, then A,” where A and B are statements

**Convex polygon** – a polygon all of whose diagonals lie inside the polygon
Coordinate plane – a plane on which points are described based on their horizontal and vertical distances from a point called the origin

Coplanar lines – lines that lie in the same plane

Corresponding angles - two angles formed by a transversal of a pair of lines that lie on the same side of the transversal and also lie on the same side of the pair of lines

Corresponding points – a pair of points, one of which is the original point and the other of which is the image of that point under a transformation

Counterexample – an example that demonstrates that a statement (conjecture) is false

Curve – a set of points drawn with a single continuous motion

Cylinder (circular) – a three-dimensional geometric object consisting of two parallel and congruent circles (and their interiors) and the parallel line segments that join corresponding points on the circles

Decagon – a polygon with exactly ten sides

Deductive reasoning – the process of coming to a conclusion based on logic

Degree – a unit of angle measure for which a full turn about a point equals 360 degrees

Diagonal – the line segment joining two non-adjacent vertices of a polygon

Diameter – a line segment through the center of a circle whose endpoints lie on the circle

Dilation – a motion of the plane in which one point P remains fixed and all other points are pushed radially outward from P or pulled radially inward toward P so that all distances have been multiplied by some scale factor

Dimension (of a real space) – the number of mutually perpendicular directions needed to describe the location of the set of points in that space

Distance (on a coordinate plane) – the size of the portion of a straight line that lies between the two points on the coordinate plane as measured by the distance formula: \( d = \sqrt{a^2 + b^2} \), where \( a \) is the horizontal distance between the points (as measured on the x-axis) and \( b \) is the vertical distance (as measured on the y-axis)

Distinct (geometric objects) - two objects that do not share all their points in common

Dodecagon – a polygon with exactly twelve sides
**Dual** (of a regular polyhedron) – the polyhedron whose vertices are exactly the midpoints of the faces of the regular polyhedron

**Edge** – the line segment (side) that is shared by two faces of a polyhedron

**Endpoint** (of a line segment) – one of two points that determines a line segment

**Equiangular polygon** – a polygon all of whose vertex angles are congruent

**Equilateral polygon** – a polygon all of whose sides are congruent

**Example** (of a definition) – a geometric object that satisfies the conditions of the definition

**Exterior angle** – the angle formed by a side of a polygon and the extension of an adjacent side

**Face** – a polygon (with interior) that forms a portion of the two-dimensional surface of a polyhedron

**Fixed point** – a point whose location remains the same under a transformation

**Generalization** – the extension of a statement (about a pattern) that is true for specific values of \( n \) (a natural number) to a statement (about that pattern) that is true for all values of \( n \)

**Geoboard** – a manipulative typically composed of a board with 25 pegs arranged in a 5 x 5 square array

**Geoboard polygon** – a polygon whose vertices are all points (pegs) on a geoboard

**Height** (of a triangle) – length of the line segment from a vertex perpendicular to the opposite side. This line segment is often called the altitude of the triangle

**Heptagon** – a polygon with exactly seven sides

**Hexagon** – a polygon with exactly six sides

**Homogeneous** (vertices in a tessellation) – vertices that have exactly the same polygons meeting in exactly the same arrangement

**Homogeneous** (vertices in a polyhedron) – vertices that have exactly the same polygon faces meeting in exactly the same arrangement

**Hypotenuse** – the side of a right triangle opposite the right angle

**Image** (of a transformation) – the set of points that result from the motion of an object by a translation, a rotation, or a reflection

**Inductive reasoning** – the informal process of coming to a conclusion based on examples

**Inscribed polygon** – the polygon inside a circle whose vertices all lie on the circle
**Interior angle** – any one of the alternate interior angles formed by a transversal to two lines

**Intersection** (of two lines) – the point the lines have in common

**Intersection** (of two sets) – the set of elements that are common to both sets

**Isosceles** – having at least one pair of congruent sides

**Justification** – an argument based on axioms, definitions, and previously proven results to show that a conjecture is true

**Leg** – a side of a right triangle opposite an acute angle

**Length** – the distance between two points on a one-dimensional curve

**Line** – an undefined one-dimensional set of points understood to cover the shortest distance and to extend in opposite directions indefinitely

**Line of reflection** – the line about which an object is reflected to form its mirror image

**Line segment** – the set of all points on a line between two given points called the endpoints

**Mass** – a concept of physics that corresponds to the intuitive idea of “how much matter there is in an object;” unlike weight, the mass of an object does not depend upon the object’s location in the universe

**Measure** – to determine the quantity of an attribute (or of a fundamental concept such as time) using a given unit

**Metric system of measurement** – the system of measurement units in which there is one fundamental unit defined for each quantity (attribute) with multiples and fractions of these units established by **prefixes based on powers of ten**

**Midpoint** – the point on a line segment that divides it into two congruent line segments

**Model** – a representation of an axiom system in which each undefined term is given a concrete interpretation in such a way that the axioms all hold

**Net** – a two-dimensional figure that can be folded into a three-dimensional object

**Nonagon** – a polygon with exactly nine sides

**Noncollinear** (points) – a set of points not all of which lie on the same line

**Non-example (of a definition)** – an example that demonstrate most of the conditions of a definition but that fails to satisfy at least one condition
Non-standard unit of measure – a unit of measure whose value is not established by reference to an accepted standard; for example, a block

Oblique prism (pyramid, cylinder) – a prism (pyramid, cylinder) that is not right

Obtuse angle – an angle with measure greater than 90 degrees

Obtuse triangle – a triangle with one obtuse angle

Octagon – a polygon with exactly eight sides

Order (of a rotational symmetry) – the number of different rotations that are a symmetry of an object

Orientation – the direction, clockwise or counterclockwise, of the reading of the vertices of a polygon in alphabetical order

Parallel lines – coplanar lines with no points in common

Parallelogram – a quadrilateral in which both pairs of opposite sides are parallel

Partition – a division of a geometric object into a set of non-overlapping objects whose union is the original object

Pentagon – a polygon with exactly five sides

Perimeter (of a plane object) – the length of the boundary of the object

Perpendicular bisector – the line through the midpoint of a line segment that is also perpendicular to the line segment

Perpendicular lines – two lines that intersect to form four right angles

Pi ($\pi$) – the exact number of times the diameter of a circle fits into its circumference (or the ratio of the circumference of a circle to its diameter); this ratio is an irrational number that is constant for all size circles and is approximately equal to 3.14159

Planar curve – a curve that lies entirely within a plane

Plane – an undefined two-dimensional set of points understood to be “flat” and to extend in all directions indefinitely

Plane of symmetry – a plane in space about which a three-dimensional object is reflected

Platonic solid – a regular polyhedron plus its interior

Point – an undefined zero-dimensional object; a location with no size

Polygon – a finite set of line segments that form a simple closed planar curve
**Polyhedron** (plural: polyhedra) – a finite set of polygon-shaped faces joined pairwise along the edges of the polygons to enclose a finite region of space within one chamber

**Prism** – a polyhedron in which two of the faces (called the bases) are congruent and lie on parallel (non-intersecting) planes and the remaining faces are parallelograms.

**Proof** – a deductive argument that establishes the truth of a claim

**Protractor** – an instrument used to measure angles

**Pyramid** – a polyhedron in which all but one of the faces are triangles that share a common vertex (called the apex); the remaining face may be any polygon and is called the base

**Pythagorean triple** – three positive integers that satisfy the Pythagorean theorem

**Quadrilateral** – a polygon with exactly four sides

**Quantifier** (in logic) – a word or phrase (such as “all” or “at least one”) that indicates the size of the set to which the statement applies

**Radius** (plural: radii) – the line segment joining a point on a circle to the center of the circle

**Ray** – the set of points on a line beginning at a given point (called the endpoint) and extending in one direction on the line from that point

**Rectangle** – a quadrilateral with four right angles

**Reflection** – (in a line $l$) is a transformation of the plane in which the image of a point $P$ on $l$ is $P$, and if $A$ is a point not on $l$ and if the image of $A$ is $A'$, then $l$ is the perpendicular bisector of $AA'$

**Reflection symmetry** (2-dimensional) – a reflection in which an object is divided by the line of reflection into two parts that are mirror images of each other

**Reflection symmetry** (3-dimensional) – a reflection in which an object is divided by the plane of reflection into two parts that are mirror images of each other

**Regular polygon** – a polygon with all sides congruent and all vertex angles congruent

**Regular polyhedron** – a polyhedron whose faces are each the same regular polygon with the same number of faces meeting at each vertex

**Regular tessellation** – a tessellation that contains only one regular polygon

**Rhombus** (plural: rhombi) – a quadrilateral with four congruent sides

**Right angle** – an angle that is exactly one fourth of a complete turn about a point
**Right prism** (pyramid, cylinder) – a prism (pyramid, cylinder) in which the line joining the centers of the bases (the apex of the pyramid to the center of its base) is perpendicular to the base

**Right triangle** – a triangle with one right angle

**Rigid motions** - transformations of the plane that preserve distances between points (they do not distort the shape or size of objects)

**Rotation** (about a point \( P \) through an angle \( \theta \)) – a transformation of the plane in which the image of \( P \) is \( P \) and, if the image of \( A \) is \( A' \), then \( \overline{PA} \cong \overline{PA'} \) and \( \angle APA' = \theta \). Point \( P \) is called the center of the rotation

**Rotation symmetry** (2-dimensional) – a rotation about a point in which the image coincides with the original object

**Rotation symmetry** (3-dimensional) – a rotation about an axis of symmetry in which the image coincides with the original object

**Scalene triangle** – a triangle none of whose sides are congruent

**Scaling** – a transformation of the plane that causes either a magnification or a shrinking of an object in which the image remains similar to the original object

**Secant** – a line that intersects a circle in two distinct points

**Sector** – the portion of a circle and its interior between two radii

**Semiregular polyhedron** – a polyhedron that contains two or more regular polygons as faces which are arranged so that all vertices are homogeneous

**Semiregular tessellation** – a tessellation that contains two or more regular polygons arranged so that the vertices are homogeneous

**Shearing** – a transformation of the plane that changes the shape of an object

**Side** – one of the line segments that make up a polygon

**Similar objects** – objects where one can be obtained from the other by composing a rigid motion with a dilation

**Simple curve** – a curve that does not intersect itself

**Slope** (of a line) – the vertical distance required to stay on a line for a one unit change in horizontal distance
**Space** – an undefined term that denotes the set of points that extends indefinitely in three dimensions

**Sphere** – the set of points in (three-dimensional) space that are equidistant from a given point, called the center

**Square** – a quadrilateral with four right angles and four congruent sides

**Standard unit of measure** – a unit of measure whose value is established by reference to an accepted standard; for example, the meter is defined to be one ten-millionth of the distance from the equator to the north pole

**Straight angle** – an angle that measures half a turn (or two right angles)

**Straightedge** – an instrument used to construct line segments

**Supplementary angles** – two angles whose measures sum to 180 degrees

**Surface** – the set of points that form the boundary of a solid three-dimensional object

**Surface area** – the sum of the areas of the faces of a closed 3-dimensional object

**Symmetry** (of an object) – a transformation of the object in which the image coincides with the original

**Tangent** (to a circle) – a line that intersects a circle in exactly one point

**Tessellation** – an arrangement of polygons that can be extended in all directions to cover the plane with no gaps and no overlaps in such a way that vertices only meet other vertices

**Theorem** – a statement that has been proven true

**Tiling** – an arrangement of polygons that can be extended in all directions to cover the plane with no gaps and no overlaps

**Time** – a measurable part of the fundamental structure of the universe, a dimension in which events occur in sequence

**Transformation** – a movement of the points of a plane that may change the position or the size and shape of objects

**Translation** (by a vector $AA'$) – a rigid motion of the plane that takes $A$ to $A'$, and for all other points $P$ on the plane, $P$ goes to $P'$ where vector $PP'$ and vector $AA'$ have the same length and direction
**Translation symmetry** – a translation of the plane such that the image corresponds to the original object

**Translation vector** – an arrow that gives the direction and distance (its length) that a point is moved during a translation

**Transversal** – a line which intersects two or more lines

**Trapezoid** – a quadrilateral with exactly one pair of parallel sides

**Triangle** – a polygon with three sides

**Trivial rotation** – the rotation of 360°; it is a rotational symmetry of every object

**Undefined term** – a term which has an intuitive meaning, but no formal definition

**Union** (of sets) – the set containing every element of each set

**Unit of measure** – an object is used for comparison with an attribute

**Unit square** – a square that is one unit by one unit and thus has an area of one square unit

**Unit cube** – a cube that is one unit by one unit by one unit and has a volume of one cubic unit

**Venn diagram** – a picture in which the objects being studied are represented as points on a plane and simple closed curves are drawn to group the points into different classifications. Venn diagrams are used to visualize relationships among sets of objects.

**Vertex** (plural: vertices) – the common endpoint of two adjacent sides of a polygon

**Vertex angle** – the angle formed by adjacent sides of a polygon

**Vertex** (of a polyhedron) – the intersection of two or more edges of a polyhedron

**Vertical angles** – a nonadjacent pair of angles formed by two intersecting lines

**Volume** – a measure of the capacity of a 3-dimensional object or, alternatively, the quantity of space enclosed by a 3-dimensional object

**Weight** – a measure of the force of gravity on an object; often used interchangeably with mass; differences in the measures of weight and mass are negligible at sea level on earth
References

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