Hyperbolic Constructions in *Geometer's Sketchpad*

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January 8, 2001

1 Introduction – Non-Euclidean Geometry

Over 2000 years ago, the Greek mathematician Euclid compiled all of the known geometry of the time into a 13-volume text known as the *Elements*. In itself, that was an impressive feat, but what made his efforts particularly astonishing was that he laid a *logical* foundation for the subject as well. He established five axioms for geometry, then showed that every result in his text could be proven from those axioms.

In modern language, Euclid's postulates can be stated as follows:

I. For every point P and every point Q not equal to P there exists a unique line ℓ that passes through P and Q.

II. For every segment AB and every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

III. For every point O and every point A not equal to O, there exists a circle with center O and radius OA.

IV. All right angles are congruent to each other.

V. For every line ℓ and point P that does not lie on ℓ, there exists a unique line m through P that is parallel to ℓ.

From the time the *Elements* was published, Euclid's axioms have been the subject of careful scrutiny, criticism and controversy. Modern texts note several deficiencies in the axioms (see, for example [3], [1]). Generally speaking, Euclid's axioms are not precise enough, and in his text, Euclid relies on diagrams and makes use of unstated assumptions. Many mathematicians, Hilbert in particular (see [2]), have devoted themselves to placing geometry on a firmer foundation than did Euclid. Nevertheless, Euclid's axioms form the basis of *Euclidean geometry*, the plane geometry studied by most high school students today.

In spite of some of the weaknesses of Euclid's axioms, they were, for the most part accepted by other mathematicians. For example, Axiom I tells us that through any two points, one and only one line can be drawn. Most people would grant that statement as a truth about our physical world. So it is with Euclid's first four axioms. However, the fifth and final axiom proved extremely controversial.

Although Euclid's fifth axiom, called the "parallel postulate", seems a reasonable assumption, geometers were skeptical from its first appearance. Specifically, many thought that Euclid V could be proven from his other four axioms. If this were true, there would be no need to assume it, and Euclid would have needed only four axioms for his *Elements*. For 2000 years, geometers attempted to prove the "parallel postulate" and met with utter failure. In every purported proof, there could be found a flaw, some statement which in turn could not be proven.

In the 19th century, mathematicians began to look at the postulate in a new way. Rather than trying to prove Euclid V, they began to question whether such attempts could really succeed. Three mathematicians (Gauss, Bolyai, and Lobachevsky) independently developed a noncontradictory geometry in which the parallel postulate is *false*. One way that Euclid's fifth axiom can fail is to have "too many parallel lines through a point":

> There exists a line ℓ and a point P not on ℓ such that at least two distinct lines parallel to ℓ pass through P.

Rather than assuming the parallel postulate, the three men assumed this axiom, which is today called the **Hyperbolic Axiom**. Using the hyperbolic axiom and Euclid's other four postulates, Gauss, Bolyai, and Lobachevsky developed the important and rich subject which has come to be known as hyperbolic geometry.

Before discussing models on hyperbolic geometry, we must point out that the hyperbolic axiom is not the only way in which Euclid V can fail. It is possible, of course, that there are "not enough parallel lines through a point":

> There exists a line ℓ and a point P not on ℓ such that there are no lines parallel to ℓ which pass through P.

If we assume this axiom, called the **Elliptic Axiom**, rather than Euclid's parallel postulate, we obtain yet another type of geometry, albeit with some restrictions. Elliptic geometry requires slight modifications to the other axioms of geometry and will not be considered here. **Non-Euclidean geometry** consists of both elliptic and hyperbolic geometry, though in the context of this report it will generally refer to hyperbolic geometry.

2 Models of Hyperbolic Geometry

One particular difficulty with non-Euclidean geometry is visualization. How can one understand a universe in which the parallel postulate does not hold? In our experience, if we construct two parallel lines, then draw a third line which intersects one of the two parallel lines, it seems clear that it must intersect the other. The solution to this difficulty again requires a change in thinking. We abstract our notions of common geometric terms such as "line", "point", and "congruent", reinterpret them and create a universe, or **model**, where Euclid's first four axioms hold, but the fifth one doesn't.

There are many models of hyperbolic geometry. Perhaps the best known are the Poincaré disk model, the Poincaré half-plane model, and the Beltrami-Klein (or Klein) model. In each of these models there are different interpretations of undefined geometric terms such as "point," "line," and "congruent." In the non-Euclidean models, the interpretations are generally different from their usual Euclidean notions, so standard Euclidean constructions do not directly apply. For example, in the Poincaré models, "lines" are interpreted to be arcs of circles, so in these models, lines cannot be constructed with just a Euclidean straightedge.

The non-Euclidean models thus provide a challenge: how does one construct typical geometric objects in hyperbolic geometry? With a straightedge and compass, one can draw circles, bisect angles, and construct midpoints in Euclidean geometry. Using the same tools, how does one perform the same constructions in the hyperbolic models?

Alexander and Finzer have written *Geometer's Sketchpad* scripts to perform ten typical constructions in the Poincaré disk model. The constructions are available at

http://forum.swarthmore.edu/sketchpad/maa96/alexander/index.html

and include scripts to

1. Construct a non-Euclidean line, given two points on the line.
2. Construct a non-Euclidean line segment, given the endpoints of the segment.
3. Measure the length of a non-Euclidean line segment.
4. Calculate the measure of an angle.
5. Construct the bisector of a given angle.
6. Construct a perpendicular to a given line through a given point on the line.
7. Construct a perpendicular to a given line through a given point not on the line.
8. Construct the perpendicular bisector of a non-Euclidean line segment.
9. Construct a circle, given its center and a point on the circle.
10. Construct a circle, given its center and two points determining the radius of the circle.

These constructions use the Euclidean tools of *Geometer's Sketchpad*, which are, in essence, computer versions of the straightedge and compass. (It should be noted that items #3 and #4 above are not actually constructions, since they require the notion of measurement.)

In this document, we describe the steps necessary to perform Constructions 1–10 for the other two well-known models of hyperbolic geometry, the Beltrami-Klein and Poincaré half-plane models. The descriptions provide a context for understanding the constructions in associated *Geometer's Sketchpad* scripts for these models. In addition, while creating the constructions for both models, it proved useful to to develop additional scripts which find the midpoint of a line segment and find the reflection of a point about a line. For example, constructing the perpendicular bisector of a given line segment becomes much simpler once we have algorithms for constructing midpoints and raising perpendiculars. These additional constructions are described as well. Note however, that in all cases, the constructions are only described - formal proofs are omitted.

The ten Klein scripts are provided first, followed by the half-plane scripts. To the best of our knowledge, the Klein scripts are original. That is, though others have certainly demonstrated the same constructions, they are an original compilation in the form of *Sketchpad* scripts. The same may be said for constructions 5–10 in the Poincaré half-plane model. Bennett has created scripts for constructions 1–4 for the half-plane model. His scripts may be found at

http://www.keypress.com/sketchpad/misc/sibley/sibley.htm.

Peil has also written several scripts for the half-plane model, which may be found at

http://classweb.moorhead.msus.edu/peil/Projects/geo.html.

Although several of the constructions described here appear to duplicate Peil's work, there are some differences. For example, the midpoint and circle constructions described here use only straightedge and compass constructions, while Peil's scripts involve coordinate geometry as well.

3 Beltrami-Klein Model Scripts

In the Beltrami-Klein model of non-Euclidean geometry, we fix a Euclidean circle γ. "Points" in this model are interpreted to be points interior to γ, and "lines" are interpreted to be open chords of the circle (i.e. chords without their endpoints). The Klein model is neither isometric nor conformal; that is, it represents neither distances nor angle measures faithfully. However, the model does provide several distinct advantages over the Poincaré models. First, it is immediately apparent that the hyperbolic axiom holds in this model. Given a chord ℓ of γ and a point P interior to the circle that is not on the chord, there are an infinitude of distinct chords passing through P which do not intersect ℓ. Second, although the model is not conformal, constructing perpendicular lines in the Klein disk proves to be somewhat easier than in the Poincaré models.

Each construction corresponds to an available *Geometer's Sketchpad* script. The name of the associated script is given in bold-face next to the number of the construction. In *Sketchpad*, the script tools assume that one is performing the constructions using a fixed circle labelled "Klein Disk," defined by a center labelled "K-Disk center," and a point on the circle labelled "K-Disk radius." The *Sketchpad* file "klnstrt.gsp" contains this startup figure. The simplest way to use the Klein tools is to create a directory, perhaps called "klein," in which all the scripts and "klnstrt.gsp" is stored. Set the script tool directory in *Sketchpad* to be "klein," and the tools will be accessible from the *Sketchpad* desktop.

For the Klein constructions discussed below, Greenberg [1] provides much useful background. Stahl's text [4] discusses the Klein model in depth as well. Note also that in the constructions, only the most general cases are considered. For example, if one wants to construct a perpendicular to a general Klein line, Constructions #6 and #7 will work nicely. However, if the Klein line happens to be a diameter of the Klein disk, the construction will fail. As noted in the introduction, in addition to the standard ten constructions, we include constructions which find the midpoint of a Klein segment (see item #8(i) below) and the reflection of a point about a Klein line (see item #9(i)). The **pole** of a Klein line is a useful construction in this model, and a description of that construction is also given (see item #5(i)). Throughout the constructions, we will use terms such as "K-line" and "K-circle" as shorthand for "Klein line" and "Klein circle". This will distinguish between the non-Euclidean constructions and their Euclidean counterparts.

1. (**kline.gss**) Construct a Klein line, given two points on the line.

Given: Two points A and B inside the Klein disk.

To Construct: K-line $\overleftrightarrow{AB}$, i.e. the Euclidean chord of the Klein disk which passes through A and B.

a) Construct the Euclidean line $\overleftrightarrow{AB}$.

b) Let P and Q be the points of intersection of $\overleftrightarrow{AB}$ with γ. (C and D are ideal points.)

c) Construct Euclidean segment PQ. By definition, this segment (excluding endpoints P and Q) is the K-line passing through A and B.

2. (**ksegment.gss**) Construct a Klein line segment, given the endpoints of the segment.

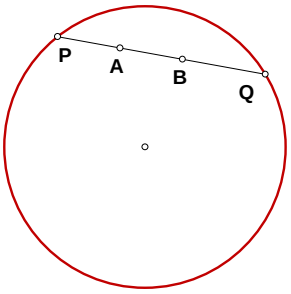

Figure 1: Calculating the length of a line segment AB in the Klein model.

Given: Two points A and B inside the Klein disk.

To Construct: K-segment AB.

a) Construct the Euclidean segment AB. By definition, this is also the K-segment joining A and B.

3. (**klength.gss**) Measure the length of a Klein line segment.

Given: Two points A and B inside the Klein disk.

To Measure: The K-length of segment AB.

a) Construct the K-line $\overleftrightarrow{AB}$. Let P and Q be the ideal points of this K-line, i.e. the ends of this chord of γ passing through A and B.

b) Find each of the following Euclidean lengths: $\overline{AP}$, $\overline{AQ}$, $\overline{BP}$, $\overline{BQ}$.

c) Define the *cross-ratio* (AB, PQ) by

$$(AB, PQ) = \frac{\overline{AP}/\overline{AQ}}{\overline{BP}/\overline{BQ}}.$$

Let $d = \frac{1}{2} \cdot |\ell n(AB, PQ)|$. Then d is the K-length of segment AB. (See [1, Theorem 7.4].) d is independent of the labelling of P and Q (i.e. their labelling could be reversed without affecting d). See Figure 1.

Note: Length is defined as one-half the natural log of the relevant cross-ratio. One could use a different log base or delete the factor of $\frac{1}{2}$. The result would still be a perfectly legitimate hyperbolic measure of length. However, our particular choice of distance allows for simpler relationships between lengths and angle measures in hyperbolic trigonometry. Specifically, this choice yields the constant $k = 1$ in Theorems 10.1 and 10.2 of [1], and allows for direct use of the hyperbolic law of sines and law of cosines ([1, Theorem 10.4]).

4. (**kangmea.gss**) Calculate the K-measure of an angle.

Given: Klein points A, B, and C.

To Calculate: the K-measure of $\angle ACB$.

a) Construct triangle ΔACB. Note that constructing the triangle in the Klein disk is the same as constructing it in the Euclidean sense.

b) Let a be the length of side BC, the side opposite $\angle A$. Let b be the length of side AC, the side opposite $\angle B$. Let c be the length of side AB, the side opposite $\angle C$. Lengths should be measured using the "standard Klein unit." See Construction #3 above.

c) According to the *hyperbolic law of cosines* (See [1, Theorem 10.4])

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C.$$

Thus

$$C = \arccos\left(\frac{\cosh c - \cosh a \cosh b}{-\sinh a \sinh b}\right).$$

5. **i)** (**kpole.gss**) Construct the pole of a given K-line (see [1, p. 239]).

Given: K-line ℓ.

To Construct: $P(\ell)$, i.e. the intersection of the tangents to the defining K-circle γ at the endpoints of the Euclidean chord making up ℓ.

a) Let Q and R be the endpoints of the Euclidean chord which constitutes K-line ℓ.

b) Draw segments OQ and OR, where O is the center of γ.

c) Construct the Euclidean line m through Q which is perpendicular to OQ. Construct the Euclidean line n through R which is perpendicular to OR. By construction, m and n are the tangents to γ through the points Q and r, respectively.

d) Assuming ℓ is not a diameter of γ, lines m and n intersect at a point $P(\ell)$ exterior to γ.

$P(\ell)$ is the *pole of* ℓ. See Figure 2.

ii) (**kangbis.gss**) Construct the bisector of a given angle.

Given: Klein points A, B, and C.

To Construct: the angle bisector of of $\angle ACB$.

(Greenberg outlines this construction in [1, p. 273].)

a) Construct K-rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Let D and E be the respective points where these rays intersect γ.

b) Let P be the pole of the Euclidean chord DE, i.e. the pole of the K-line defined by the ideal points D and E.

c) Draw Euclidean line $\overleftrightarrow{PC}$. Let Q and R be the points of intersection of this line with γ.

d) Euclidean chord QR is the K-line which bisects $\angle ACB$ in the sense of Klein.

See Figure 3.

6. (**kprpon.gss**) Construct a perpendicular to a given Klein line through a given point on the line.

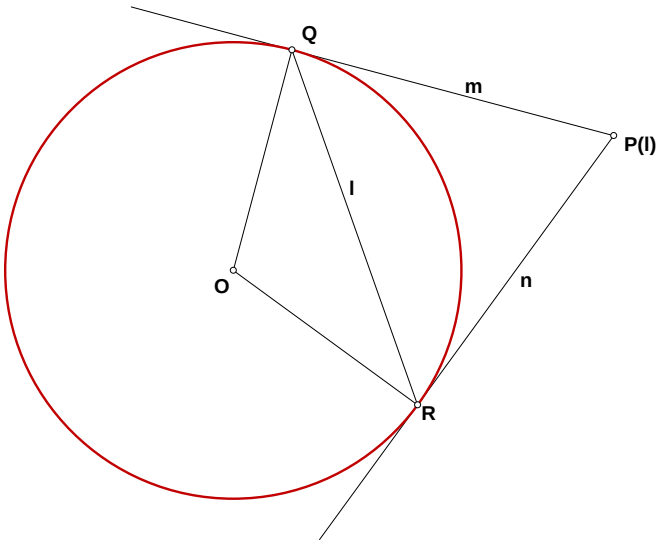

Figure 2: Constructing the pole $P(\ell)$ of line ℓ in the Klein model.

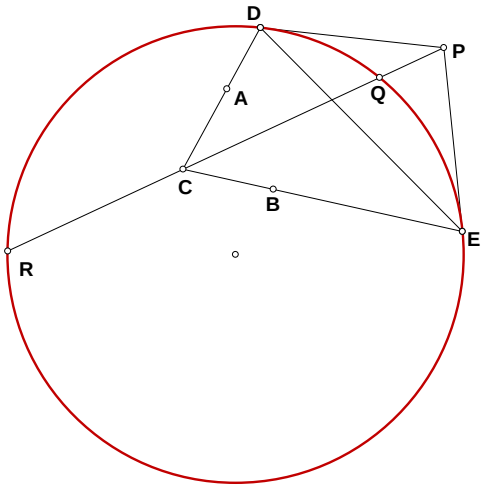

Figure 3: Constructing the angle bisector of an angle in the Klein model.

Given: Klein points A and B.

To Construct: The K-line through B which is perpendicular (in the Klein sense) to the K-line $\overleftrightarrow{AB}$.

a) Construct K-line $\overleftrightarrow{AB}$ and construct the pole P of this line (See Construction #5 (i) above).

b) Draw Euclidean line $\overleftrightarrow{PB}$. Let Q and R be the points where this line meets circle γ. Then the chord of γ defined by QR is the K-line through B which is perpendicular (in the Klein sense) to $\overleftrightarrow{AB}$.

7. (**kprpoff.gss**) Construct a perpendicular to a given Klein line through a given point not on the line.

Given: Klein points A, B and C, not collinear.

To Construct: The K-line through C which is perpendicular (in the Klein sense) to the K-line $\overleftrightarrow{AB}$.

a) Construct K-line $\overleftrightarrow{AB}$ and construct the pole P of this line (See Construction #5 (i) above).

b) Draw Euclidean line $\overleftrightarrow{PC}$. Let Q and R be the points where this line meets circle γ. Then the chord of γ defined by QR is the K-line through C which is perpendicular (in the Klein sense) to $\overleftrightarrow{AB}$.

8. **i)** (**kmidpt.gss**) Construct the midpoint of a Klein line segment.

Given: Klein points A and B.

To Construct: The point M on K-segment AB such that $AM \cong BM$ in the Klein sense.

(This construction is Exercise K-7 on p. 273 of [1]. It follows from discussion on p. 262 and 263 of that text.)

a) Construct the K-line $\overleftrightarrow{AB}$ (i.e. the Euclidean chord of γ which passes through A and B).

b) Using Construction #5 (i) above, let P be the pole of Klein line $\overleftrightarrow{AB}$.

c) Draw Euclidean lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$.

d) The lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ each intersect circle γ in two points. Let S and S' be the points of intersection of γ and $\overleftrightarrow{PA}$. Let T and T' be the points of intersection of γ and $\overleftrightarrow{PB}$. Then S, S', T, and T' are all ideal points.

e) Construct chords ST' and $S'T$ of circle γ. Let M be the intersection point of these chords.

By construction, M lies on segment AB and $AM \cong BM$. See Figure 4.

ii) (**kprpbis.gss**) Construct the perpendicular bisector of a Klein line segment.

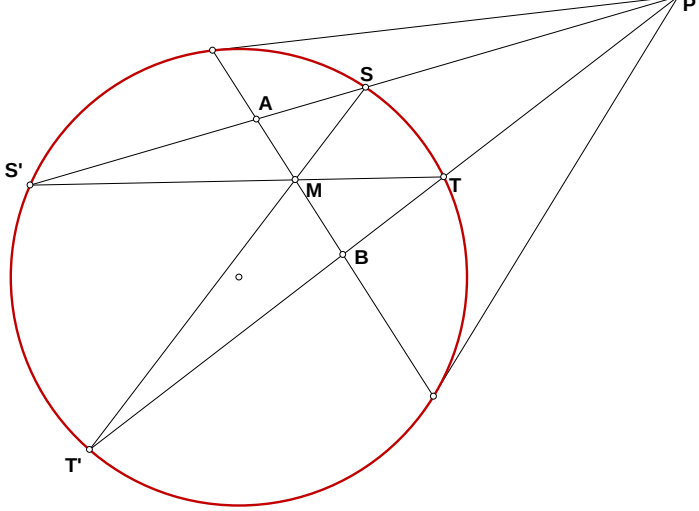

Figure 4: Constructing the midpoint of a line segment in the Klein model.

Given: Klein points A and B.

To Construct: The K-line ℓ perpendicular (in the Klein sense) to the K-line $\overleftrightarrow{AB}$ and passing through the midpoint M of K-segment AB.

a) Using the construction outlined in Construction #8 (i) above, construct the midpoint M of segment AB.

b) Using Construction #6 above, construct the line ℓ through M which is perpendicular to K-line $\overleftrightarrow{AB}$. Then ℓ is the required perpendicular bisector of AB.

9. **i)** (**kreflpt.gss**) Construct the reflection of a point about a Klein line.

Given: Klein points A and M.

To Construct: The point A' (distinct from A) on K-line $\overleftrightarrow{AM}$ such that $AM \cong A'M$ in the Klein sense. A' is thus the image of A under reflection about the K-line through M which is perpendicular (in the Klein sense) to $\overleftrightarrow{AM}$.

(Greenberg outlines this construction on p. 262 of [1].)

a) Construct the K-line passing through A and M. Call this line t.

b) Construct the pole Q of t using Construction #5 (i) given above.

c) Draw Euclidean line $\overleftrightarrow{QA}$. This line intersects γ at two points, S and S'.

d) Draw Euclidean line $\overleftrightarrow{QM}$. The portion of this line which is a chord of γ is a K-line through M which is perpendicular to t. Call this line m.

e) Construct the pole P of m using Construction #5 (i) given above.

f) Draw the Euclidean lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{PS'}$. Each of these lines intersects γ at another point on γ. Call these additional ideal points T and T', respectively.

g) Draw Euclidean chord TT'. TT' intersects K-line t at a point A'.

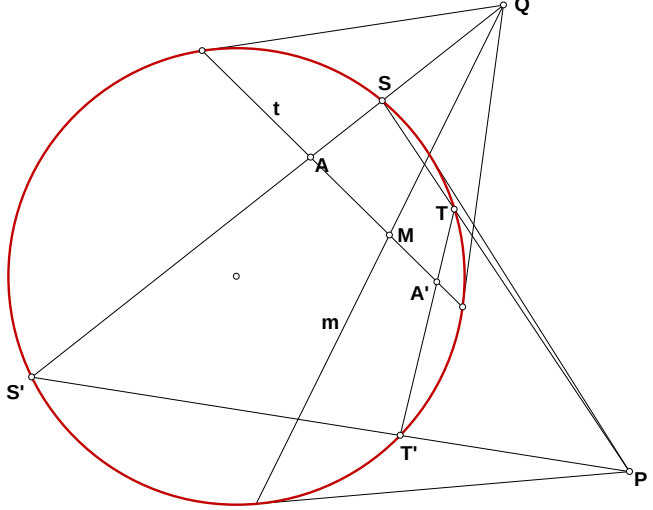

Figure 5: Constructing the reflection of a point about a line in the Klein model.

A' is the desired reflection of A about line m, with $AM \cong A'M$ in the Klein sense. See Figure 5.

ii) (**kcntrpt.gss**) Construct a Klein circle, given its center and a point on the circle.

Given: Klein points O and P.

To Construct: The K-circle centered at O with radius OP.

a) Construct the Euclidean circle c_1 centered at O with Euclidean radius OP.

b) Choose a point Q on circle c_1.

c) Draw line $\overleftrightarrow{OQ}$.

d) Construct the reflection of point P about ray $\overrightarrow{OQ}$. To do this, drop a K-perpendicular from P to line $\overleftrightarrow{OQ}$ and let M be the point of intersection. Reflect point P using Construction #9 (i) above. Let P' be the image of this reflection.

e) By construction, $OP \cong OP'$ in the sense of Klein.

The Klein circle c_2 centered at O with radius OP is the locus of points P' as Q travels around the Euclidean circle c_1. Note that c_2 is *not* a Euclidean circle. See Figure 6.

10. (**kcntrrd.gss**) Construct a Klein circle, given its center and two points, the segment between which determines the radius of the circle.

Given: Klein points O, A, and B.

To Construct: The K-circle centered at O with radius OP, where $OP \cong AB$ in the Klein sense.

a) Draw segment OA. Let K-line ℓ be the perpendicular bisector of segment OA, using Construction #8 (ii) above.

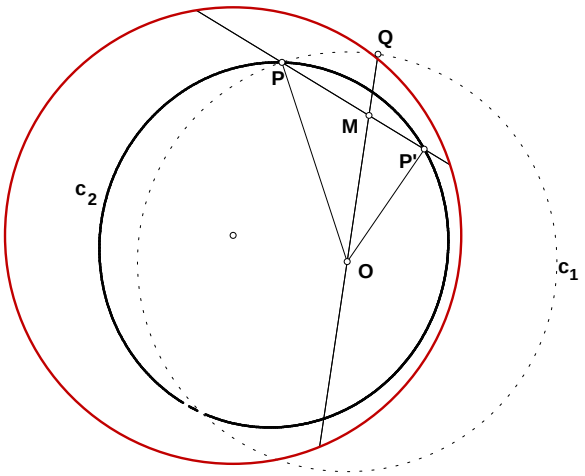

Figure 6: Constructing the Klein circle centered at O and with radius OP

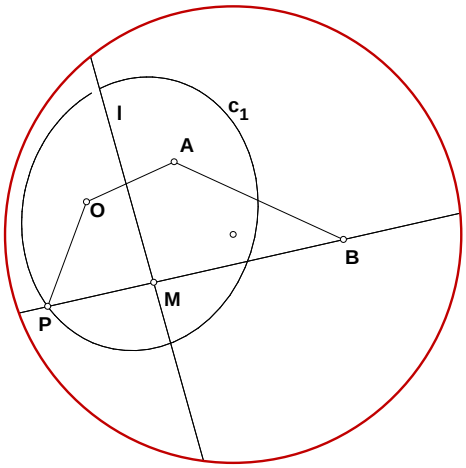

Figure 7: Constructing the Klein circle centered at O and with radius OP congruent to AB.

b) Drop a K-perpendicular from point B to line ℓ at a point M, using Construction #7 above.

c) Reflect point B about line ℓ using Construction #9 (i) above. Let P be the image of this reflection. By construction, $OP \cong AB$ in the sense of Klein.

d) Construct the K-circle c_1 centered at O with radius OP, using Construction #9 (ii) above.

Circle c_1 is our desired circle. See Figure 7.

4 Poincaré Half-Plane Model Scripts

In the Poincaré Half-plane model of hyperbolic geometry, we fix a (Euclidean) line and arbitrarily choose one side of that line to be the "upper half-plane." For intuitive purposes, we generally use the x-axis as our line and let the set of points whose y-coordinate is positive be the upper half-plane. "Points" in this model are interpreted to be points in the upper half-plane (not including the x-axis), and "lines" are interpreted to be either vertical rays with endpoint on the x-axis, or, more generally half circles whose center lies on the x-axis. (The constructions below work in the more general case, not for the special case of the lines which are rays.)

Below are descriptions of basic constructions 1–10 in the Poincaré half-plane model. The descriptions supplement the associated *Geometer's Sketchpad* scripts for the constructions. Although I give descriptions for all ten constructions, others have produced scripts for these constructions previously. As noted earlier, Bennett has created scripts for items 1–4 in the half-plane model. His scripts may be found at

http://www.keypress.com/sketchpad/misc/sibley/sibley.htm.

These scripts are titled **hyp_line.gss**, **hyp_seg.gss**, **hyp_dist.gss**, and **hyp_angl.gss**, and construct hyperbolic lines and segments, and measure hyperbolic distances and angles, respectively, in the Poincaré half-plane model. My description of "Construction" #3 is slightly different from Bennett's. I choose a slightly different unit for measuring length, for ease of use of hyperbolic trigonometry. As such, I give a description of my construction/calculation script (**phlength.gss**), rather than his. The remaining scripts given here have been written by this author.

Peil has also written several scripts for the half-plane model, which may be found at

http://classweb.moorhead.msus.edu/peil/Projects/geo.html.

Although several of the constructions described here appear to duplicate Peil's work, it should be noted that several constructions differ in their fundamentals (Peil uses coordinate geometry for some of his constructions).

As in the Beltrami-Klein case, each construction corresponds to an available *Geometer's Sketchpad* script. The name of the associated script is given in bold-face next to the number of the construction. In *Sketchpad*, the script tools assume that one is performing the constructions using a fixed line defined by two points labelled "A" and "B". Although one can typically use any line so defined, it is helpful to use the x-axis as the defining line. In

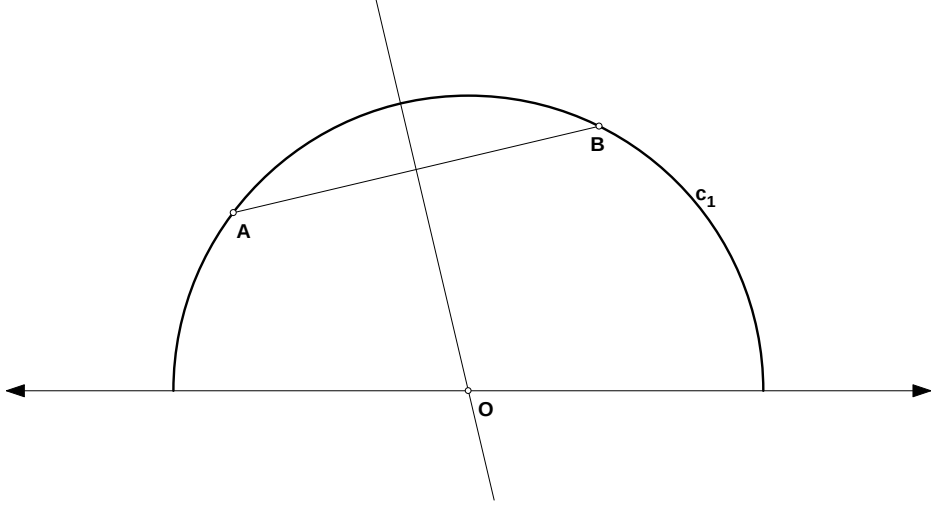

Figure 8: Constructing the line through A and B in the Poincaré Half-plane model.

particular, *Sketchpad* can become confused as to which side of line $\overleftrightarrow{AB}$ is the "upper half-plane," an ambiguity which is less troublesome when $\overleftrightarrow{AB}$ is the x-axis. The *Sketchpad* file "poinhalf.gsp" contains a startup figure in which A and B are points on the x-axis. The simplest way to use the script tools is to create a directory, perhaps called "poinhalf," in which all the scripts and "poinhalf.gsp" is stored. Set the script tool directory in *Sketchpad* to be "poinhalf," and the tools will be accessible from the *Sketchpad* desktop.

As noted in the introduction, in addition to the standard ten constructions, we include constructions which find the midpoint of a half-plane segment (see item #8(i) below) and the reflection of a point about a half-plane line (see item #9(i)). In all the constructions we will use terms such as "PH-line" and "PH-circle" as shorthand for "Poincaré half-plane line" and "Poincaré half-plane circle". This will distinguish between the non-Euclidean constructions and their Euclidean counterparts.

1. (Bennett's **hyp_line.gss**) Construct a PH-line, given two points on the line.

Given: Two points A and B in the PH-plane.

To Construct: PH-line $\overleftrightarrow{AB}$, i.e. the half-circle centered on the x-axis and passing through A and B.

a) Construct Euclidean segment AB.

b) Let ℓ be the Euclidean perpendicular bisector of AB. Assuming that A and B do not lie on the same vertical line, let O be the intersection of ℓ and the x-axis.

c) Let c_1 be the circle centered at O with radius OA.

The half of circle c_1 which lies in the upper half plane is the PH-line through A and B. See Figure 8.

2. (Bennett's **hyp_seg.gss**) Construct a PH-line segment, given the endpoints of the segment.

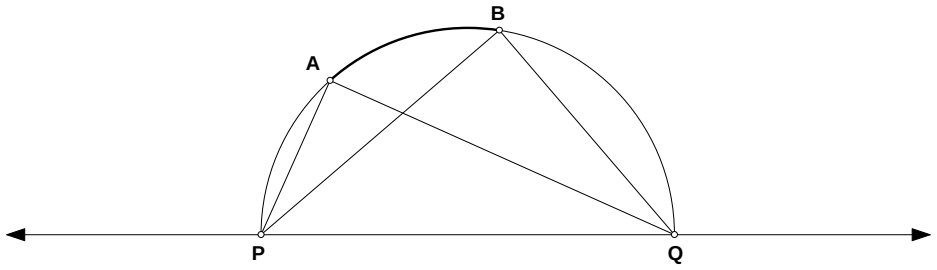

Figure 9: Calculating the length of a line segment AB in the Poincaré Half-plane model.

Given: Two points A and B in the PH-plane.

To Construct: PH-segment $\overleftrightarrow{AB}$, i.e. the arc of the circle centered on the x-axis, contained entirely in the upper half-plane, and with endpoints A and B .

a) Construct the PH-line $\overleftrightarrow{AB}$ as described in Construction #1 above.

b) Let PH-segment AB be the arc of the $\overleftrightarrow{AB}$ which is contained entirely in the upper half-plane, and has endpoints A and B.

3. (**phlength.gss**) Measure the length of a PH-line segment.

Given: Two points A and B in the Poincaré half-plane.

To Measure: The PH-length of segment AB.

a) Construct the PH-line $\overleftrightarrow{AB}$. Let P and Q be the ideal points of this PH-line, i.e. the points of intersection of the x-axis and the circle centered on the x-axis which passes through A and B.

b) Find each of the following Euclidean lengths: $\overline{AP}$, $\overline{AQ}$, $\overline{BP}$, $\overline{BQ}$.

c) Define the *cross-ratio* (AB, PQ) by

$$(AB, PQ) = \frac{\overline{AP}/\overline{AQ}}{\overline{BP}/\overline{BQ}}.$$

Let $d = |\ell n(AB, PQ)|$. Then d is the PH-length of segment AB. d is independent of the labelling of P and Q (i.e. their labelling could be reversed without affecting d). See Figure 9.

Note: As in the length definition for the Klein model, other definitions (using a different log base or scaling factor) are possible, but this one allows for easy use of hyperbolic trig formulas.

4. (Bennett's **hyp_angl.gss**) Calculate the measure of an angle.

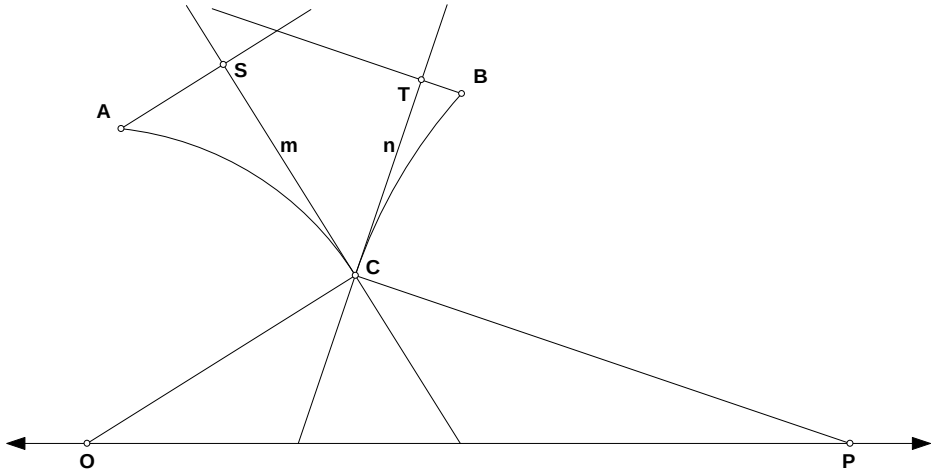

Figure 10: Calculating the measure of angle $\angle ACB$ in the Poincaré Half-plane model.

Given: Points A, B, and C in the Poincaré half-plane.

To Calculate: the PH-measure of $\angle ACB$.

a) Construct PH-segments AC and BC.

b) Construct the Euclidean tangent lines to the arcs in step (a) at C: Let O and P be, respectively, the centers on the x-axis of the circles which define the Euclidean arcs given by PH-segments AC and BC. Draw Euclidean radius segments OC and PC. Let m and n be the Euclidean lines through C which are perpendicular to OC, and PC, respectively. Then m and n are the desired tangent lines.

c) Euclidean lines m and n define the angle we wish to measure. We need to ensure the interior of that angle is well-defined. Drop Euclidean perpendicular $\overleftrightarrow{AS}$ to line m at S and Euclidean perpendicular $\overleftrightarrow{BT}$ to line n at T.

d) Find the Euclidean measure of Euclidean angle $\angle SCT$.

See Figure 10.

5. (**phangbis.gss**) Construct the bisector of a given angle.

Given: Points A, B, and C in the Poincaré half-plane.

To Construct: the PH-line which bisects PH-angle $\angle ACB$.

a) Using steps (a)-(c) in Construction #4 above, construct Euclidean angle $\angle SCT$.

b) Let $\overrightarrow{r}$ be the Euclidean angle bisector of $\angle SCT$.

c) Let ℓ be the Euclidean line through C which is perpendicular to r.

d) Assuming that r is not a vertical ray, ℓ will intersect the x-axis at some point U.

e) Let c_1 be the Euclidean circle centered at U with radius UC.

The upper half of circle c_1 is the PH-line which bisects $\angle ACB$. See Figure 11.

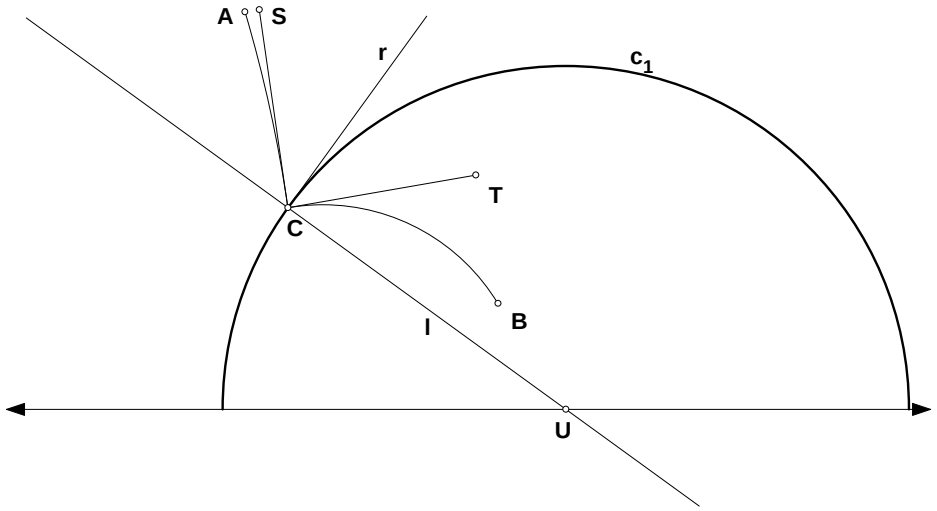

Figure 11: Constructing the angle bisector of an angle in the Poincaré Half-plane model.

6. (**phprpon.gss**) Construct a perpendicular to a given PH-line through a given point on the line.

Given: Points A and B in the Poincaré half-plane.

To Construct: the PH-line (i.e. Euclidean half-circle) through A which is perpendicular (in the PH-sense) to the PH-line joining A and B.

a) Construct the PH-line $\overleftrightarrow{AB}$ as described in Construction #1 above. As a Euclidean half-circle, let this PH-line have center O (on the x-axis).

b) Construct Euclidean segment OA.

c) Construct the Euclidean line ℓ through A which is perpendicular to segment OA. Line ℓ is thus the tangent to the PH-line $\overleftrightarrow{AB}$ at point A.

d) Assuming OA is not a vertical line segment, ℓ intersects the x-axis at some point P.

e) Construct the Euclidean half-circle m centered at P and passing through point A.

PH-line m is the desired line. Note that by construction, Euclidean line $\overleftrightarrow{PA}$ is perpendicular to segment OA. On the other hand, Euclidean lines $\overleftrightarrow{PA}$ and OA are tangent to the PH-lines $\overleftrightarrow{AB}$ and m, respectively. By the definition of perpendicularity in the half-plane model, then, $m \perp \overleftrightarrow{AB}$. See Figure 12.

7. (**phprpoff.gss**) Construct a perpendicular to a given PH-line through a given point not on the line.

Given: Points A and B defining PH-line $\overleftrightarrow{AB}$ and point C not on $\overleftrightarrow{AB}$.

To Construct: PH-line (i.e. Euclidean half-circle) ℓ through C which is perpendicular (in the PH-sense) to the PH-line joining A and B.

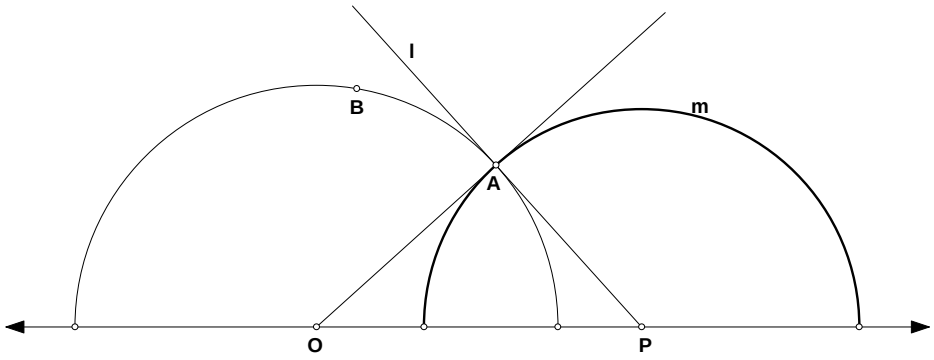

Figure 12: Constructing the PH-line through B which is perpendicular to $\overleftrightarrow{AB}$ in the Poincaré Half-plane model.

a) Construct the PH-line $\overleftrightarrow{AB}$ as described in Construction #1 above. As a Euclidean half-circle, let this PH-line have center O. Extend the half-circle to a full-circle c_1.

b) Construct Euclidean line $\overleftrightarrow{OC}$. Let D be the point above the x-axis where the PH-line $\overleftrightarrow{AB}$ meets Euclidean line $\overleftrightarrow{OC}$.

c) Construct the Euclidean circle c_2 centered at O and passing through point C.

d) Construct the Euclidean line through O which is perpendicular to Euclidean line $\overleftrightarrow{OC}$. Let this line intersect circle c_1 at point E and circle c_2 at point F, where E and F lie above the x-axis.

e) Construct Euclidean line $\overleftrightarrow{DF}$.

f) Construct the line through E which is parallel to $\overleftrightarrow{DF}$. Let this line intersect Euclidean line $\overleftrightarrow{OC}$ at point G.

g) Construct the PH-line $\overleftrightarrow{CG}$ as described in Construction #1 above.

PH-line $\overleftrightarrow{CG}$ is the desired line. The Euclidean half-circles $\overleftrightarrow{CG}$ and $\overleftrightarrow{AB}$ intersect orthogonally, so the lines are perpendicular in the Poincaré half-plane sense. See Figure 13.

8. **i)** (**phmidpt.gss**) Construct the midpoint of a PH-line segment.

Given: Points C and D.

To Construct: the point M lying on PH-segment CD which is the midpoint (in the PH-sense) of CD.

a) Construct PH-segment (i.e. Euclidean circle arc) CD. Extend this arc to a Euclidean circle c_1, centered at the point F.

b) Construct Euclidean line $\overleftrightarrow{CD}$. Assuming $\overleftrightarrow{CD}$ is not parallel to the x-axis, let G be the point of intersection of $\overleftrightarrow{CD}$ and the x-axis.

c) Let H be the Euclidean midpoint of Euclidean segment FG which is part of the x-axis.

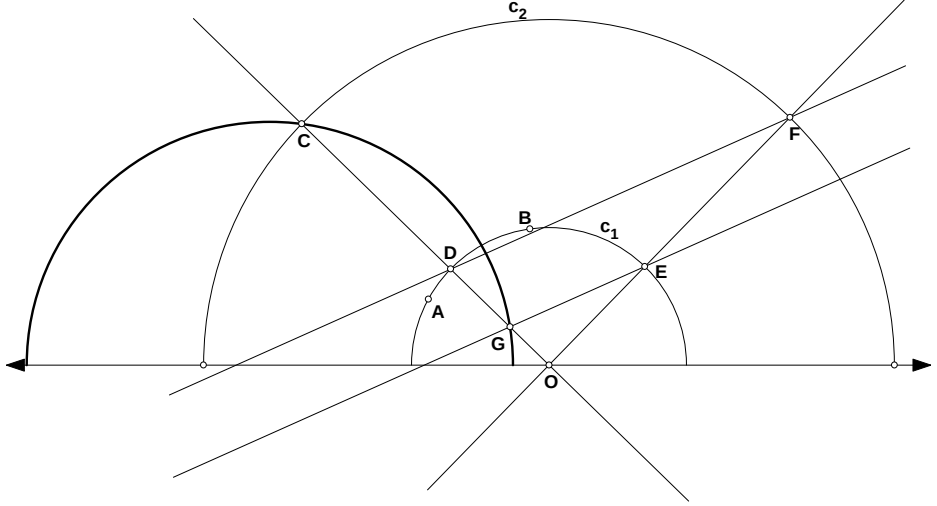

Figure 13: Constructing the PH-line through C which is perpendicular to $\overleftrightarrow{AB}$ in the Poincaré Half-plane model.

d) Let c_2 be the Euclidean circle centered at H and passing through G.

e) Circles c_1 and c_2 intersect in two points on opposite sides of the x-axis. The intersection point M which is above the x-axis is the midpoint of PH-segment CD.

This completes the essence of the construction; however, *Geometer's Sketchpad* has some difficulty in deciding which intersection point in step (e) "is above the x-axis." The remaining construction steps select the correct point without resorting to coordinate geometry:

f) Call the intersection points from step (e) points I and J. Construct Euclidean line segment IJ.

g) $\overleftrightarrow{IJ}$ is perpendicular to the x-axis at some point K. In addition, $\overleftrightarrow{IJ}$ intersects Euclidean segment CD at some point L.

h) Construct Euclidean ray $\overrightarrow{KL}$. By construction, this ray emanates from a point on the x-axis and "points in the positive y-direction."

i) Let M be the intersection of ray $\overrightarrow{KL}$ with circle c_2.

By construction, M will be either point I or point J, whichever has positive y-coordinate. See Figure 14.

ii) (**phprpbis.gss**) Construct the perpendicular bisector of a PH-line segment.

Given: Points C and D.

To Construct: the perpendicular bisector (in the PH-sense) of the PH-segment CD.

a) Construct the midpoint M of PH-segment CD as in Construction #8 (i) above.

b) Construct the PH-line ℓ through M which is perpendicular to PH-segment CD (see Construction # 6 above).

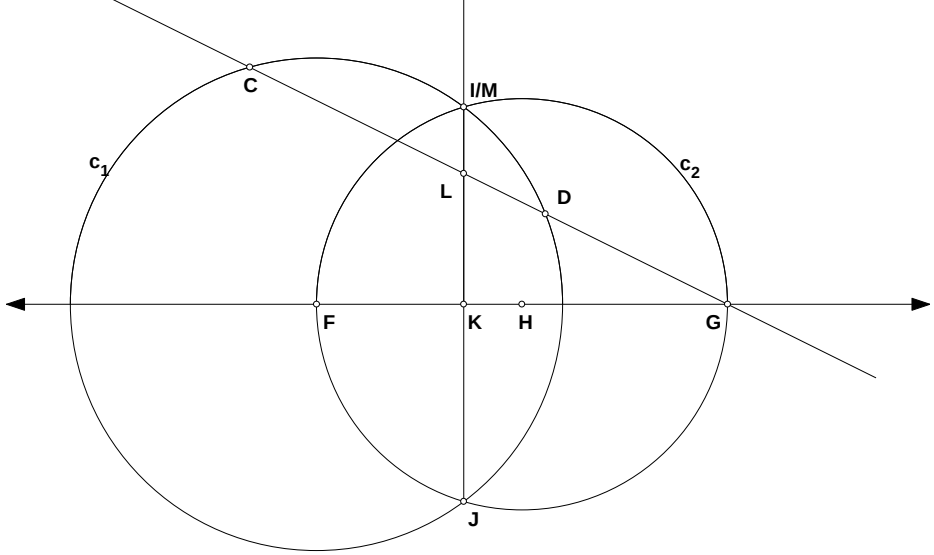

Figure 14: Constructing the midpoint of PH-line segment AB in the Poincaré Half-plane model.

9. **i)** (**phreflpt.gss**) Construct the reflection of a point about a PH-line.

Given: Points A and M in the Poincaré half-plane.

To Construct: The point A' on PH-line $\overleftrightarrow{AM}$ such that $AM \cong A'M$ in the Poincaré sense. A' is thus the image of A under reflection about the PH-line through M which is perpendicular to $\overleftrightarrow{AM}$.

a) Construct the PH-line $\overleftrightarrow{AM}$. Let c_1 be the Euclidean half-circle which is this PH-line.

b) Using Construction #6 above, construct the PH-line m through M which is perpendicular to PH-line $\overleftrightarrow{AM}$. As a Euclidean half-circle, let this PH-line have center O.

c) Draw Euclidean line $\overleftrightarrow{OA}$. $\overleftrightarrow{OA}$ intersects the half-circle c_1 at two points, A and another point A'.

A' is the desired reflection of A about line m with $AM \cong A'M$. See Figure 15.

This completes the essence of the construction; however, *Geometer's Sketchpad* has some difficulty in deciding which point of intersection of line $\overleftrightarrow{OA}$ with circle c_1 is A and which is the desired reflection A'. To resolve this difficulty, I include the following additional constructions:

d) After constructing line $\overleftrightarrow{OA}$, let C and D be the points of intersection of this line with circle c_1. (One of C and D is point A, and one is A'.)

e) Construct Euclidean segment CD and find its Euclidean midpoint E.

f) Draw ray $\overrightarrow{AE}$. This ray will intersect circle c_1 at either C or D. This point of intersection is our desired reflection A'.

ii) (**phcntrpt.gss**) Construct a PH-circle, given its center and a point on the circle.

Given: Points O and P in the Poincaré half-plane.

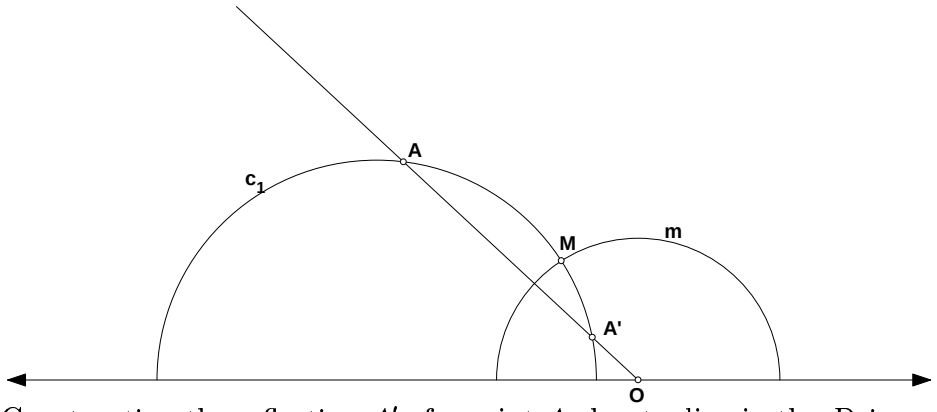

Figure 15: Constructing the reflection A' of a point A about a line in the Poincaré Half-plane model.

To Construct: the PH-circle centered at O with PH-radius OP.

Note: A PH-circle is also a Euclidean circle, albeit with a different "center." The Euclidean center of our desired circle will lie on the same vertical line as O.

a) Construct the Euclidean line ℓ through O which is perpendicular to the x-axis.

b) Using Construction # 9(i) above, let P' be the image of P under PH-reflection about the line perpendicular to PH-line $\overleftrightarrow{OP}$. Thus $OP \cong OP'$ in the PH-sense and P' is a second point on the desired circle.

c) Draw Euclidean segment PP' and construct its Euclidean perpendicular bisector m.

d) Assuming m is not a vertical line, let O' be the intersection of lines m and ℓ.

e) Let c_1 be the Euclidean circle centered at O' and passing through P (and P' as well).

Circle c_1 is our desired circle. See Figure 16.

10. (**phcntrrd.gss**) Construct a PH-circle, given its center and two points, the distance between which determines the radius of the circle.

Given: Points O, A, and B in the Poincaré half-plane.

To Construct: The PH-circle centered at O with radius OP, where $OP \cong AB$ in the PH-sense.

a) Draw PH-segment OA. Let PH-line ℓ be the perpendicular bisector of segment OA, using Construction #8 (ii) above.

b) Drop a PH-perpendicular from point B to line ℓ at a point M, using Construction #7 above. Call this line m.

c) Reflect point B about line ℓ using Construction #9 (i) above. Let P be the image of this reflection, noting that P lies on PH-line m.

d) Construct the PH-circle c_1 centered at O with radius OP, using Construction #9 (ii) above.

Circle c_1 has center O and radius OP. By construction, $OP \cong AB$. See Figure 17.

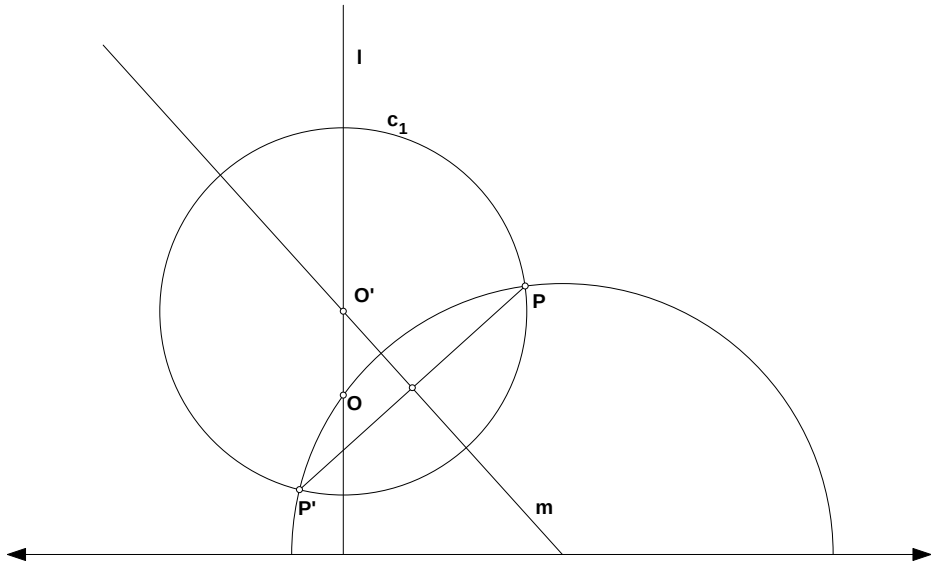

Figure 16: Constructing PH-circle centered at O with PH-radius OP in the Poincaré Half-plane model.

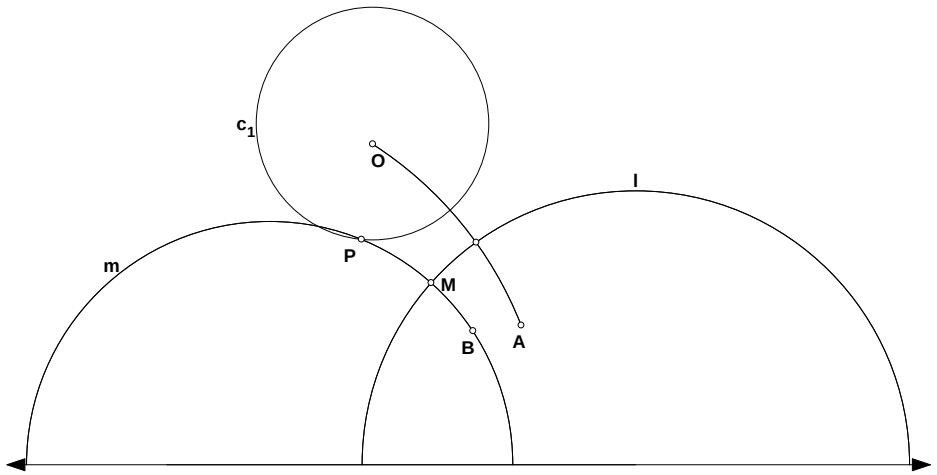

Figure 17: Constructing PH-circle centered at O with PH-radius OP congruent to AB in the Poincaré Half-plane model.

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