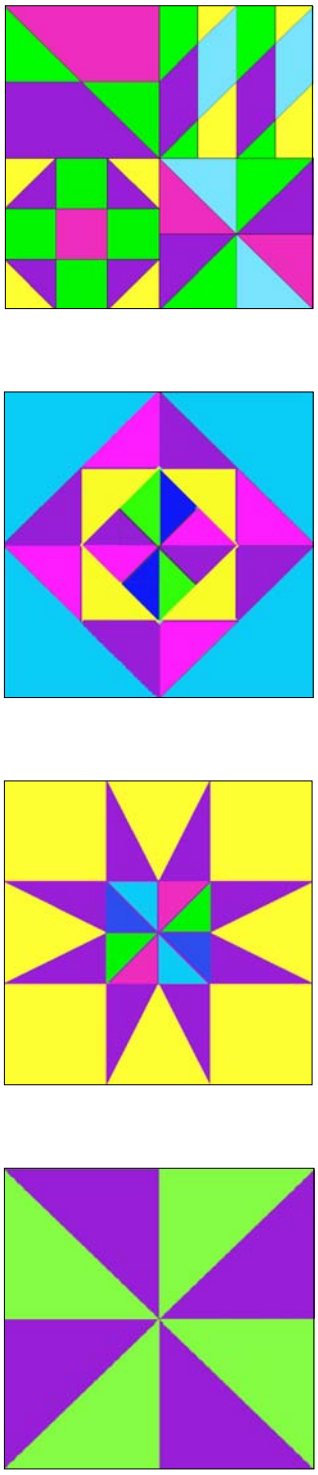

Learning Mathematics in the Intermediate Grades

Teaching and Learning—Math Division

Madison Metropolitan School District

"Our mission is to assure that every student has the knowledge and skills needed for academic achievement and a successful life."

Strategic Plan
Madison Metropolitan School District
2004

Executive Director, Teaching and Learning: *Lisa Wachtel*

Superintendent of Schools: *Daniel Nerad*

Assistant Superintendent for Elementary Schools: *Susan Abplanalp*

Madison Metropolitan School District 545 West Dayton Street Madison, WI 53703

www.madison.k12.wi.us

Math-ing

A poem for two voices.

Teachers,

focusing on the big ideas
and knowing student understandings,

recognizing the power of diversity,

pose the problem.

Noticing the starting points,

observe the strategies,
reflecting on students' hows
and asking whys
and what ifs.

Bringing the community together,

build the conversation,

clarifying the concepts and computations
and
encouraging the connections.

Challenge to reason further.
Understand how each has
figured out the math.

Confidently, competently, successfully
teaching
and learning mathematics
together.

Students,

looking forward to the next challenge
or maybe
worrying about the next challenge,
trusting the community for support,

accept the challenge.

Analyzing the details,
comprehending the question,

construct the models—
drawing pictures,
jotting notes,
making tables,
writing symbols.

Determine solutions.

Reflecting on the hows,
and asking whys and what ifs,
join the conversation.

Clarifying their concepts and computations,
evaluating and affirming accuracy,
extending their understandings,

figure out the math.

Confidently, competently, successfully
engaging
in learning mathematics.

m. jensen
may 2007

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Pamela Ferrill	Melissa Paton
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Karen Lenoch	Carrie Valentine
Andrea Lodato	Wendy Zucker
Julie Melton	Susan Roehlk, alt

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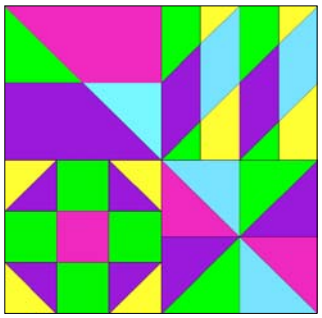
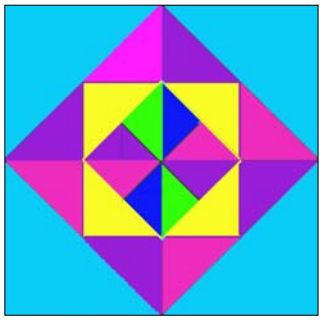
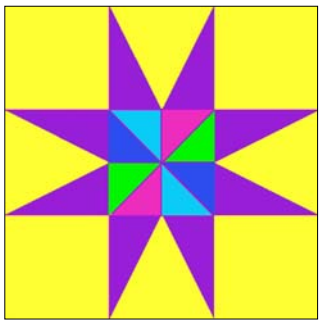
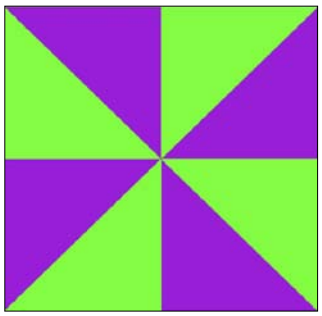




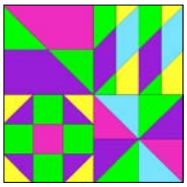


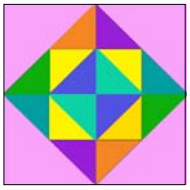


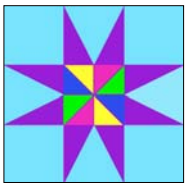
Chapter 1

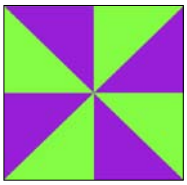
Learning Mathematics in the Intermediate Grades

Learning Mathematics at the Intermediate Level









Learning mathematics at the intermediate level builds on early number, geometry, measurement, and data concepts taught beginning in kindergarten.

During the intermediate grades, students expand their knowledge of number facts to include all four operations. They build their knowledge of the base-ten system and use it to solve increasingly more complex problems while also learning to compute accurately, flexibly, and efficiently. They make connections between whole numbers, fractions, and decimals and begin to develop proportional reasoning. They increase their skill at using a variety of representations to show their thinking and they use number sense when they justify their solution strategies to each other in small groups and to the whole class.

Intermediate students build on their intuition about number to make conjectures about number properties. They use connections between number concepts and geometry to show how number properties work. They refine their ideas about geometry as they work with manipulatives to classify shapes by geometric attribute, perceive shapes from different perspectives, and learn to transform shapes mentally through space.

During the intermediate grades students increase their facility with measurement and number concepts as they estimate, choose the appropriate tools and units for measuring, convert measurements, and develop measurement formulas.

Intermediate grade students use their data collection experiences from the primary grades to make predictions based on data in charts and graphs and to identify important features of data such as range, mean, median, and mode.

As students in the intermediate grades deepen their math knowledge in all areas of mathematics they become more confident and adept in their abilities to solve a wider variety of problems with unfamiliar as well as familiar contexts.

At the intermediate level, students:

- ☑ explore and solve a wide-variety of problems in number, geometry, measurement, and data
- ☑ develop flexibility, accuracy, and efficiency working with numbers
- ☑ use invented and conventional mathematical notations
- ☑ use mathematical vocabulary
- ☑ explain their solutions

At the intermediate level, the math curriculum:

- ☑ focuses on problem solving and the connections between number, algebra, geometry, measurement, data and probability
- ☑ provides problems that allow students to develop new insights, methods, and skills
- ☑ encourages the use of models and symbolic representations for solutions to problems
- ☑ fosters mathematical thinking by encouraging explanations and demonstrations of solution strategies

"Mathematically proficient students have the necessary skill to carry out procedures flexibly, accurately, efficiently, and appropriately."

Adding It Up, 2001

At the intermediate level, math teachers:

- ☑ consider each student's diverse background when choosing or designing problems

  Math problems should:

  - have familiar settings or contexts
  - have accessible language
  - build new knowledge and skills

- ☑ provide ample opportunity for each student to learn new concepts and build proficiency in mathematical reasoning and computation

- ☑ assess individual progress regularly and re-direct instruction accordingly

- ☑ design, select, or adjust and supplement textbook materials into a coherent sequence to address district standards

- ☑ provide a minimum of sixty minutes of math instruction every day

At the intermediate level, teachers support a healthy learning community.

Teachers:

Equity and accessibility requires instruction that assures every student is successful.

- ☑ provide classroom structures and curriculum that invite all students to contribute to the learning community
- ☑ differentiate instruction so that all students have opportunities to engage in learning through individual, small, and whole group instruction
- ☑ develop classroom norms that treat every student's ideas as teaching and learning opportunities
- ☑ recognize and build on the experiences, skills, and knowledge that each child brings to the community
- ☑ provide many opportunities for students to:
    - ask questions
    - develop strategies and mathematical explanations
    - share ideas
    - listen to, value, and interpret the thinking of their classmates

In this classroom, mistakes are accepted, respected, and inspected as a natural part of learning

A. Andrews, 2004

At the intermediate level, students:

- ☑ communicate their thinking to:
  - make thoughts explicit
  - make solution strategies explicit
  - provide record of progress
- ☑ learn to communicate mathematical thinking:
  - verbally through many opportunities to practice
  - using physical materials (e.g. base-ten blocks, geometric solids)
  - using grade-appropriate math symbols and language (e.g. empty number line, arrow language ratio table, equations)
- ☑ decide which materials, models, or symbols best convey their thoughts and methods
- ☑ learn that physical materials can convey different meanings when used in different situations (e.g. a base-ten "flat" could have a "value" of 100 when working with whole numbers or "1" when working with decimals)

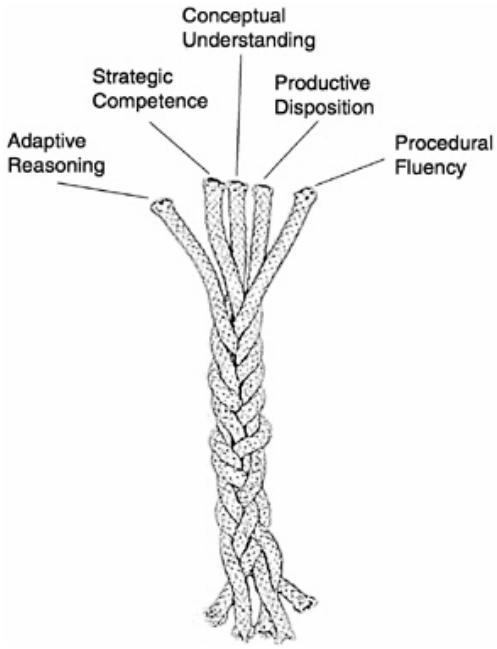
At the intermediate level, instruction assures that each student:

- ☑ has the opportunity to achieve the "Proficient" level as defined in the MMSD K-5 Grade-level Mathematics Standards

- ☑ develops the mathematical proficiency needed to cope with daily life and to continue to study and use mathematics in high school and beyond.

  Mathematical proficiency includes:

  - *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
  - *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
  - *strategic competence*—ability to formulate, represent, and solve mathematical problems
  - *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
  - *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.



The Strands of Mathematical Proficiency
Adding It Up, 1999

For More Information:

Fuson, K. C., Kalchman, M. & Bransford, J. D. (2005). Mathematical understanding: An introduction. In Donovan, M. S. & Bransford, J. D. (Eds.), *How Students Learn: Mathematics In the Classroom* (pp. 215-256). Washington, DC: The National Academies Press.

Hiebert, J. et al. (1997). *Making sense: Teaching and learning math with understanding*. Portsmouth NH: Heinemann.

Kilpatrick, J., Swafford, J. & Findell, B. (Eds.). (2001). *Adding it up*. Washington, DC: National Academy Press.

Kilpatrick, J., Swafford, J. & Findell, B. (Eds.). (2002). *Helping students learn mathematics*. Washington, DC: National Academy Press.

Malloy, C. E. (2004). Equity in mathematics education is about access. In Rubenstein, R. N. & Bright, G. W. (Eds.), *Perspectives on the Teaching of Mathematics Sixty-sixth Yearbook* (pp. 1-14). Reston, VA: National Council of Teachers of Mathematics.

Smith, M. (2004). Beyond presenting good problems: How a Japanese teacher implements a mathematics task. In Rubenstein, R. N. & Bright, G. W. (Eds.), *Perspectives on the Teaching of Mathematics Sixty-sixth Yearbook* (pp. 98-106). Reston, VA: National Council of Teachers of Mathematics.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Think

mathematically…

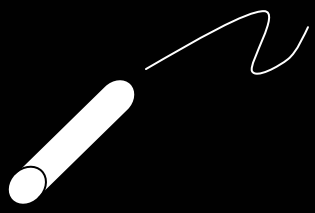
Solve problems

Represent

Communicate

Reason

Make connections



Chapter 2

Math Content and Processes

Math Content

All content strands include number concepts.

The Madison Metropolitan School District K–5 Grade Level Mathematics Standards organize math concepts and knowledge into four content strands: number, operations & algebraic relationships; geometry; measurement; and data analysis & probability. Many aspects of math experiences, no matter in which content strand, are based on using number concepts. During the intermediate grades students:

- ☑ solve problems that involve number, geometry, measurement, data analysis and probability
- ☑ solve multi-step problems with including rate, price, and multiplicative comparisons
- ☑ work with large numbers and decimals to understand the base-ten system
- ☑ solve a variety of problems that involve fractions including sharing/partitioning, ratio, equivalence, and operations
- ☑ increase use of mathematical vocabulary and symbolic language to convey mathematical reasoning
- ☑ develop fluency with computation and estimation
- ☑ work with equations to learn that the equal sign indicates a relationship rather than a signal to compute
- ☑ make conjectures about basic number properties that emerge from discussions about T/F and open number sentences

Math Processes

"Students become more proficient when they understand the underlying concepts of math, and they understand the concepts more easily if they are skilled at computational procedures." (Helping Children Learn Mathematics, p.12).

THINKING MATHEMATICALLY INVOLVES

Solving Problems

Representing Problems and Solutions

Communicating

Reasoning and Proof

Making Connections

Mathematically proficient students have a "productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile…"
Adding It Up, 2001

Math knowledge cannot be limited to procedures. Today's students must know how to use their knowledge and skills in flexible and efficient ways. They must learn to engage willingly to solve problems whether the context is familiar or unfamiliar and learn to construct models that represent their interpretation of a problem. They must learn to communicate their thinking and mathematical ideas clearly. They need to use mathematical reasoning to solve problems and convince others their solutions make sense. They need to make connections within and across content strands.

Today's students need to develop proficiencies in using all five processes in order to think mathematically!

Students need more skill and more understanding along with the ability to apply concepts, solve problems, reason logically, and see math as sensible, useful, and doable.

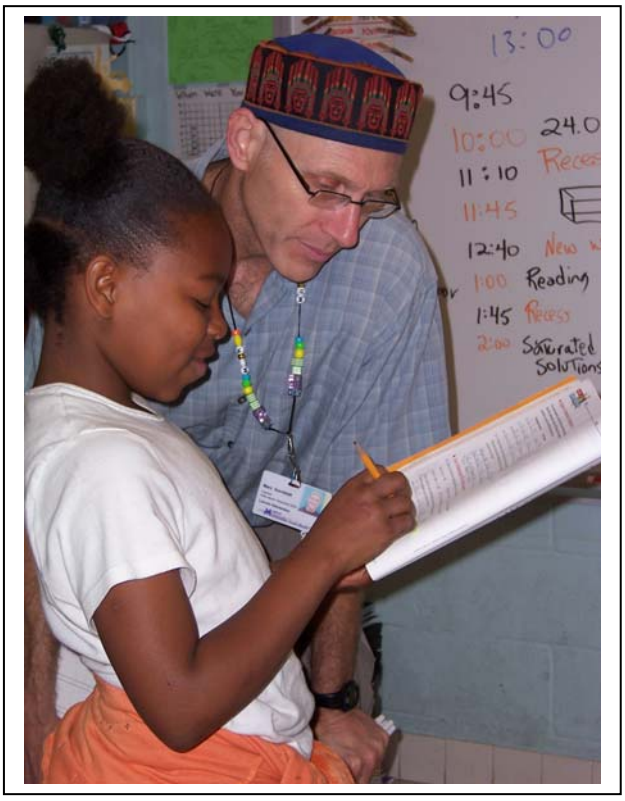
Anything less leads to knowledge that is fragile, disconnected, and weak.

Adding It Up, 2001

The five math **processes** are:

- **problem solving** – develop and adapt strategies to solve problems that arise in mathematics and in other contexts
- **representation** – use representations (e.g. objects, drawings, words and symbols) to organize one's thinking and record the steps taken to solve a problem
- **communication** – use the language of mathematics to express and explain mathematical ideas
- **reasoning and proof** – develop, test, represent, and justify conjectures about mathematical relationships
- **connections** – see the connections among ideas within mathematics and between mathematics and everyday experiences

Madison Metropolitan School District's *MMSD K-5 Grade Level Mathematics Standards* contain specific grade-level content and process expectations.



For More Information:

Barnberger, H. & Oberdorf, C. (2007). *Introduction to Connections, Grades 3-5*. Portsmouth, NH: Heinemann.

Ennis B. & Witeck, K. S. (2007). *Introduction to Representation, Grades 3-5*. Portsmouth, NH: Heinemann.

Kilpatrick, J., Martin, G. W. & Schifter, D. (Eds.). (2003). *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Madison Metropolitan School District. (2004). *MMSD mathematics grade level K-5 standards*. Madison, WI: Retrieved April 30, 2009 from http://mathweb.madison.k12.wi.us/node/268

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: An on-line version retrieved April 30, 2009 from http://standards.nctm.org/

National Research Council. (2001). *Helping children learn mathematics*. Kilpatrick, J., Swafford, J. & Findell, B. (Eds.). Mathematics Learning Study Committee, Center for Education, Division for Behavioral and Social Sciences and Education, Washington DC: National Academy Press.

O'Connell, S. (2007). *Introduction to Communication, Grades 3-5*. Portsmouth, NH: Heinemann.

O'Connell, S. (2007). *Introduction to Problem Solving, Grades 3-5* (2nd ed.). Portsmouth, NH: Heinemann.

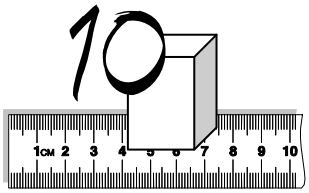
Schultz-Ferrell, K., Hammond, B. & Robles, J. (2007). *Introduction to Reasoning and Proof, Grades 3-5*. Portsmouth, NH: Heinemann.

Sowder, J. & Schapelle, B. (Eds.). (2002). *Lessons learned from research*. Reston, VA: National Council of Teachers of Mathematics.

Wisconsin Department of Public Instruction. *Wisconsin model academic standards*. Retrieved April 30, 2009 from http://dpi.wi.gov/standards/matintro.html

Wisconsin Department of Public Instruction. (2005). *Wisconsin knowledge and concepts examination assessment framework*. Retrieved April 30, 2009 from http://dpi.wi.gov/oea/pdf/math_framework.pdf

Intermediate Grade Mathematics



Assess

Evaluate

Plan

Teach

Chapter 3

Teaching and Learning Cycle

and the

Role of the Teacher

The Teaching and Learning Cycle

"The key to teaching students is figuring out what and how they are thinking while the teaching and learning are actually happening. Teaching and learning occur in a social context as a dynamic process rather than as a preconceived one." Lev Vygotsky's work is based on this idea. The basic premise of his theory is that, if we want to study how students learn, to assess their potential to learn, and to improve instruction, we must analyze their performance and their thinking while they are engaged in learning activities. This is what effective teachers do daily."

Vygotsky in the Classroom
Dixon-Krauss

Assessment provides an entry into the Teaching and Learning Cycle. Both formal and informal assessments at the beginning of the year are essential to establish the point at which a teacher begins instruction. Daily, ongoing observation and assessment continue throughout the year to identify strengths and needs and ascertain how each child's knowledge has changed. Being a skilled observer of children is one of the teacher's most important skills.

Assessing each child's knowledge provides evidence for the effectiveness of the instruction and informs planning. Instruction, based on thoughtful planning, is differentiated to support learning for each student. This cycle continues as the teacher reassesses learning, evaluates the assessment results, reflects on lessons taught, and plans and teaches new lessons.

The Teaching and Learning Cycle diagram that follows illustrates the recursive nature of assessing, evaluating, planning, and teaching.

Teaching and Learning Cycle for Math Instruction

Assess

Observe and assess daily

- *What do my students know about math concepts and procedures?*
- *What do my students' solution strategies tell me about their mathematical thinking?*

Evaluate

Identify strengths and needs

- *Did my teaching effectively address the strengths and needs of each student?*
- *How did the student's math reasoning or thinking change?*
- *What does this student understand now compared to: prior knowledge, other students in my class, and the MMSD standards?*

Plan

Design lessons to meet needs

- *What is the likely next step for this child?*
- *What tasks (discussion, problems, or practice) will address this child's needs?*
- *Which instructional configuration (whole class, small group, or individual) will serve the student best?*

Teach

Provide differentiated instruction

- *What knowledge does the student communicate?*
- *What might help this group of students make sense of a concept or record their thinking?*
- *Have I provided for the range of learners in the class?*

Based on the Teaching and Learning Cycle by Esser, D., Gleason, C., Kolan, T., Lucas, P. & Rohde, J. (2005). Primary Literacy Notebook. Madison, WI: Madison Metropolitan School District.

The Role of the Teacher

What is the teacher's role during problem solving activities?

Most of math instruction focuses on solving problems. During problem-solving activities, the teacher's role includes "listener," "mathematical thinker," "observer," "problem poser," and "questioner."

A teacher begins by providing a problem or set of problems and observing student efforts to solve them. The teacher supports and guides each student's mathematical growth by asking questions about their thinking and encouraging reflection rather than modeling, thinking for the student, or demonstrating methods to solve the problem.

During problem solving, the teacher:

- ☑ presents a problem or set of problems for students to solve.
- ☑ expects students to solve problems in ways that make sense to them and explain their thinking.
- ☑ observes how each student proceeds in finding solutions rather than "thinking for the student" or demonstrating how to solve a problem.
- ☑ assists students to develop representations that match their strategies and to show each step of their process when appropriate.
- ☑ helps students clarify explanations and identify connections between mathematical ideas.
- ☑ engages students in comparing and contrasting solutions to help expand and extend their knowledge and to develop fluency (flexible, accurate, and efficient strategies.)
- ☑ asks students to reflect on the important mathematical concepts and knowledge they have learned during the problem-solving experience.
- ☑ shows confidence in each student's ability and initiative.
- ☑ assesses each student's progress as compared to other students in the classroom and the MMSD standards.
- ☑ designs or chooses the next set of problems to help students increase their knowledge of particular math concepts and move toward more advanced strategies.

What tends to happen if a teacher models his/her thinking before students have a chance to work on solving the problem?

When a teacher regularly begins by thinking out-loud or modeling solution strategies, students:

- wonder if the teacher has a preconceived expectation of the steps or procedures they should use to solve a given problem.
- focus on trying to remember procedures and information instead of using mathematical concepts and knowledge they know in flexible ways.
- wait for a teacher to prompt or guide their thinking and acknowledge the accuracy of their solutions
- lose confidence in their own abilities.
- become less engaged in the mathematical discussion and lose the opportunity to learn.

Is it ever appropriate to teach a mathematical idea explicitly?

From time to time in math instruction, students need to learn math conventions such as:

- how to write 1/3
- what number comes 5 numbers before 10,000
- what symbols represent multiplication or division
- how to use an empty number-line, arrow language, or a ratio table

When teaching mathematical conventions, the teacher uses the same teaching strategies used when introducing new literacy concepts or processes. For example, the teacher models and thinks aloud about a symbol's attributes or the process of constructing a conventional representation. Gradually, the teacher releases responsibility and expects the students to become more and more independent in their use of these mathematical conventions. The teacher may want to find out what students think a particular symbol means or why and when to use it before expecting appropriate use.

To deepen their own mathematical knowledge, teachers should frequently ask themselves, "Why does this work?"

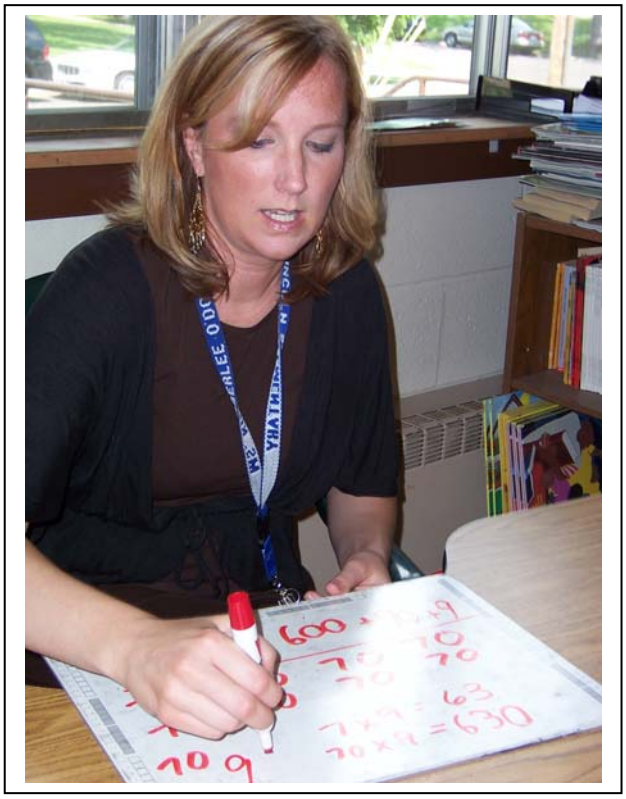
What do teachers consider when selecting problems?

Teachers use their knowledge of mathematics and each student's mathematical thinking when selecting problems. Problems should invite participation and be mindful of the prior knowledge, language, culture, and interests of each child. Teachers select problems that most effectively provide access to learning particular math concepts and skills.

When selecting problems to pose, teachers consider each student's:

- ☑ knowledge and skills related to the specific math topic
- ☑ strategies and reasoning
- ☑ use of models and symbolic representation
- ☑ language skills

Intermediate math teachers provide a series of problem-solving experiences to foster growth in all areas of mathematics by focusing on the goals outlined in the Madison Metropolitan School District K–5 Grade Level Standards.



For More Information:

Chappell, M. F., Schielack, J. F. & Zagorski, S. (2004). *Empowering the beginning teacher of mathematics: Elementary school*. Reston, VA: National Council of Teachers of Mathematics.

Dixon-Krauss, L. (1996). *Vygotsky in the classroom: Mediated literacy instruction and assessment*. Boston: Allyn and Bacon.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D. & Hanlie, M. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Ma, L. (1999). *Knowing and teaching elementary mathematics Chapter 5*. Mahwah, NJ: Lawrence Erlbaum Associates.

Chapter 4

Assessment

Teacher Observations

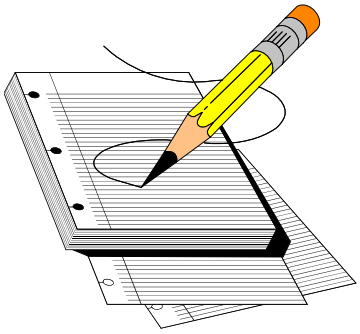
Work Samples

Informal Inventories

Fact Interviews

Problem Solving Interviews

Intermediate Math Assessments



Assessment

Assessment can serve many purposes ranging from program evaluation (e.g. WKCE-CRT) to learning the proficiency level of an individual student (e.g. fact fluency interviews). Teachers use assessment to determine instructional goals and report to colleagues and parents about progress.

All assessments are set in time, place, and form. A student may demonstrate proficiency in one setting (e.g. interview with a teacher) and not in another (e.g. whole class discussion). For this reason, it is important to use multiple forms of assessment to inform instruction and when reporting on student progress. Using multiple forms gives a more accurate picture of proficiency.

There are many ways for teachers to collect information about each student's progress in achieving mathematical proficiency. Assessment information includes daily observations, samples of student work, inventories, brainstorming lists, formal and informal pre- and post- tests, interviews, and pencil and paper tasks.

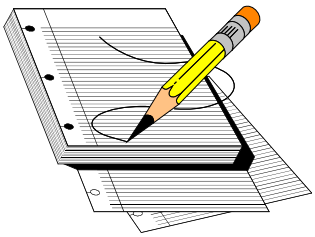
Responding to daily observations is at the core of good math instruction. As teachers notice what their students know and can do and how they interact with teachers and peers, the teacher makes instructional decisions that build on each student's mathematical knowledge. These instructional decisions make math accessible to all students and allow a range of learners to develop their knowledge and understanding.

The assessments included in this chapter provide the teacher information about what students know and can do. The assessments are intended to inform instruction in conjunction with other assessment data chosen by the teacher. It is essential to determine the purpose for assessment before deciding which assessment method to use and data to collect.

The following table offers examples of the purposes and ways to assess student learning. The MMSD "Learning Math in the Primary Grades" binder offers additional assessments for students who may need to develop proficiency on primary-level concepts before moving to intermediate-level concepts.

Purposes and Ways to Collect Assessment information

For the purpose of:	The teacher collects data from:
Having the most current information for specific day to day planning and instruction	**Daily observations** on any of the following: – individual student strategies including how students represent and explain their solution strategies – individual student responses to questions about their solutions – numbers used in the problems
Showing growth over time	*Samples* of student work.
Gathering information on prior knowledge and skills about a specific topic	*Inventories, student-brainstorming lists, and formal and informal pre-tests.*
Recording growth as a result of differentiated instruction for a specific topic	*Inventories, student brainstorming lists, and informal post-tests.*
Illuminating number sense and thinking strategies	**Fact Interviews** found in this chapter. *Completed Fact Interviews are also used to record student progress over time. Teachers pass completed Fact Interviews on to the following teacher each year.*
Determining ability to solve story problems	**CGI Story Problem Assessment** and related tasks found in this chapter.
Determining knowledge of base ten concepts	**Components of Place-value Understanding** found in this chapter. **Quick Tasks** assessments found in this chapter.
Determining knowledge of fraction concepts	**Fraction Concepts** assessment found in this chapter. **Quick Tasks** assessments found in this chapter.
Determining knowledge and use of equality relationships	Use Inspecting Equations activities, Chapter 8. See Table 8.1 **Benchmarks for understanding the equal sign.**



Fact Interviews

1. **Addition Fact Interview A**
   Sums within and to Ten
   (First Grade MMSD Math Standard)
   Available in the Learning Mathematics in the Primary Grades
2. **Addition Fact Interview B**
   Sums to 20
   (Second Grade MMSD Math Standard)
3. **Subtraction Fact Interview C**
   Differences Less Than, Equal To, and Greater Than 3
   (Third Grade MMSD Math Standard)
4. **Multiplication Fact Interview C**
   2, 5, 4, and 3 As Multiplier or Multiplicand
   (Third Grade MMSD Math Standard)
5. **Multiplication Fact Interview D**
   All Multiplication Facts
   (Fourth Grade MMSD Math Standard)
6. **Division Fact Interview E**
   All Division Facts
   (Fifth Grade MMSD Math Standard)

Within each interview, the individual computations (facts) are organized developmentally from those requiring the least amount of number sense to compute mentally to those requiring the most flexibility in working with number relationships mentally.

Five interviews are included in *Learning Mathematics in the Intermediate Grades*. Intermediate teachers will use some or all of the Fact Interviews depending on what is needed to learn about individual students. All six interviews are available on the district-wide web.

First grade teachers use **Addition Fact Interview A** to record the first stages of fact development. **Addition Fact Interview B** is included in this chapter for students who need to develop number relationship strategies. This development begins in kindergarten when students most often count everything, sometimes using fingers to show and count each set, in order to do single-digit calculations. In first grade, they develop counting on strategies.

By second grade, most students use strategies based on number relationships. However, there will always be some intermediate students working on 0-20 number relationships, in particular "across ten" facts.

Teachers who want to monitor a student's development of number sense and strategy use for computing basic single-digit subtraction facts should use **Subtraction Fact Interview C**. Students will be most successful with subtraction when they have a solid knowledge of part-whole relationships (for addition) and understand that subtraction can mean both the "difference" between two numbers as well as "taking away" one quantity from another.

By third grade, students begin to develop a multiplicative understanding of number. They use number relationships to multiply instead of repeatedly adding a given number or skip counting. Teachers use **Multiplication Fact Interview C** and **Multiplication Fact Interview D** to identify a student's facility in working with groups of the same number.

When students understand the inverse relationship between multiplication and division use **Division Fact Interview E** to assess the strategies a student uses for basic division facts.

Purposes of fact interview assessments include

- ☑ illuminating a child's sense of number relationships and the thinking strategies he/she uses to calculate single-digit computations
- ☑ identifying the size of numbers that a student can compute mentally (a student's mental computation level)
- ☑ identifying the number sizes to use in number work, problem solving and inspecting equations activities
- ☑ identifying which facts to use for a student's fluency and maintenance independent practice
- ☑ communicating a student's progress with parents and future teachers
- ☑ recording a student's progress over time (Schools keep the Fact Interviews in the blue PLAA binder.)

Materials for fact interview assessments:

- ☑ a student copy of the interview (without codes at the bottom)
- ☑ a teacher copy of the interview (with codes at the bottom)
- ☑ one pencil for the teacher, no pencil for the student
- ☑ no counters! (This interview determines a student's mental computation level.)

Conducting the interview

The teacher begins with an interview that is at a student's independent level and continues until the student consistently uses a counting by ones strategy or a repeated addition or skip counting strategy. (This is similar to continuing a running record reading until reaching a level that challenges a student's word identification and comprehension strategies.)

Note:

For sums within 10, a child might count fingers as a matter of habit. Check a few sums across 20 to see if the child can use a counting on strategy.

- ☑ Find a place where you and the student can sit next to each other.
- ☑ Have a copy of the interview in front of the student and a copy in front of you.
- ☑ Say: *"Please keep your hands above the table. I can learn about your math thinking if I can see when and how you use your fingers to help you."*
- ☑ Say: *"Look at each equation. All you have to say out loud is the number that goes on each line."*
- ☑ Say: *"Begin here,"* (point to the equation at the top of the column on the left) *and go down the column."*
- ☑ Do not read the equation aloud. If the student reads the equation aloud, remind the student that the only thing to say out loud is the number that goes on the line. Reading the entire equation can be very tiring for the student!
- ☑ Write the number the student says on the line in each equation.
- ☑ Use the coding system (see the coding guides for each interview) to indicate the student's strategies.
- ☑ Continue the interview as long as the student consistently uses number relationship strategies and consistently responds within 3-4 seconds. Occasionally, stop at specific facts and ask how the student is thinking about the numbers to be able to compute so quickly. Note the strategy with a code (see the coding guides)
- ☑ Stop the interview when you see a student consistently using counting strategies (all four operations) or a repeated addition strategy (for multiplication).
- ☑ If a student uses more than 3-4 seconds of thinking time, watch the student very carefully. If you can determine that the student is using a counting strategy (mouthing numbers, nodding, tapping a foot, moving fingers slightly), code it as a counting on strategy.
- ☑ Finish the interview by saying, "Thank you for sharing your thinking with me."
- ☑ Total the facts that have no coding as well as those that indicate a strategy other than a counting strategy. Write the total and date on the line in the lower right corner.
- ☑ Complete the class roster to aid in planning for instruction and recording growth over time.

Note:

Teachers learn students' number relationship strategies by asking them to explain their thinking on selected facts. See the coding guides for suggestions.

Interview B (All addition facts) – Coding counting strategies

Use the following codes to record a student's strategies. The codes indicate one of three things:

1. a student's counting strategy
2. using more than 3-4 seconds of thinking time
3. if the solution is inaccurate.

As you listen to the student's responses, use the following codes to mark each equation.

Counting Strategies (All addition facts)		
Code	**Example**	**Student**
ce	ce **4 + 5 = 9**	*counts everything* student puts up fingers for each addend and counts total
sb	sb **4 + 5 = 9**	*shows both sets* student puts up group of fingers for each addend and does not count total
f	f **4 + 5 = 9**	*fingers* student counts up from number with f written above it and puts up fingers to track counting
c	c **4 + 5 = 9**	*counts* student counts on from the number with a c above it, perhaps tapping, nodding, or mouthing the count
•	**5 + 4 = 9•**	student uses thinking time to respond, more than 3-4 seconds
—	**4 + 5 = $\overline{8}$**	response is incorrect (line above student's response)

Interview B (All addition facts) – Coding number relationship strategies

Teachers want to know which number relationship strategies a student uses. If you cannot see any evidence that a student is using a counting strategy, you should ask the student to explain his/her thinking. Select a near double (e.g., 6 + 5), an adding 9, (e.g., 7 + 9), or a making a 10 (e.g., 8 + 6). Ask the student: *"Tell me how you are thinking about the numbers in order to do this one,"* pointing to a specific equation**.** If the student shares a counting strategy, note it with the codes on the previous page. If the student shares a number relationship strategy, such as decomposing addends, compensating, or using a known fact, write the symbols that show the student's thinking above the equation.

Here are three examples:

Number Relationship Strategies (All addition facts)	
Example	**Student**
6+6+2 **6 + 8 = <u>14</u>**	use a doubles fact (6+6+2)
8+2+4 **6 + 8 = <u>14</u>**	decompose an addend (8+2+4)
10 + 6 - 1 **6 + 9 = <u>14</u>**	change an addend and then subtract to compensate for the change

Definitions:

Addend—any numbers that are added together

Sum—total or whole amount, the result of adding

Compensating strategy—changes the numbers in the equation, computes, and then adjusts the solution to correct for the change made at the beginning. E.g., given 7 + 9, the student shares "7 + 10 is 17. I have to take away 1 from the 17 because I added 1 to the 9."

Decomposing an addend—uses another name for the quantity to make the computation more easily accomplished. E.g., given 6 + 8, the student shares " 6 is 2 + 4. 8 + 2 is 10. 10 and 4 more is 14."

Interview C (All subtraction facts) – Coding counting strategies

Use the following codes to record a student's strategies. The codes indicate one of three things:

1. a student's strategy
2. using more than 3-4 seconds of thinking time
3. if the solution is inaccurate.

As you listen to a student's responses, use the following codes to mark each equation:

Counting Strategies (All subtraction facts)		
Code	**Example**	**Student**
ce	ce **9 – 6 = 3**	***c**ounts **e**verything* student puts up fingers for minuend and puts down a finger at a time for subtrahend student
sb	sb **9 – 6 = 3**	***s**hows **b**oth* student puts up group of fingers for minuend, puts down group of fingers for subtrahend, does not count remaining fingers
fb	fb **9 – 6 = 3**	***f**ingers **b**ack* student counts back the quantity indicated by the subtrahend, putting up fingers to track the counting (e.g. student put fingers up while saying 8, 7, 6, 5, 4, 3)
cb	cb **9 – 6 = 3**	***c**ounts **b**ack* student counts back the quantity indicated by the subtrahend (e.g. student said 8, 7, 6, 5, 4, 3)
f	f **9 – 6 = 3**	***f**ingers* student counts down from minuend to subtrahend or up from number with "f" written above it and puts up fingers to track counting (e.g. student put fingers up while saying 7, 8, 9)
c	c **9 – 6 = 3**	***c**ounts* student counts down or up from number with "c" written above it, perhaps tapping, nodding, or mouthing the count (e.g. student said 7, 8, 9)
•	**9 – 6 = 3•**	student uses thinking time to respond, more than 3-4 seconds
—	**9 – 6 = $\overline{4}$**	response is incorrect (line above student's response)

Interview C (All subtraction facts) – Coding number relationship strategies

Students who have moved beyond counting strategies may have learned number relationship strategies. When there is no evidence of a counting strategy, ask the student to explain how they determined the solution. Select an equation that might be solved using a known fact (e.g. 9 - 5). Ask the student: "*Tell me how you are thinking about the numbers in order to do this one*," pointing to the specific equation. If the student shares a counting strategy, note it with the codes on the previous page. If the student shares a number relationship strategy, such as one of those listed below, write the symbols that show the student's thinking above the equation. Be sure to check subtractions that cross ten (e.g. 14 - 6).

Number Relationship Strategies (All subtraction facts)	
Example	**Student**
6 + 8 = 14 **14 – 6 = 8**	uses a known addition fact
6 + ? = 14 **14 – 6 = 8**	determines a missing addend
14 - 4 - 2 **14 – 6 = 8**	decomposes the subtrahend
10 - 6 + 4 **14 – 6 = 8**	decomposes the minuend

Definitions:

Minuend—the number being subtracted from

Subtrahend—the number being subtracted

Difference—the number that is left after one quantity is taken away from another, the result of subtracting

Interviews C & D (Multiplication) and E (Division) – Coding counting and number relationship strategies

Use the codes on the tables that follow to record a student's strategies. The codes indicate one of three things:

1. a student's strategy (counting or number relationship)
2. using more than 3-4 seconds of thinking time
3. if the solution is inaccurate.

If you cannot see any evidence that the student is using a counting strategy, you should ask the student to explain they solved the problem. Point to an equation and ask the student: *"Tell me how you are thinking about the numbers in order to do this one."* If the student shares a number sense strategy, such as halving then doubling, using partial products, working from a known fact, or some other strategy, write the code or symbols that show the student's thinking above the equation.

For **Interview C** – Multiplication, you might want to ask about 6 x 5, 8 x 4, and 9 x 3.

For **Interview D** – Multiplication, you might want to ask about 8 x 6, 9 x 7, and 7 x 5.

Adding and Counting Strategies (Multiplication)		
Code	**Example**	**Student**
rc	rc (18) **8 x 6 = <u>48</u>**	*<u>r</u>epeatedly adds a few, then <u>c</u>ounts on* student repeatedly adds a few groups (i.e. 6, 12, 18), then counts on by ones (i.e. 19, 20, 21, 22, 23, 24 and so on)
r	r **8 x 6 = <u>48</u>**	*<u>r</u>epeatedly adds* student adds groups without counting
sf	sf **8 x 6 = <u>48</u>**	*<u>s</u>kip counts, <u>f</u>ingers* student skip counts, keeping track of groups with fingers
s	s **8 x 6 = <u>48</u>**	*<u>s</u>kip counts* student skip counts
•	**8 x 6 = <u>48</u> •**	uses thinking time to respond, more than 3-4 seconds
—	**8 x 6 = <u>45</u>**	response is incorrect (line above student's response)

Number Relationship Strategies (Multiplication)		
Code	**Example**	**Student**
write strategy	24 + 24 **8 x 6 = <u>48</u>**	explains how to half and then double (i.e. student computed 4 x 6 and then doubled that product)
write strategy	30 + 18 **8 x 6 = <u>48</u>**	explains how to add partial products (i.e. student computed 5 x 6 and 3 x 6 and added their products)
write strategy	60 - 6 **9 x 6 = <u>54</u>**	explains how to use a known fact

Strategies (Division)		
Code	**Example**	**Student**
rs	r **24 ÷ 6 = <u>4</u>**	**<u>r</u>**epeatedly **<u>s</u>**ubtracts the divisor
ra	ra **24 ÷ 6 = <u>4</u>**	**<u>r</u>**epeatedly **<u>a</u>**dds the divisor
s	s **24 ÷ 6 = <u>4</u>**	**<u>s</u>**kip counts
•	**24 ÷ 6 = <u>4</u>•**	uses thinking time to respond, more than 3-4 seconds dot
—	**24 ÷ 6 = $\overline{\underline{3}}$**	response is incorrect (line above student's response)
write strategy	? ÷ 6 = 24 **24 ÷ 6 = <u>4</u>**	explains "inverse" thinking (e.g. "What times 6 is 24?")
write strategy	6 x 4 **24 ÷ 6 = <u>4</u>**	explains how to use a known multiplication fact

Definitions:

Dividend—the quantity to be divided

Divisor— the quantity by which another quantity is to be divided

Quotient—the result of dividing one quantity by another

Factor—one of the whole numbers multiplied to get a given number; an integer that divides evenly into an integer

Multiplicand—the number (factor) being multiplied

Multiplier—the number (factor) being multiplied by

Product—the result of multiplying

Interview B - Addition

5 + 1 = ____	3 + 3 = ____	4 + 6 = ____	6 + 8 = ____	5 + 7 = ____
1 + 7 = ____	3 + 4 = ____	5 + 5 = ____	3 + 9 = ____	5 + 9 = ____
1 + 9 = ____	3 + 6 = ____	3 + 7 = ____	9 + 3 = ____	8 + 6 = ____
4 + 2 = ____	0 + 9 = ____	2 + 8 = ____	8 + 7 = ____	9 + 9 = ____
2 + 5 = ____	3 + 5 = ____	6 + 6 = ____	7 + 7 = ____	4 + 9 = ____
3 + 2 = ____	4 + 4 = ____	2 + 9 = ____	6 + 7 = ____	8 + 4 = ____
2 + 4 = ____	5 + 3 = ____	7 + 4 = ____	7 + 9 = ____	6 + 9 = ____
2 + 6 = ____	4 + 3 = ____	5 + 6 = ____	9 + 8 = ____	7 + 8 = ____
2 + 3 = ____	4 + 5 = ____	3 + 8 = ____	8 + 8 = ____	9 + 7 = ____
2 + 7 = ____	6 + 4 = ____	7 + 5 = ____	8 + 5 = ____	5 + 8 = ____

Addition Fact Interview B - Sums to 20—Within, To, and Across 10 Name_____

5 + 1 = ____	3 + 3 = ____	4 + 6 = ____	6 + 8 = ____	5 + 7 = ____
1 + 7 = ____	3 + 4 = ____	5 + 5 = ____	3 + 9 = ____	5 + 9 = ____
1 + 9 = ____	3 + 6 = ____	3 + 7 = ____	9 + 3 = ____	8 + 6 = ____
4 + 2 = ____	0 + 9 = ____	2 + 8 = ____	8 + 7 = ____	9 + 9 = ____
2 + 5 = ____	3 + 5 = ____	6 + 6 = ____	7 + 7 = ____	4 + 9 = ____
3 + 2 = ____	4 + 4 = ____	2 + 9 = ____	6 + 7 = ____	8 + 4 = ____
2 + 4 = ____	5 + 3 = ____	7 + 4 = ____	7 + 9 = ____	6 + 9 = ____
2 + 6 = ____	4 + 3 = ____	5 + 6 = ____	9 + 8 = ____	7 + 8 = ____
2 + 3 = ____	4 + 5 = ____	3 + 8 = ____	8 + 8 = ____	9 + 7 = ____
2 + 7 = ____	6 + 4 = ____	7 + 5 = ____	8 + 5 = ____	5 + 8 = ____

Coding: **ce** – counted everything; **sb** - showed both sets, doesn't count all; **f** – used fingers to count on from; **c** - counted on from; **dot** after sum – used thinking time; **line** above sum – incorrect

Notes:

Recorder _____Date _____ Score ____/50
Recorder _____Date _____ Score ____/50

Addition Fact Interview B Data Second Grade Teacher _____Date _____

Name	Within and To 10 Facts						Across 10 Facts								
	Counts both sets	Fingers		Counts on		**Recalls**	Fingers		Counts on		**Uses Number Relationship Strategies**				**Recalls**
		From First	From largest	From first	From largest		From first	From largest	From first	From largest	Uses a double	Decomposes to make 10	Compensates	Uses known fact	

Knows Number Sense Strategies for Addition Facts (Level B)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0 + 0	1 + 0	2 + 0	3 + 0	4 + 0	5 + 0	6 + 0	7 + 0	8 + 0	9 + 0	10 + 0
1	0 + 1	1 + 1	2 + 1	3 + 1	4 + 1	5 + 1	6 + 1	7 + 1	8 + 1	9 + 1	10 + 1
2	0 + 2	1 + 2	2 + 2	3 + 2	4 + 2	5 + 2	6 + 2	7 + 2	8 + 2	9 + 2	10 + 2
3	0 + 3	1 + 3	2 + 3	3 + 3	4 + 3	5 + 3	6 + 3	7 + 3	8 + 3	9 + 3	10 + 3
4	0 + 4	1 + 4	2 + 4	3 + 4	4 + 4	5 + 4	6 + 4	7 + 4	8 + 4	9 + 4	10 + 4
5	0 + 5	1 + 5	2 + 5	3 + 5	4 + 5	5 + 5	6 + 5	7 + 5	8 + 5	9 + 5	10 + 5
6	0 + 6	1 + 6	2 + 6	3 + 6	4 + 6	5 + 6	6 + 6	7 + 6	8 + 6	9 + 6	10 + 6
7	0 + 7	1 + 7	2 + 7	3 + 7	4 + 7	5 + 7	6 + 7	7 + 7	8 + 7	9 + 7	10 + 7
8	0 + 8	1 + 8	2 + 8	3 + 8	4 + 8	5 + 8	6 + 8	7 + 8	8 + 8	9 + 8	10 + 8
9	0 + 9	1 + 9	2 + 9	3 + 9	4 + 9	5 + 9	6 + 9	7 + 9	8 + 9	9 + 9	10 + 9
10	0 + 10	1 + 10	2 + 10	3 + 10	4 + 10	5 + 10	6 + 10	7 + 10	8 + 10	9 + 10	10 + 10

Name _____ Date Started _____ Goal Met _____

Interview C - Subtraction

2 – 1 = ____	8 – 6 = ____	8 – 3 = ____	14 – 7 = ____	16 – 8 = ____
4 – 3 = ____	9 – 7 = ____	10 – 5 = ____	11 – 7 = ____	12 – 5 = ____
6 – 5 = ____	10 – 8 = ____	10 – 4 = ____	12 – 3 = ____	18 – 9 = ____
8 – 8 = ____	6 – 3 = ____	10 – 6 = ____	12 – 8 = ____	16 – 9 = ____
9 – 8 = ____	9 – 6 = ____	10 – 3 = ____	13 – 5 = ____	13 – 9 = ____
8 – 7 = ____	7 – 4 = ____	12 – 6 = ____	17 – 9 = ____	17 – 8 = ____
4 – 2 = ____	8 – 5 = ____	11 – 9 = ____	13 – 6 = ____	14 – 9 = ____
5 – 3 = ____	10 – 7 = ____	11 – 6 = ____	12 – 4 = ____	12 – 7 = ____
7 – 5 = ____	8 – 4 = ____	11 – 5 = ____	14 – 8 = ____	15 – 8 = ____
6 – 4 = ____	9 – 4 = ____	11 – 8 = ____	15 – 7 = ____	16 – 7 = ____

Subtraction Fact Interview C - Differences Less Than, Equal To, and Greater Than 3 Name_____

2 – 1 = ____	8 – 6 = ____	8 – 3 = ____	14 – 7 = ____	16 – 8 = ____
4 – 3 = ____	9 – 7 = ____	10 – 5 = ____	11 – 7 = ____	12 – 5 = ____
6 – 5 = ____	10 – 8 = ____	10 – 4 = ____	12 – 3 = ____	18 – 9 = ____
8 – 8 = ____	6 – 3 = ____	10 – 6 = ____	12 – 8 = ____	16 – 9 = ____
9 – 8 = ____	9 – 6 = ____	10 – 3 = ____	13 – 5 = ____	13 – 9 = ____
8 – 7 = ____	7 – 4 = ____	12 – 6 = ____	17 – 9 = ____	17 – 8 = ____
4 – 2 = ____	8 – 5 = ____	11 – 9 = ____	13 – 6 = ____	14 – 9 = ____
5 – 3 = ____	10 – 7 = ____	11 – 6 = ____	12 – 4 = ____	12 – 7 = ____
7 – 5 = ____	8 – 4 = ____	11 – 5 = ____	14 – 8 = ____	15 – 8 = ____
6 – 4 = ____	9 – 4 = ____	11 – 8 = ____	15 – 7 = ____	16 – 7 = ____

Coding: **ce** – counted whole, took away, then counted remaining set; **sb** – showed sets, didn't count; **fb** – used fingers to count back subtrahend; **cb** counted back subtrahend; **f** – used fingers to count difference to or from; **c** – counted difference up from or down to; **dot** after difference – used thinking time; **line** above difference – incorrect

Notes:

Recorder _____Date _____ Score ____/50
Recorder _____Date _____ Score ____/50

Subtraction Fact Interview C Data Third Grade Teacher _____Date _____

Name	Facts With a Difference of 1, 2, or 3								Facts With a Difference of 4 or more											
	Counts all	Counts						Recalls	Fingers			Counts			**Uses Number Relationship Strategies**					Recalls
		Back from	Back to	Up from	Back from	Back to	Up from		Back from	Back to	Up from	Back from	Back to	Up from	Uses known addition fact	Thinks of missing addend	Decom-poses minuend	Decom-poses subtra-hend	Compen-sates	

Knows Number Sense Strategies for Subtraction Facts (Level C)

Difference of 0	Difference of 1	Difference of 2	Difference of 3	Difference of 4	Difference of 5	Difference of 6	Difference of 7	Difference of 8	Difference of 9	Difference of 10
0 – 0	1 – 0	2 – 0	3 – 0	4 – 0	5 – 0	6 – 0	7 – 0	8 – 0	9 – 0	10 – 0
1 – 1	2 – 1	3 – 1	4 – 1	5 – 1	6 – 1	7 – 1	8 – 1	9 – 1	10 – 1	11 – 1
2 – 2	3 – 2	4 – 2	5 – 2	6 – 2	7 – 2	8 – 2	9 – 2	10 – 2	11 – 2	12 – 2
3 – 3	4 – 3	5 – 3	6 – 3	7 – 3	8 – 3	9 – 3	10 – 3	11 – 3	12 – 3	13 – 3
4 – 4	5 – 4	6 – 4	7 – 4	8 – 4	9 – 4	10 – 4	11 – 4	12 – 4	13 – 4	14 – 4
5 – 5	6 – 5	7 – 5	8 – 5	9 – 5	10 – 5	11 – 5	12 – 5	13 – 5	14 – 5	15 – 5
6 – 6	7 – 6	8 – 6	9 – 6	10 – 6	11 – 6	12 – 6	12 – 6	14 – 6	15 – 6	16 – 6
7 – 7	8 – 7	9 – 7	10 – 7	11 – 7	12 – 7	13 – 7	14 – 7	15 – 7	16 – 7	17 – 7
8 – 8	9 – 8	10 – 8	11 – 8	12 – 8	13 – 8	14 – 8	15 – 8	16 – 8	17 – 8	18 – 8
9 – 9	10 – 9	11 – 9	12 – 9	13 – 9	14 – 9	15 – 9	16 – 9	17 – 9	18 – 9	19 – 9
10 – 10	11 – 10	12 – 10	13 – 10	14 – 10	15 – 10	16 – 10	17 – 10	18 – 10	19 – 10	20 – 10

Name _____ Date Started _____ Goal Met _____

Interview C - Multiplication

1 x 5 = ____	6 x 2 = ____	5 x 5 = ____	0 x 6 = ____	4 x 9 = ____
2 x 3 = ____	2 x 9 = ____	5 x 3 = ____	4 x 4 = ____	6 x 3 = ____
3 x 0 = ____	2 x 7 = ____	6 x 5 = ____	7 x 4 = ____	7 x 3 = ____
2 x 4 = ____	7 x 2 = ____	5 x 6 = ____	6 x 4 = ____	4 x 8 = ____
7 x 1 = ____	8 x 2 = ____	5 x 4 = ____	4 x 3 = ____	8 x 3 = ____
5 x 2 = ____	2 x 8 = ____	5 x 8 = ____	4 x 6 = ____	3 x 6 = ____
2 x 6 = ____	9 x 2 = ____	9 x 5 = ____	4 x 7 = ____	3 x 8 = ____
2 x 5 = ____	3 x 5 = ____	5 x 9 = ____	8 x 4 = ____	9 x 3 = ____
3 x 2 = ____	4 x 5 = ____	5 x 7 = ____	3 x 3 = ____	3 x 7 = ____
4 x 2 = ____	8 x 5 = ____	7 x 5 = ____	9 x 4 = ____	3 x 9 = ____

Multiplication Fact Interview C - 2, 5, 4, and 3 as Multiplier or Multiplicand Name_____

1 x 5 = ____	6 x 2 = ____	5 x 5 = ____	0 x 6 = ____	4 x 9 = ____
2 x 3 = ____	2 x 9 = ____	5 x 3 = ____	4 x 4 = ____	6 x 3 = ____
3 x 0 = ____	2 x 7 = ____	6 x 5 = ____	7 x 4 = ____	7 x 3 = ____
2 x 4 = ____	7 x 2 = ____	5 x 6 = ____	6 x 4 = ____	4 x 8 = ____
7 x 1 = ____	8 x 2 = ____	5 x 4 = ____	4 x 3 = ____	8 x 3 = ____
5 x 2 = ____	2 x 8 = ____	5 x 8 = ____	4 x 6 = ____	3 x 6 = ____
2 x 6 = ____	9 x 2 = ____	9 x 5 = ____	4 x 7 = ____	3 x 8 = ____
2 x 5 = ____	3 x 5 = ____	5 x 9 = ____	8 x 4 = ____	9 x 3 = ____
3 x 2 = ____	4 x 5 = ____	5 x 7 = ____	3 x 3 = ____	3 x 7 = ____
4 x 2 = ____	8 x 5 = ____	7 x 5 = ____	9 x 4 = ____	3 x 9 = ____

Coding: **rc**-repeatedly added, then counted on; **r**-repeatedly added; **sf**-used fingers to track skip counting, **s**-skip counted; **dot** after product-used thinking time; **line** above product-incorrect

Notes:

Recorder _____Date _____ Score ____/50
Recorder _____Date _____ Score ____/50

Multiplication Fact Interview C Data Third Grade Teacher _____Date _____

Name	**2, 5, 4 and 3 as Multiplier or Multiplicand**								
	Counts everything	Adds		Skip Counts		**Uses Number Relationship Strategies**			Recalls
		Adds a few, then counts on	Repeatedly adds	Uses fingers to track skip counting	Uses memorized sequence	Uses doubling	Works from a known fact	Adds partial products	

Knows Number Sense Strategies for Multiplication Facts (Level C)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0 x 0	1 x 0	2 x 0	3 x 0	4 x 0	5 x 0	6 x 0	7 x 0	8 x 0	9 x 0	10 x 0
1	0 x 1	1 x 1	2 x 1	3 x 1	4 x 1	5 x 1	6 x 1	7 x 1	8 x 1	9 x 1	10 x 1
2	0 x 2	1 x 2	2 x 2	3 x 2	4 x 2	5 x 2	6 x 2	7 x 2	8 x 2	9 x 2	10 x 2
3	0 x 3	1 x 3	2 x 3	3 x 3	4 x 3	5 x 3	6 x 3	7 x 3	8 x 3	9 x 3	10 x 3
4	0 x 4	1 x 4	2 x 4	3 x 4	4 x 4	5 x 4	6 x 4	7 x 4	8 x 4	9 x 4	10 x 4
5	0 x 5	1 x 5	2 x 5	3 x 5	4 x 5	5 x 5	6 x 5	7 x 5	8 x 5	9 x 5	10 x 5
6	0 x 6	1 x 6	2 x 6	3 x 6	4 x 6	5 x 6					
7	0 x 7	1 x 7	2 x 7	3 x 7	4 x 7	5 x 7					
8	0 x 8	1 x 8	2 x 8	3 x 8	4 x 8	5 x 8					
9	0 x 9	1 x 9	2 x 9	3 x 9	4 x 9	5 x 9					
10	0 x 10	1 x 10	2 x 10	3 x 10	4 x 10	5 x 10					

Name _____ Date Started _____ Goal Met _____

Interview D - Multiplication

1 x 5 = ____	3 x 5 = ____	6 x 4 = ____	8 x 3 = ____	9 x 4 = ____
2 x 4 = ____	5 x 3 = ____	3 x 4 = ____	3 x 8 = ____	8 x 8 = ____
7 x 1 = ____	4 x 5 = ____	4 x 3 = ____	3 x 9 = ____	6 x 9 = ____
3 x 0 = ____	8 x 5 = ____	4 x 6 = ____	4 x 7 = ____	7 x 7 = _____
6 x 2 = ____	5 x 4 = ____	7 x 4 = ____	5 x 9 = ____	7 x 8 = ____
2 x 8 = ____	5 x 5 = ____	4 x 8 = ____	6 x 6 = ____	9 x 9 = ____
7 x 2 = ____	5 x 7 = ____	3 x 3 = ____	4 x 9 = ____	8 x 9 = ____
2 x 9 = ____	6 x 5 = ____	6 x 3 = ____	7 x 5 = ____	6 x 7 = ____
8 x 2 = ____	5 x 8 = ____	3 x 7 = ____	8 x 4 = ____	9 x 7 = ____
0 x 6 = ____	4 x 4 = ____	7 x 3 = ____	8 x 7 = ____	8 x 6 = ____

Multiplication Fact Interview D Data - All Multiplication Facts Name _____

1 x 5 = ____	3 x 5 = ____	6 x 4 = ____	8 x 3 = ____	9 x 4 = ____
2 x 4 = ____	5 x 3 = ____	3 x 4 = ____	3 x 8 = ____	8 x 8 = ____
7 x 1 = ____	4 x 5 = ____	4 x 3 = ____	3 x 9 = ____	6 x 9 = ____
3 x 0 = ____	8 x 5 = ____	4 x 6 = ____	4 x 7 = ____	7 x 7 = _____
6 x 2 = ____	5 x 4 = ____	7 x 4 = ____	5 x 9 = ____	7 x 8 = ____
2 x 8 = ____	5 x 5 = ____	4 x 8 = ____	6 x 6 = ____	9 x 9 = ____
7 x 2 = ____	5 x 7 = ____	3 x 3 = ____	4 x 9 = ____	8 x 9 = ____
2 x 9 = ____	6 x 5 = ____	6 x 3 = ____	7 x 5 = ____	6 x 7 = ____
8 x 2 = ____	5 x 8 = ____	3 x 7 = ____	8 x 4 = ____	9 x 7 = ____
0 x 6 = ____	4 x 4 = ____	7 x 3 = ____	8 x 7 = ____	8 x 6 = ____

Coding: **rc**-repeatedly added, then counted on; **r**-repeatedly added; **sf**-used fingers to track skip counting; **s**-skip counted; **dot** after product-used thinking time; **Line above** product-incorrect

Notes:

Recorder _____Date _____ Score ____/50

Recorder _____Date _____ Score ____/50

Multiplication Fact Interview D Data - All Multiplication Facts Teacher _____Date _____

Name	**All Multiplication Facts**								
	Counts Everything	Adds		Skip Counts		**Uses Number relationship Strategies**			Recalls
		Adds a few, then counts on	Repeatedly adds	Uses fingers to track skip counting	Uses memorized sequence	Uses doubling	Works from a known fact	Adds partial products	

Knows Number Sense Strategies for Multiplication Facts (Level D)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0 x 0	1 x 0	2 x 0	3 x 0	4 x 0	5 x 0	6 x 0	7 x 0	8 x 0	9 x 0	10 x 0
1	0 x 1	1 x 1	2 x 1	3 x 1	4 x 1	5 x 1	6 x 1	7 x 1	8 x 1	9 x 1	10 x 1
2	0 x 2	1 x 2	2 x 2	3 x 2	4 x 2	5 x 2	6 x 2	7 x 2	8 x 2	9 x 2	10 x 2
3	0 x 3	1 x 3	2 x 3	3 x 3	4 x 3	5 x 3	6 x 3	7 x 3	8 x 3	9 x 3	10 x 3
4	0 x 4	1 x 4	2 x 4	3 x 4	4 x 4	5 x 4	6 x 4	7 x 4	8 x 4	9 x 4	10 x 4
5	0 x 5	1 x 5	2 x 5	3 x 5	4 x 5	5 x 5	6 x 5	7 x 5	8 x 5	9 x 5	10 x 5
6	0 x 6	1 x 6	2 x 6	3 x 6	4 x 6	5 x 6	6 x 6	7 x 6	8 x 6	9 x 6	10 x 6
7	0 x 7	1 x 7	2 x 7	3 x 7	4 x 7	5 x 7	6 x 7	7 x 7	8 x 7	9 x 7	10 x 7
8	0 x 8	1 x 8	2 x 8	3 x 8	4 x 8	5 x 8	6 x 8	7 x 8	8 x 8	9 x 8	10 x 8
9	0 x 9	1 x 9	2 x 9	3 x 9	4 x 9	5 x 9	6 x 9	7 x 9	8 x 9	9 x 9	10 x 9
10	0 x 10	1 x 10	2 x 10	3 x 10	4 x 10	5 x 10	6 x 10	7 x 10	8 x 10	9 x 10	10 x 10

Name _____ Date Started _____ Goal Met _____

Interview E - Division

4 ÷ 2 = ____	40 ÷ 5 = ____	21 ÷ 3 = ____	56 ÷ 7 = ____	36 ÷ 9 = ____
10 ÷ 2 = ____	15 ÷ 5 = ____	32 ÷ 4 = ____	35 ÷ 7 = ____	48 ÷ 8 = ____
20 ÷ 4 = ____	30 ÷ 5 = ____	27 ÷ 3 = ____	27 ÷ 9 = ____	42 ÷ 7 = ____
14 ÷ 2 = ____	45 ÷ 5 = ____	24 ÷ 3 = ____	54 ÷ 6 = ____	64 ÷ 8 = ____
25 ÷ 5 = ____	12 ÷ 3 = ____	36 ÷ 4 = ____	36 ÷ 6 = ____	72 ÷ 8 = ____
18 ÷ 2 = ____	16 ÷ 4 = ____	30 ÷ 6 = ____	40 ÷ 8 = ____	63 ÷ 9 = ____
5 ÷ 5 = ____	24 ÷ 4 = ____	18 ÷ 6 = ____	56 ÷ 8 = ____	63 ÷ 7 = ____
16 ÷ 2 = ____	18 ÷ 3 = ____	28 ÷ 7 = ____	49 ÷ 7 = ____	81 ÷ 9 = ____
12 ÷ 6 = ____	9 ÷ 3 = ____	32 ÷ 8 = ____	42 ÷ 6 = ____	48 ÷ 6 = ____
35 ÷ 5 = ____	28 ÷ 4 = ____	21 ÷ 7 = ____	45 ÷ 9 = ____	54 ÷ 9 = ____

Division Fact Interview E - All Division Facts

Name_____

4 ÷ 2 = ____	40 ÷ 5 = ____	21 ÷ 3 = ____	56 ÷ 7 = ____	36 ÷ 9 = ____
10 ÷ 2 = ____	15 ÷ 5 = ____	32 ÷ 4 = ____	35 ÷ 7 = ____	48 ÷ 8 = ____
20 ÷ 4 = ____	30 ÷ 5 = ____	27 ÷ 3 = ____	27 ÷ 9 = ____	42 ÷ 7 = ____
14 ÷ 2 = ____	45 ÷ 5 = ____	24 ÷ 3 = ____	54 ÷ 6 = ____	64 ÷ 8 = ____
25 ÷ 5 = ____	12 ÷ 3 = ____	36 ÷ 4 = ____	36 ÷ 6 = ____	72 ÷ 8 = ____
18 ÷ 2 = ____	16 ÷ 4 = ____	30 ÷ 6 = ____	40 ÷ 8 = ____	63 ÷ 9 = ____
5 ÷ 5 = ____	24 ÷ 4 = ____	18 ÷ 6 = ____	56 ÷ 8 = ____	63 ÷ 7 = ____
16 ÷ 2 = ____	18 ÷ 3 = ____	28 ÷ 7 = ____	49 ÷ 7 = ____	81 ÷ 9 = ____
12 ÷ 6 = ____	9 ÷ 3 = ____	32 ÷ 8 = ____	42 ÷ 6 = ____	48 ÷ 6 = ____
35 ÷ 5 = ____	28 ÷ 4 = ____	21 ÷ 7 = ____	45 ÷ 9 = ____	54 ÷ 9 = ____

Coding: **rs** – repeatedly subtracted number, **ra** - repeatedly added number, **sf**-used fingers to track skip counting, **s** – skip counted, **dot** after quotient – used thinking time, **line** above quotient – incorrect

Notes:

Recorder _____Date _____ Score ____/50
Recorder _____Date _____ Score ____/50

Division Fact Interview E Data Fifth Grade Teacher _____ Date _____

Name	All Division Facts							
	Adds		Skip Counts		**Uses Number Relationship Strategies**			Recalls
	Adds a few, then counts on	Repeatedly adds	Uses fingers to track skip counting	Uses memorized sequence	Uses inverse multiplication fact	Works from a known division fact	Halves and then doubles	

Knows Number Sense Strategies for Division Facts (Level E)

Dividend of 0	**Dividend of 1**	**Dividend of 2**	**Dividend of 3**	**Dividend of 4**	**Dividend of 5**	**Dividend of 6**	**Dividend of 7**	**Dividend of 8**	**Dividend of 9**	**Dividend of 10**
1 ÷ 0	1 ÷ 1	2 ÷ 1	3 ÷ 1	4 ÷ 1	5 ÷ 1	6 ÷ 1	7 ÷ 1	8 ÷ 1	9 ÷ 1	10 ÷ 1
2 ÷ 0	2 ÷ 2	4 ÷ 2	6 ÷ 2	8 ÷ 2	10 ÷ 2	12 ÷ 2	14 ÷ 2	16 ÷ 2	18 ÷ 2	20 ÷ 2
3 ÷ 0	3 ÷ 3	6 ÷ 3	9 ÷ 3	12 ÷ 3	15 ÷ 3	18 ÷ 3	21 ÷ 3	24 ÷ 3	27 ÷ 3	30 ÷ 3
4 ÷ 0	4 ÷ 4	8 ÷ 4	12 ÷ 4	16 ÷ 4	20 ÷ 4	24 ÷ 4	28 ÷ 4	32 ÷ 4	36 ÷ 4	40 ÷ 4
5 ÷ 0	5 ÷ 5	10 ÷ 5	15 ÷ 5	20 ÷ 5	25 ÷ 5	30 ÷ 5	35 ÷ 5	40 ÷ 5	45 ÷ 5	50 ÷ 5
6 ÷ 0	6 ÷ 6	12 ÷ 6	18 ÷ 6	24 ÷ 6	30 ÷ 6	36 ÷ 6	42 ÷ 6	48 ÷ 6	54 ÷ 6	60 ÷ 6
7 ÷ 0	7 ÷ 7	14 ÷ 7	21 ÷ 7	28 ÷ 7	35 ÷ 7	42 ÷ 7	49 ÷ 7	56 ÷ 7	63 ÷ 7	70 ÷ 7
8 ÷ 0	8 ÷ 8	16 ÷ 8	24 ÷ 8	32 ÷ 8	40 ÷ 8	48 ÷ 8	56 ÷ 8	64 ÷ 8	72 ÷ 8	80 ÷ 8
9 ÷ 0	9 ÷ 9	18 ÷ 9	27 ÷ 9	36 ÷ 9	45 ÷ 9	54 ÷ 9	63 ÷ 9	72 ÷ 9	81 ÷ 9	90 ÷ 9
10 ÷ 0	10 ÷ 10	20 ÷ 10	30 ÷ 10	40 ÷ 10	50 ÷ 10	60 ÷ 10	70 ÷ 10	80 ÷ 10	90 ÷ 10	100 ÷ 10

Name _____ Date Started _____ Goal Met _____

Fact Computation Development **Student: _____** **Date: _____**

Addition Facts	**Uses counting strategies**	**Uses known number relationships**
Sums to 10		
Sums to 20		
Subtraction Facts		
Differences of 1, 2, 3		
Differences greater than 4		

Multiplication Facts	**Uses counting or repeated addition strategies**	**Uses known number relationships**
Multiples of 2, 3, 4, 5		
Multiples of 6, 7, 8, 9		
Division Facts		
Divisors of 2, 3, 4, 5		
Divisors of 6, 7, 8, 9		

Use Number Relationships to Help Compute Basic Facts (Van de Walle, J. A. (2001). *Elementary and middle school mathematics*. New York, NY: Longman.)

Big Ideas to Remember:

1. Number relationships can be used to help remember basic facts.
2. There are patterns and relationships in basic facts. You can figure out new or unknown facts from the ones you already know.
3. All the facts can be learned with the help of efficient strategies.

Steps to Take to Learn the Facts:

1. Develop a strong understanding of the operations and of number relationships.
2. Think of efficient strategies to help recall the facts.
3. Decide which strategies work best for particular numbers.
4. Practice using these strategies.

Addition

- Use a double. If you know 7 + 7, use it to help you know 8 + 7 and 6 + 7.
- Look for doubles that might be hiding. For instance, 5 + 7 can be thought of as 5 + 5 + 2.
- Make it easier to add 9. Change the 9 to 10 by adding 1 to it (in your head). Then add the other set. To finish, take away 1 from the sum (to undo that change you made when you added 1 to 9 to change it to 10).
- Here's another way to add 9 quickly. Look at 9 + 5. Think of a name for 5 that uses a 1 (1 + 4)! Then add 9 + 1 to make 10 and finish by adding 4 to the 10.
- Make it easier to add 8. Look at 8 + 6. Think of a name for 6 that uses a 2 (2 + 4). Then add 8 + 2 to make 10 and finish by adding 4 to the 10.

Subtraction

- Use what you know about addition. Look at 11 – 6 and think, "What can I add to 6 to make a total of 11?"
- Think about subtraction as comparing two numbers. Think of how much more one number is than the other. For 10 – 4, think about how much more you need to count to get to 10.
- Think of subtraction as the difference between two numbers. Put the two numbers on a "pretend" number line. Think about the difference between those numbers.
- If ten is in between the numbers, think back down through 10. For 15 – 8, think 15 – 5 is ten, then take off 3 more to get to 7.

Multiplication

- Practice skip counting by every single-digit number! Kids who follow football often know the x 7's without any effort!
- Look for ways to double. You know 2 x 8 = 16. 4 x 8 is the double of 2 x 8, so 4 x 8 will be 16 + 16.
- See the patterns. Multiples of 5 end in either a zero or a 5!
- Use the facts you know. If you know that 6 x 4 = 24, then you can reason that 7 x 4 is just one more group of 4.
- Think of the facts you know in another way. For 6 x 7, think about 3 x 7. 3 x 7 = 21 so 6 x 7 will be 21 + 21.
- Think 10! You know what 10 x 8 is. Use that to help you do 9 x 8 quickly. Think 10 x 8 and then subtract 8.

Division

- Use the multiplication facts. For 32 ÷ 8, think what times 8 makes 32.
- Get as close as you can. For 56 ÷ 7, think about what you know that's close, perhaps 49 ÷ 7. Then adjust by adding or taking away another group of 7. 49 ÷ 7 will be 7 so 56 ÷ 7 will be 8!
- Think halves! For 64 ÷ 8, think 32 ÷ 4. Then double it.

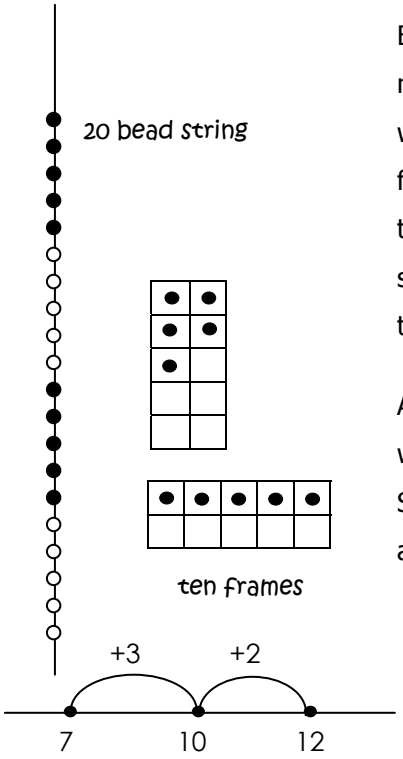
Try out these strategies!

Helping Students Develop Fact Fluency

Addition Facts

Most students in the intermediate grades have developed fluency with addition facts and readily access addition facts to solve problems. Intermediate students who are not yet fluent with addition facts will need deliberate focused attention on learning them.

Power tools to visualize number relationships:



Before working on developing fluency, students need lots of practice with numbers to 20. They need time to reflect on number relationships and discuss ways to use number relationships to compute. This practice will help them move from counting by ones to using part-part-whole relationships, and combinations to make 10. Number relationships are strengthened with the use of a 20-bead string, Rekenrek, ten frames, and empty number lines. These are powerful tools to help students visualize number relationships.

After many experiences using objects to support counting, students begin to use what they know about numbers to develop mental strategies for computing. Students develop these strategies in individual ways, but there does seem to be a general sequence of development shared by many students as follows:

- count on 1, 2, or 3 without using fingers or objects
- learn about the zero property (0 + 8 = 8)
- understand the commutative property (1 + 6 = 6 + 1)
- recall doubles (5 + 5 = 10)
- learn part-, part-, whole relationships to make 10
- use known doubles facts to reason about a computation close to a double (E.g. Because I know that 5 + 5 = 10, since 4 is one less than 5 then 5 + 4 is one less than 10. It must be 9.)
- understand the value of working with, to, and through 10
  - by decomposing one addend (For 8 + 6, think of 6 as 2 + 4. Then 8 + 2 is 10. Add the 4 to make 14.)
  - by compensating for a number close to 10 (For 9 + 7, think of 9 as 10, add the 7, and then correct the sum by taking away 1.)
- use known facts to reason about a computation (For 7 + 5, I know 7 + 3 is 10, 7 + 4 is 11, so 7 + 5 is 12.)

Rekenrek

Subtraction Facts

Subtraction is a challenging operation for students. They need plenty of time to construct physical models to support their development of the number relationships involved in "taking away." They also need to broaden and deepen their understanding of subtraction to see it as comparing to find the difference.

Ideas about comparing begin to develop when students learn to count on, usually in first grade.

Students' concepts about differences and their ability to determine differences tend to become more evident in second grade. Teachers facilitate learning about finding the difference between two numbers by asking students to solve comparison problems, analyze bar graphs, and by using the empty number line. The following sequence describes a typical development of subtraction concepts and skills:

- ☑ count back 1, 2, or 3 without using fingers
- ☑ use part-part-whole relationships (I know 8 is made of 5 and 3. If I take away 5, I know I have 3 left.)
- ☑ compare two numbers and count on to determine the difference
- ☑ understand the value of working with, to and through 10 by decomposing the subtrahend (For 14 – 6, first work to 10 by taking away 4, then take away 2 to make 8.)
- ☑ use known facts to reason about a computation (For 15 – 7, think I know 14 – 7 is 7 so 15 – 7 must be 8.)

Multiplication Facts

Students begin modeling multiplication story problems in kindergarten. They construct physical models to hold the sets in order to count them by ones. As students learn counting sequences such as counting by 2's or by 5's, they start to use those counting sequences to think about the total of two or three groups. Primary students work on multiplication by building fluency with their *count by* strategies.

Intermediate students learn to relate one set of facts to another. (e.g. 8x5 is the same as two times 4x5). Typically, students learn multiplication facts for 2s and 5s first and then become fluent with other multiplication facts through repeated practice using a known fact to compute an unknown fact. Eventually, students recall most multiplication facts and use derived fact strategies for just a few.

The following concepts and skills are needed for fluency with multiplication facts:

- ☑ count by single-digit numbers efficiently (see Count By assessment in this chapter)
- ☑ relate addition doubles to x2s
- ☑ derive what is not known from what is known, efficiently. (e.g. three groups of six can be thought of as two groups of six plus one more six, 3x6=2x6+6)
- ☑ recall facts

Division Facts

Students learn that division "reverses" or "un-does" multiplication through many experiences with multiplication and division story problems. Students will often solve division problems by building up groups of the divisor to reach the dividend and intuitively understand the inverse operations. For these reasons, fluency with division facts is dependent on understanding the inverse relationship with multiplication and knowing multiplication facts.

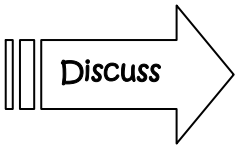
The following concepts and skills needed for fluency with division facts:

- ☑ mastery of multiplication facts
- ☑ reverse thinking (e.g. 36 ÷ 9 as , "nine times what is thirty-six")
- ☑ practice with "near facts" such as 50 ÷ 6 to develop fluency

Factors and Multiples

Knowing factors of numbers 1-100 and multiples of all single-digit numbers extends and deepens knowledge of multiplication and division facts. Students should have repeated experiences exploring number patterns for factors and multiples (e.g. List the first ten multiples of 5. What do you notice about the numbers?) Alternatively, provide a list of numbers for students to analyze with question such as: Which of these numbers does not have 8 as a factor? How do you know? Circle all of the multiples of 9. How did you decide? Provide plenty of class discussion about patterns and student strategies.

Developing Mental Computation Strategies



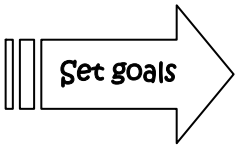
Begin working on developing mental computation strategies by asking students to share ideas about number relationships. Facilitate discussions that ask students to reflect on how they use number relationships to solve problems. Ideas they might share include:

- ☑ Plus 1 just means going to the next number when you count.
- ☑ Minus 1 just means going back one number.
- ☑ You don't have to count when you add to 10. It is just that many past ten.
- ☑ When I add 6 + 5, I think about 5 + 5 and just add one more because 6 is one more than 5.
- ☑ It is easy to add 9. You just pretend that it is 10, add the rest, and then take one away, 'cuz you added 1 to make the 10.
- ☑ To add 9, just take 1 from the other number. That makes 10. Then just add the rest.
- ☑ I know that 7 + 4 is 11 so 7 + 5 has to be one more than 11.
- ☑ To subtract 8 from 14, I first take away 4 to get to 10 and then take away 4 more, to get to 6.
- ☑ I know that 2 times 6 is 12, so to get 4 times six, I just double 12, 'cuz 4 is the double of 2.

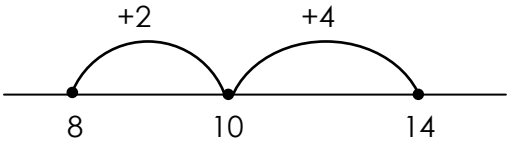
Write students' ideas for using number relationships on a chart posted in the classroom. Add to it as students think of other ideas. Expect students to explain their strategies.

Allow students to work without time pressure until their number sense is very well developed. If students feel time pressure, they often revert to counting on or using fingers to track counting.

When students use well developed number sense strategies, teachers:



- ☑ help them identify the facts that they know and those that they still use counting strategies to compute
- ☑ ask students to set personal goals to find number sense strategies for unknown facts
- ☑ have students make fact cards for their unknown facts
  - keep the number of cards small and focused, 8 – 10 cards at a time
  - write the numbers horizontally, 6 + 8 = ____
  - create an empty number line representation on the back of each card



- ☑ provide short periods of instruction to develop the number relationships and strategies for those facts

Developing Fluency with Fact Computations

"Fortunately, we know quite a bit about helping students develop fact mastery, and it has little to do with quantity of drill or drill techniques."

-John A. Van de Walle

John A. Van de Walle, in *Teaching Student-Centered Mathematics* (page 94) states, "Fortunately, we know quite a bit about helping students develop fact mastery, and it has little to do with quantity of drill or drill techniques. Van de Walle identifies three components or steps to develop fluency with fact computation. They are:

1. Help students develop a strong understanding of the operations and of number relationships.
2. Develop efficient strategies for fact retrieval.
3. Then provide practice in the use and selection of those strategies."

After students have developed strategies for using number relationships to compute basic facts, they are ready to work on developing fluency and automaticity. Card games, computer games, and worksheets using numbers that are at the student's independent mental calculation level (as determined by the Fact Interviews) are all ways to foster fluency. Short periods of practice sustained over weeks and months are effective in building proficiency in basic fact computations. If games have a competitive element, create groups of students who have similar fact proficiencies.

Fact fluency progress should be personal information. Instruction to support individual needs is more effective than posting students' progress on class charts for comparison between classmates.

The following table provides grade-level expectations for fact fluency.

Fact Fluency Targets K—5

Grade	**Number Sense Development** (Based on MMSD standards) **Bold face type** indicates a new expectation at this grade level.
K	Not Applicable (See following grades if student demonstrates number sense at those levels)
1	**Demonstrate fluency with addition facts for:** • **'within' ten facts (sums less than 10)** • **combinations to make 10** Available Assessment: *Fact Interview A* — Sums Within and To 10 *(see Learning Math in the Primary Grades)*
2	Demonstrate fluency with **all** addition facts: • **doubles** • **doubles ± 1** • **'across ten' facts (sums greater than ten)** • 'within' ten facts (sums less than 10) • combinations to make 10 Available Assessment: Fact Interview B — Sums to 20 Within, To and Across 10
3	**Demonstrate fluency using part-whole relationships, comparison, the concept of difference, or recall to determine the results of subtraction for facts.** Available Assessment: *Fact Interview C* — Differences Less Than, Equal To, and Greater Than 3 **Demonstrate fluency with multiplication facts for (2, 5, 4, and 3 as multiplier or multiplicand)** Available Assessment: *Fact Interview C* — 2, 5, 4, and 3 as Multiplier or Multiplicand
4	**Demonstrate fluency with all multiplication facts (**2, 5, 4, 3, **9, 6, 7 8 as multiplier or multiplicand)** Available Assessment: *Fact Interview D* — All Multiplication Facts **Use the inverse relationship to determine the results of a division using multiplication facts.**
5	**Know all division facts and the first ten multiples of 2, 3, 4, 5, 6, 7, 8, 9, 10, and 25.** Available Assessment: *Fact Interview E* — All Division Facts Use the inverse relationship to determine the results of a division using multiplication facts.

CGI Problem Type Assessment

The following problems can be used to assess each student's ability to solve seven basic story problem types. The numbers are at a level that intermediate students should be able to solve mentally and in more than one way.

However, for the purposes of this assessment only one solution strategy is required. If the student uses a standard algorithm to solve the problem you may want to ask the student to show a second strategy in order to assess the student's number sense.

This assessment form asks students to write a sentence that answers the question in the problem rather than an equation. This supports students in reflecting on their solution as it connects to the original problem. The assessment does not ask students to write an equation that "goes with" the problem as there are many equations that can be used to solve most problems.

When students can solve these basic problems readily (using number sense) try similar problems with different numbers (see Chapter 6: Number Domains) Some students may find the meaning of the problem situation becomes difficult with unfamiliar numbers which is important assessment information.

Teachers may also use the basic problem types to assess student knowledge about new number domains such as fractions and decimals.

To assess students who have difficulty solving the basic story problem types, use the Story Problem Interviews in the *MMSD Learning Math in the Primary Grades* binder.

See Chapter 6: Problem Types for more information on problem types.

Name_____ Date_____

1. Marina has 43 marbles. How many more marbles does she need to have 65 in all?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

2. Connie had 62 coins. She gave some to her friend, Marko. Now she has 48. How many coins did she give to Marko?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

Name_____ Date_____

3. Martin has 57 stamps. 29 are from Mexico. The rest are from Canada. How many are from Canada?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

4. Akamu has 86 shells. Lani has 49 shells. How many more shells does Akamu have than Lani?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

Name_____ Date_____

5. Nakato has 6 bags of bananas to sell at the market. Each bag has 15 bananas. How many bananas does Nakato have?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

6. Mbali has 45 carrots to put in bags for the market. He wants to put 9 carrots into each bag. How many bags will he need for all of his carrots?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

Name_____ Date_____

7. Paki has 36 potatoes to sell at the market. He has 4 bags. He wants to put the same number of potatoes into each bag. How many potatoes will he put into each bag?

   Show one way to solve the problem.

_____

Write a sentence that answers the question in this math story problem.

CGI Problem Type Assessment Data Teacher _____ Date _____

Basic Problem Type Assessment							
Student Names	Join Change Unknown	Separate Change Unknown	Part-Part-Whole Part Unknown	Compare Difference Unknown	Multiplication	Measurement Division	Partitive Division

Record the strategy used for each problem. See Chapter 6: Strategies

Assessing Place-Value Understanding

Assessment of place-value knowledge requires much more than simply asking what the place-value names mean or what number is in a given place. Place-value knowledge includes knowing that a single unit (digit) can have multiple meanings or different values. The 3 in 1,367 has the value of 3 hundreds (3x100) because of its place but can also mean 30 tens or 300 ones while in that same place.

Most importantly, understanding place-value requires multiplicative reasoning rather than additive reasoning. Many students think of multiplication as repeated addition. They see 3x5 and think 5+5+5. However multiplication is different than addition because it involves hierarchical thinking. Research suggests students who recognize that the "4" in 4x5 refers to "4 fives" and that the "5" is one group of five rather than 5 ones can "think multiplicatively." The "5" is a "higher order unit" and has a different meaning than the "4" (Kamii, 2003).

There are many components to place-value understanding. Place-value knowledge is complex and takes years to develop. Children may be inconsistent for a long time and it takes a careful observer to notice changes in a student's place-value knowledge. Teachers assess place-value knowledge informally during Problem Solving, Inspecting Equations, and during Number Work.

For students who struggle with computations for numbers greater than 10:

- ☑ assess fact fluency using the Fact Interviews in this chapter (mental computations for single-digits)
- ☑ use the Kamii place-value task in the Quick Tasks section of this chapter

The following table outlines components of place-value knowledge, what students need to know and be able to do, assessment questions, and instructional implications. The instructional implications include suggestions for activities or the number range to develop place-value knowledge.

Components of Place-value Understanding

Place-value Concept or Skill:	Students with this understanding:	Assessment questions	Instructional implications
Verbal counting sequence	count by 10s, 100s, 1,000 from any number.	Where in the counting sequence does the student get confused? Can the student cross a decade, hundred, or thousand easily?	Use numbers that provide opportunities for students to: ◆ count on 10s starting at 10, 100, 1,000, or starting at any decade. ◆ count on 10s, 100s, 1,000s starting at any decade, hundred or thousand, or starting at any number.
Build/Count Mixed Quantities	group objects in 10s, 100s, 1,000s and change units when counting groups containing 100s, 10s, 1s.	Does the student change units as they count? (e.g. three 10-sticks and 2 ones are counted as: *10, 20, 30, 40, 50…instead of, 10, 20, 30, 31, 32…)* Child can easily talk about 100 flat as 1 hundred, 10 tens, and 100 ones?	Provide opportunities for students to count base-ten blocks that include 1s, 10s, 100s with more than ten of at least one value. (e.g. 3-100s, 15-10s, 23-1s) Try the Count and Compare activity in Chapter 7: Number Work.
Read & write numbers	read and write numbers correctly.	Which kinds of numbers seem the most challenging? Does the student know about "hidden zeros? (e.g. one hundred and fifteen is written 115 not 10015). Does the student know the value and place-value name for each place?	Pair reading and writing with use of base-ten materials to model quantities. Place-value arrow cards can help students see the "hidden zeros." Try the Nickname, Real name activity in Chapter 7: Number Work. Provide opportunities to discuss and compare quantities. (e.g. 51 is more than 15 because the left most place is key).
Sequence Numbers	strategically locate numbers on a 100 grid. can use an empty number line to solve problems. can explain how numbers relate to ten.	Does the student see a pattern on the hundreds chart and use it or randomly search? Can the student use an empty number line to show how they solved a problem?	Provide a variety of hundreds charts (see Appendix) to support flexibility in number order and seeing relationships. Sequence numbers from smallest to largest. Use numbers that may challenge students to think about place value (e.g. 7068, 6807, 6078, 7608, 76.08, 7.60, 7.06).

Components of Place-value Understanding (cont.)

Place-value Concept or Skill:	Students with this understanding:	Assessment questions	Instructional implications
Understanding value and quantity	explain the different meanings of a digit within a number explain the possible meanings of the numbers "4" and "5" in 4 x 5.	Can the student maintain place-value concepts while explaining solutions to problems?	Ask students to name the value and the quantity (units) when explaining solutions to problems or when using algorithms. For example: The 3 in 1,345 is 3 hundreds. The expression 4 x 5 can be thought of as 4 groups of 5 or 5 groups of 4.
Represent number sequences on an empty number line and use arrow language	mentally increase and decrease numbers by 10s, 100s, 1,000s, etc. from any number represent solutions to problems on an empty number line or use arrow language to represent their thinking when appropriate.	Which numbers are most difficult? Can the student easily cross a decade, hundred, or thousand and continue an accurate count?	Use numbers that provide opportunities for students to: ◆ increase or decrease by 10s starting at 10, 100, 1,000 then starting at any decade. ◆ increase or decrease by any decade, hundreds, starting at any decade, hundred or thousand then starting at any number.
Tens & tens within tens	readily compose and decompose ten. compare a non-decade number to the decade immediately above and below it (e.g. 164 is 6 away from 170 or 4 more than 160).	Does the student know all sums to make ten? Can the student easily decompose any number under ten mentally? Can the student mentally add/subtract a single-digit number to/from any decade? (e.g. 30 + 8, 56-6)	Use numbers that provide opportunities for students to: ◆ make a ten within a ten (63+7) ◆ compare a number to a decade (356 compared to 360) Use number games to build mental math skills and fluency.
Group decades, hundreds, and thousands	mentally group and regroup 10s, 100s, 1000s.	Which numbers are most challenging to mentally compose and decompose?	Use multiplication and measurement division problems with groups of 10. Ask questions about groups of 10s or 100s (e.g. *What number is 7 hundreds? How many is 12 tens? How many 10s would be needed to make 342? What is 5 hundreds, 25 tens and 2 ones?*)

Components of Place-value Understanding (cont.)

Place-value Concept or Skill:	Students with this understanding:	Assessment questions	Instructional implications
Decompose and compose numbers	decompose and compose numbers in many ways.	Does the student always use the same method to add and subtract or consider number relationships before choosing a strategy? What number sizes can the student work with mentally or model with base-ten blocks or use written representations?	Use numbers that provide many opportunities for students to: ◆ write equations horizontally ◆ work with numbers near a decade (78, 89), hundred (197, 203), or thousand (996, 1985) ◆ practice finding the difference between two numbers ◆ decompose a number into a decades (120 into 70 and 50)
Multiplicative relationships **(Magnitude)**	know that that each place is 10 times the value of the place to the right and that each place is 1/10 value of the place to the left. know how the magnitude of a number changes with a given operation and number. know the relative size (10s, 100s, 1000s) of a number and predict the result of a computation.	What does the magnitude of the student's estimate indicate about the student's number sense and each operation? What conceptions does the student's estimate indicate about how numbers change through the operations? Can the student explain how each place is 10 times the value of the place to its right and that each place is 1/10 value of the place to its left? Does the student use doubling or multiplying by ten when using a ratio table?	Make estimation a routine before computation. Use all problem types and operations with numbers ending in one or more zeros and decimals for older students. Provide many opportunities for students to compare additive and multiplicative number relationships through Problem Solving and Number Work. Use, "How many times bigger (or smaller)?" when comparing numbers. Teach students how to use a ratio table for grouping and partitioning problems. See pg._____ for practice ideas. Study a series of problems that include working with numbers ending in one or more zeros. Use inspecting equations activities.

Fraction Concepts

1. **Naming fractions** problems involve naming parts of wholes or portions of a group of objects.

2. **Sharing/partitioning** problems involve equally sharing or partitioning quantities resulting in fractional parts.

3. **Fractions as operator** problems involve using fractions to describe a part of a whole or group of objects.

4. **Ratio** problems involve the relationships between whole numbers expressed as fractions.

5. **Equivalence** problems involve equivalent fractions.

6. **Operations** problems involve adding, subtracting, multiplying, or dividing fractions using student-derived methods.

Each concept category has **several levels** of difficulty depending upon the fraction knowledge necessary to understand the situation or solve the problem.

Use these paper & pencil assessments with individual students, in small groups, or whole class settings.

Ask students to explain their thinking processes in writing (using words, drawings, or symbols) as if they are explaining to someone who doesn't know much about fractions.

Be sure to "interview" students who have incomplete or unclear rationale or diagrams. Proficient students should (minimally) be able to explain their reasoning orally using mathematical language.

Purposes of Fraction Concept Assessments:

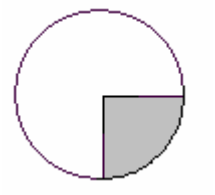
- ☑ document the ways a student works with fractions
- ☑ determine the kinds of models a student creates
- ☑ monitor a student's progress over time
- ☑ communicate a student's progress to family and future teachers
- ☑ illuminate an individual student's conceptual knowledge

Name_____ Date_____

The drawings below represent three kinds of pizza. Each pizza has a different shape.

A very hungry Cameron ate the shaded part of each pizza. Give a fraction name to the pieces she ate and explain how you named it.

Cheese Pizza

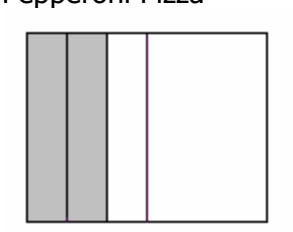


Cameron ate _____ of the Cheese Pizza.
Explain your thinking.

Sausage Pizza

Cameron ate _____ of the Sausage Pizza.
Explain your thinking.

Pepperoni Pizza



Cameron ate _____ of the Pepperoni Pizza.
Explain your thinking.

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

1. Twelve (12) people equally share three (3) sub sandwiches. How much does each person get?_____

2. Eight (8) people equally share three (3) sub sandwiches. How much does each person get?_____

3. Nine (9) people equally share twelve (12) sub sandwiches. How much does one person get?_____

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

Damon has 360 coins in his collection.

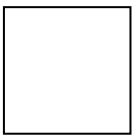
1. $\frac{1}{4}$ of the coins are pennies. How many coins are pennies? _____

2. $\frac{3}{10}$ of the coins are from Canada. How many coins are from Canada? _____

3. $\frac{5}{6}$ of the coins are older than Damon. How many coins are older than Damon?_____

A local market sells cheese and meat for sandwiches. Add the missing part to each drawing.

A. The rectangle below represents $\frac{1}{3}$ of a block of cheese. How big was the whole block of cheese?



B. The rectangle below represents $\frac{5}{6}$ of a block of cheese. How big was the whole block of cheese?

C. The rectangle below represents $\frac{2}{5}$ of a block of cheese. How big was the whole block of cheese?

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

A. You are at a party. You could either sit at a table where 4 friends equally share a small cake or you could sit at a table where 5 friends equally share a small cake. At which table would you get more cake? _____

B. Abena walks $\frac{1}{4}$ of a mile to school. Bevin walks $\frac{3}{8}$ of a mile to school.
Who has the longest walk? _____

C. Jemiah lives $\frac{3}{4}$ of a mile from the swimming pool. Hikari lives $\frac{2}{3}$ of a mile from the swimming pool.
Who lives the closest to the pool? _____

D. Lara walks $\frac{2}{5}$ of a mile to the library. Johan walks $\frac{7}{10}$ of a mile to the library.
Who has the shortest walk to the library? _____

E. Adrian ran $\frac{4}{7}$ in gym class. Berta ran $\frac{5}{8}$ of a mile in a gym class.
Who ran the farthest? _____

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

A. Breezy used $\frac{1}{2}$ of a bottle of paint for an art project. Bao used $\frac{1}{4}$ of a bottle of paint for his project. How much did Breezy and Bao use altogether? _____

B. Hakim bought $\frac{5}{6}$ of a yard of material. He used $\frac{2}{3}$ of a yard to make a flag. How many yards does he have left? _____

C. Choua decided to visit his grandmother who lives 10-kilometers from his house. He ran $2\frac{5}{8}$ kilometers and then walked the rest of the way. How many miles did Choua walk? _____

D. Olivia mixed $\frac{3}{5}$ of a bottle of red paint with $\frac{7}{8}$ of a bottle of blue paint. How much paint did she use in all?_____

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

Mrs. Long needs to order supplies for next year's art classes.

A. Each child in a class of 21 needs $\frac{1}{3}$ of a ball of yarn.
How much yarn will Mrs. Long need for the whole class? _____

B. Each child in a class of 17 will use $\frac{1}{5}$ of a bag of beads.
How much will Mrs. Long need for the whole class? _____

C. Each child in a class of 18 will need $\frac{3}{4}$ of a bar of clay.
How much will Mrs. Long need for the whole class? _____

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

1. Ariana has 8 yards of material. He wants to make kites to sell. One kite takes $\frac{3}{4}$ of a yard of material. How many kites can he make? _____ How much material is left over? _____

2. Bryene has $5\frac{5}{6}$ yards of string. She wants to make string games. Each game takes $\frac{1}{3}$ of a yard. How many string games can she make? _____ How much string is leftover? _____

3. Carlos has 12 yards of ribbon. He wants to make bows. One bow takes $1\frac{2}{3}$ of yards of ribbon. How many bows can he make? _____ How much ribbon is leftover? _____

4. Mom makes apple tarts. She uses $\frac{3}{4}$ of an apple for each tart. If mom has 20 apples, how many apple tarts can she make? _____ How much apple is leftover? _____

Name_____ Date_____

Use drawings, words, or symbols to explain your thinking.

The students from St. Clair Elementary School are going on a field trip to the art museum. There are 3 chaperones needed for every 24 students.

1. How many chaperones are needed for a group of 36 students? _____

2. If there are 5 chaperones available, how students could go on the trip? _____

3. If there are 18 students, how many chaperones are needed? _____

Name_____ Date_____

Use pictures and words to show how you solve the problems. Label your answers carefully.

A recipe calls for 10 teaspoons of lemonade mix to make 32 ounces of lemonade.

1. How many teaspoons of mix would you need to make 64 ounces of lemonade? _____

2. How many teaspoons of mix would you need to make 16 ounces of lemonade? _____

3. How many teaspoons of mix would you need to make 48 ounces of lemonade? _____

Quick Tasks

Use these assessments for:

> ***Base-Ten Concepts*** – These tasks address understanding base-ten concepts including number order and operations
>
> ***Fractions Concepts*** – These tasks address representation, relative size of fractions, and equivalence without a story context.

These tasks can be used as interviews to assess individual students or as pencil/paper assessments in small group or whole class settings.

Each task has a range of suggested numbers to get started. Use only a few at a time as needed.

Ask students to explain their thinking processes. Some tasks work best as "interviews."

Be sure to ask students who have unclear rationale or diagrams to explain further orally. Proficient students should (minimally) be able to explain their reasoning orally and in written form using drawings, words, or symbols.

Purposes of quick tasks:

- ☑ illuminate the base ten or fraction concepts that a student understands
- ☑ document how a student engages with these concepts
- ☑ determine the kinds of models or explanations that a student creates
- ☑ monitor a student's progress over time
- ☑ communicate a student's progress to family and future teachers

Early Base-Ten Assessment
Adapted from Kamii, C. Young Children Reinvent Arithmetic: Implications of Piaget's Theory, 1985

This assessment can provide insight into beginning base-ten concepts for students who struggle with multi-digit computations.

1. Ask the student to draw twenty-five tally marks.
2. Ask the student to write, "twenty-five with numbers" on the same sheet, to show that there are 25 tally marks.
3. As you circle the "5" in 25, ask the student to draw a circle that shows "that part" of the tally marks.
4. As you circle the "2" in 25 ask the student to draw a circle that shows "that part" of the tally marks.

If the student draws a circle around 2 tally marks, ask why they didn't write a number for the left over tally marks. Then circle the 25 and ask the student to explain the whole number in relation to the drawing.

Some possible responses follow. The last response indicates the level of knowledge about representation needed to understand base ten.

Level	Response
1	Child thinks that "25" stands for the whole quantity, but that the individual digits have no numerical meaning.
2	Child thinks that "25" stands for the whole quantity, but invents numerical meanings for the individual digits. For example, the child thinks the "5" means groups of 5 and the "2" means groups of 2.
3	Child thinks that the "25" stands for the whole quantity and that the individual digits have meanings related to groups of tens or ones but has only partial or confused idea of how the system works. The sum of the parts need not equal the whole. For example, the child thinks that both individual digits mean ones, or that "5" stands for tens and "2" stands for ones.
4	The child thinks that "25" stands for the whole quantity, that the "5" stands for ones, that the "2" stands for tens, and that the whole must equal the sum of the parts.

Base-Ten Number Order Interview
Adapted from S, Griffin, Number Worlds

This assessment provides insight into a student's knowledge of number order and difference.
Ask the student what they thought or did to answer each question. (For example: "I thought about the tens and knew the next number would be in the 800s.") Stop the interview when the student uses counting by one strategies or cannot easily answer the question. Numbers can be adjusted up or down to learn more about the child's independent or instructional level.

1. What number comes 4 numbers before 60?_____
2. What number comes 10 numbers after 99?_____
3. What number comes 9 numbers after 999?_____
4. What is 10 more than 3794?_____
5. What is 100 less than 2037?_____
6. What is 301–7?_____
7. What is 36–18?_____
8. What is the bigger difference, the difference between 99 and 92 or the difference between 25 and 11?_____
9. What is the bigger difference, the difference between 48 and 36 or the difference between 84 and 73?_____
10. Which is closer to 1; -0.2 or 1.8?_____
11. Which is closer to 1; -1.4 or 3.7?_____
12. Is 148.26 closer to 150 or 149?_____
13. How much is 126 divided by 6? _____
14. How much is 248 divided by 4? _____

Base-Ten Related Computations Interview
From International Perspectives on Learning and Teaching

This assessment provides insight about how students relate one computation to another.

Write the first equation 16 +27 = 43, then ask the student or a small group of students to explain how they would use it to solve each of the following equations. Record student answers and reasoning.

For example, a student thinking about 16+26 might say, "The answer would be 42 because 26 is one less than 27 and they both have 16." This series of questions could also be given in a written format with follow-up to clarify unclear responses.

"How could you use 16 + 27 = 43 to help you think about the solution to the following"

16 + 26 =_____, _____

27 + 16 =_____, _____

160 + 270 = _____, _____

15 + 27 + 15 = _____, _____

43 – 16 = _____, _____

16 + 16 + 27 + 27 = _____, _____

17 + 26 = _____, _____

Count-by Interview

This assessment provides insight about number order when skip counting.

Ask the student to tell which numbers are easiest to count by. Record the count for those numbers first. Then ask the student to tell you which one they would like to try next. When a student takes more than a few seconds between numbers, ask the student tell what strategy they use to get to the next number. If the student counts by ones, stop and try another number. If, the student uses number sense strategies, record the strategy and continue.

Draw a small arch (between two numbers) to indicate when a student slows down. Ask the child to tell you how they figured out the next number. Record the strategy. Write the date above the last number in the count. For example:

5/17

7 *7 14 21 28 35 42 49 56 counts by 1s to cross decades*

Student _____ Interviewer_____

2 _____

3 _____

4 _____

5 _____

6 _____

7 _____

8 _____

9 _____

Base-Ten Estimation Interview

This assessment provides insight about how students estimate computations.

Write or show the equation, then ask the student if the expression is in the tens (10-99), hundreds (100-999), or thousands (1000-9999) without computing. For example, a student thinking about 724 + 302 might say, "Thousands, because 700 and 300 is 1,000 and the rest of the tens and one make it more than 1,000."

Record the answer and student's reasoning. Try:

724 + 302 _____, _____

36+ 54 _____, _____

243 + 679 _____, _____

134 + 979 _____, _____

249 + 457 + 391 _____, _____

301 – 198 _____, _____

3027 – 283 _____, _____

11 × 256 _____, _____

638 × 5 _____, _____

2415/10 _____, _____

278/10 _____, _____

47,609/100 _____, _____

Fraction Paper Folding Assessment

This assessment provides insight about how students visualize fractional parts in a two-dimensional paper model.

Give each student several sheets of 8½" x 11" paper. Review folding techniques for halves, thirds, fourths, and fifths. Then give a series of folding directions such as the following.

- Fold your paper in half, then in thirds. How many equal parts will your paper have when you open it up? Write the fractional name of the parts on the outside of the folded paper.
- Fold your paper in fifths, then in thirds, then in half. How many equal parts will your paper have when you open it up? Write the fractional name of the parts on the outside of the folded paper.

Try any combination of folds. Notice who can mentally conceptualize how the folds change the number of parts that will result after folding and what fractional name to give each part.

Fraction Size/Order Assessment

This assessment provides insight about student's knowledge of fraction notation, relative size of fractions, and relationships to benchmarks of 1 & ½ without using a model or common denominators. Write or show the fraction pair and ask the student to tell or write which is larger and explain why. Use some or all as needed.

Try these. Which number in each pair is larger?

1 $\frac{4}{3}$ _____

$\frac{3}{10}$ $\frac{1}{2}$ _____

$\frac{4}{7}$ $\frac{5}{7}$ _____

$\frac{1}{6}$ $\frac{1}{3}$ _____

$\frac{5}{8}$ $\frac{5}{3}$ _____

$\frac{3}{4}$ $\frac{4}{5}$ _____

$\frac{3}{5}$ $\frac{5}{8}$ _____

$\frac{3}{7}$ $\frac{5}{8}$ _____

$\frac{9}{8}$ $\frac{4}{3}$ _____

$\frac{4}{6}$ $\frac{7}{12}$ _____

$\frac{8}{9}$ $\frac{9}{8}$ _____

$\frac{5}{8}$ $\frac{6}{10}$ _____

Analysis

Look for explanations that use 0, ½, or 1 as convenient anchors or benchmarks that include awareness of the following concepts:

- More of the same-size parts such as $\frac{4}{7}$ vs. $\frac{5}{7}$
- Same number of parts but different sizes such as $\frac{5}{8}$ vs. $\frac{5}{3}$
- More or less than one-half or one whole such as $\frac{3}{7}$ vs. $\frac{5}{8}$ or $\frac{8}{9}$ vs. $\frac{9}{8}$
- Distance from one-half or one whole $\frac{4}{6}$ vs. $\frac{7}{12}$ or $\frac{9}{8}$ vs. $\frac{4}{3}$

Fraction Estimation Assessment

This assessment provides insight about what students know about computation with fractions without using a model or common denominators.

Which of the following expressions is less than one? Explain your thinking.

$\frac{5}{10} + \frac{3}{6}$ _____

$\frac{3}{8} + \frac{5}{10}$ _____

$\frac{1}{2} + \frac{3}{5}$ _____

$\frac{5}{8} + \frac{5}{6}$ _____

These expressions are close to a whole number. What number is that? Explain your thinking.

$3\frac{1}{8} + 2\frac{5}{6}$ _____, _____

$\frac{9}{10} + 2\frac{7}{8}$ _____, _____

$6\frac{1}{2} - 2\frac{1}{3}$ _____, _____

$3\frac{1}{2} - \frac{9}{10}$ _____, _____

$\frac{12}{13} + \frac{7}{9}$ _____, _____

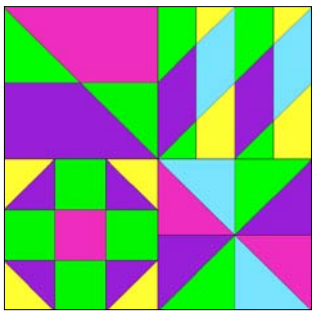
For More Information:

Kamii, C. (2004). *Young Students Continue to reinvent arithmetic* (2nd ed.). New York: Teachers College Press.

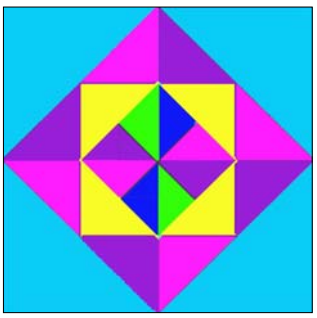
Stenmark, J. K. & Bush, W. S. (2001). *Mathematics assessment: A practical handbook K-2*. Reston, VA: National Council of Teachers of Mathematics.

Sutton, J. & Krueger, A. (2002). *EDThoughts: What we know about mathematics teaching and learning*. Aurora, CO: Mid-continent Research for Education and Learning.

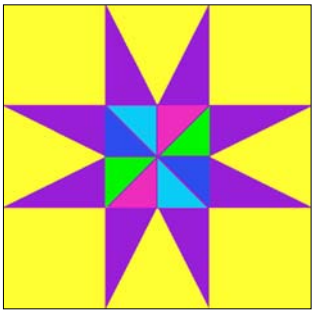
Van de Walle, J. A. & Lovin, L. H. (2006). *Teaching student-centered mathematics: Grades 3-5*. Boston: Allyn and Bacon Pearson Education.



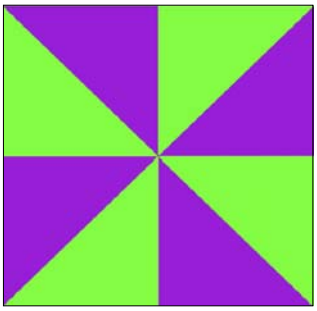
PROBLEM SOLVING



NUMBER WORK



INSPECTING EQUATIONS



FLUENCY & MAINTENANCE

CHAPTER 5

Organizing for Instruction

Organizing for Instruction

Teachers can provide needed opportunity for students to be proficient in mathematics by planning a sequence of instruction that includes all strands of mathematics as required by the MMSD Elementary Math Standards.

Planning instruction ensures a systematic approach that supports a cohesive and connected curriculum and allows students to see connections between number, geometry, measurement, and data.

For this document, we base this instruction on the following diagram.

Instruction	**Problem Solving** (Differentiated small group instruction followed by independent work or whole group investigations) **30-45 minutes/day**	**Number Work** (Whole Class) *or* **Inspecting Equations** (Whole class or small group) **10-15 minutes/day**
Practice	**Fluency & Maintenance** (Differentiated individual practice) **10-15 minutes/day (may be assigned as homework)**	

The four-part math block keeps problem solving at the forefront of instruction. Students learn and practice number concepts throughout the school year during Number Work and Inspecting Equations. Fluency & Maintenance reminds students of past learning and provides practice with new concepts. Teachers can organize the block in any sequence that best meets student needs and the class schedule.

Organizing for the School Year

The MMSD standards provide the basis for planning curriculum across the school year. The standards are linked to the Wisconsin State Assessment Frameworks and align with NCTM 2000 Principles & Standards for School Mathematics.

Organizing for the school year involves:

- ☑ knowing the grade-level MMSD standards (see Chapter 6: Problem solving)
- ☑ choosing or creating materials that ensure students meet the standards
- ☑ recognizing and using the interrelationships between topics as part of a teaching plan (e.g. fractions can be studied in number, measurement, and geometry)
- ☑ assessing student learning before, during, and after studying a concept
- ☑ anticipating curriculum that addresses standards in a timely way

There are no recommendations about the order of topics for study. However, those topics that require more number-knowledge (such as "range" or "mean" in the data strand) may require placement later in the school year. Those topics that take a long time to grasp (such as fractions and ratios) will require yearlong attention.

Yearlong planning requires frequent evaluation of progress and alignment of instruction as it relates to district and state goals for student learning. While many students will have just reached proficient benchmarks, it is important that teachers address individual students learning needs. Some students will need adjustments in the content beyond the standards while others will need more study to meet the standards.

Planning for Daily Instruction

Teachers in the intermediate grades plan for *everything* that impacts instruction. They consider such things as student interests, room arrangements and daily schedules as they make plans for each day's mathematics instruction. Additionally, teachers plan for the year in advance to be sure to address all topics based on the standards and assessments.

Three critical things to plan for are:

1. *goals and activities most effective for learning the mathematical content*
2. *each student's needs based on assessments*
3. *addressing all standards*

Effective daily instruction depends on planning for materials, groupings, and allocation of time. When planning, teachers consider the following:

- ☑ **manipulatives** and other materials needed (e.g. journals, white boards & dry erase markers, math reference guides) and ways to organize the materials so students can access them
- ☑ **organization** of the kinds and sizes of groups and the ways students will move from group to group
- ☑ **time allocations** for focusing on *Problem Solving, Number Work, Inspecting Equations and Fluency & Maintenance*

Manipulatives for Instruction in the Intermediate Grades

Manipulatives give access and ownership to all students for developing models that support their mathematical explanations. Teachers rely on ready access to manipulatives to help students make their thinking transparent especially when language skills are emerging. Having manipulatives on hand is essential for building students' knowledge throughout intermediate math instruction. Students use manipulatives to:

- model numbers (whole numbers, fractions, and decimals)
- talk about and justify solutions
- explore geometry, measurement, and number relationships

Manipulatives used on a daily basis must be available in each classroom. Other materials that students use occasionally, such as balances, geometric shape sets, or class sets of measuring tools, can be stored in a central place to share as classrooms need them. Consider creating a student math tool kit. The following recommendations for intermediate grade manipulatives are based on a class size of 30 students.

Manipulatives help students build mental models K-12.

During the intermediate grades, manipulatives provide fundamental models for developing base-ten knowledge and for understanding geometry and measurement.

Manipulatives for Number, Operations, & Algebraic Relationships:

- ☐ number charts in a variety of forms (See appendix for examples of tenths, hundredths, starting at 1, 0, and vertical)
- ☐ number cubes with dots and numerals
- ☐ place-value arrows – 1 teacher set per classroom & 5 student sets (See Chapter 6: Representation)
- ☐ place-value blocks (transparent and interlocking)
  - 50 ones (units) per student
  - 20 tens (rods) per student
  - 5 hundred (flats) per student
  - several thousand cubes (for class)
- ☐ number cards (one deck for every student with 4 sets of 1-10)
- ☐ coins and dollar bills
- ☐ calculators (one per student)
- ☐ optional
  - Clock-o-Dial (one per class)
  - Mini Judy-clocks with gears (3rd grade only)
  - Rekenrek (3 per class)
  - ¾" red and white 20-bead string (one per class)

The National Library of Virtual Manipulatives, a digital library containing Java applets and activities for K-12 mathematics, can provide easy access to manipulatives for students using computers.

http://nlvm.usu.edu/en/nav/vlibrary.html

Manipulatives for Geometry
**indicates a useful item for a student tool kit*

- ☐ Attribute blocks (8 sets) – 3rd grade
- ☐ Manipulatives for exploring 2-D shapes include:
  - Geoboards (1 per student + overhead set)
  - geometric shapes template (one per student)
  - pentominoes (1 set per student + overhead set)
  - Power Polygons (3 buckets)
  - tangrams (1 set per student + overhead set)
  - optional
    - Isotiles (several sets for small group or individual instruction)
    - Trigram set (several sets for small group or individual instruction)
- ☐ Manipulatives for exploring 3-D shapes include:
  - Geoblocks (3 kits per class)
  - plastic (fillable) geometric solids (8 sets)
  - Polydron frames (3 class sets)
  - snap cubes or multi-link cubes (1,000 per class)
  - wooden geometric solids (8 sets)
- ☐ *Geometry template (1 per student)

Manipulatives for Measurement
**indicates useful items for a student tool kit*

- ☐ *angle ruler (one per student)
- ☐ *circle compass (one per student)
- ☐ *protractor (one per student)
- ☐ *ruler(s) (1/8″ and cm) (one per student)
- ☐ yard and meter sticks (15 per class)
- ☐ square tiles (7-8 buckets)
- ☐ 1″ cubes (7-8 buckets)
- ☐ liquid measurement (1 demonstration set)
- ☐ thermometers (Celsius & Fahrenheit) (one per student)
- ☐ scientific balance and personal scale

Other Materials

- ☐ student journals (square-grid or lined)
- ☐ folders to store example of student work
- ☐ white boards and markers (to show solutions during number work or problem solving)
- ☐ math reference guides

Managing Manipulatives

Mathematical tools are "amplifiers of human capacities"

-Bruner, 1966

☑ Teachers organize manipulatives to:

- allow students' independent access and use
- bring focus to mathematical ideas
- emphasize the value of models for understanding math concepts

☑ Communicate clearly how students should get and put away manipulatives.

- Does each child get the needed things or is there a person whose job it is to pass out and collect manipulatives?
- When are students allowed or expected to get manipulatives?
- How many students will share a specific manipulative?
- Are materials organized so that all students are using the manipulatives to model and solve problems?

☑ Store manipulatives to allow for individual or small group use.

☑ Label the locations for storing each kind of manipulative.

☑ Store the manipulatives that support the current learning activities in easily accessible places.

☑ Be sure students know which manipulatives they may use.

Organizing Supplies to Bring Focus to Mathematical Ideas

- ☑ Store base-ten blocks in trays so that each student can efficiently access what they need. (See labels in Appendix.)

- ☑ Put all of the geometry manipulatives together and label the storage area Geometry. Do the same with the manipulatives for Measurement and for Number. This structure indicates connections between tools and reinforces academic mathematical vocabulary. (See labels in Appendix.)

Organization that Supports Individual, Small, and Large Group Work in the Intermediate Grades

Students in all grades depend on routine. Routines allow students to maximize instructional time and focus on mathematical thinking involved in each activity instead of on behavioral and social situations that may arise.

- ☑ Plan a sequence to the math hour so students can anticipate activities and expectations.
- ☑ Determine ways for students to transition from large group to small group to independent activities.
- ☑ Have clear expectations for how each area of the room is used during the math hour.
- ☑ Organize individual work so students can independently access it and hand it in. Many teachers place individual work in pocket folders, math boxes, or hanging files.

Time Allocation for the Math Hour

*A **minimum** of one hour of every school day is used for math instruction.*

In the intermediate grades, students:

- begin by relying on their informal intuitive understandings to help them make sense of new problem situations
- expand their vocabulary to learn to reflect on their work and share their thinking
- become aware of patterns when they work with numbers and shapes
- learn new math symbols and begin to understand that there are conventions in the ways those symbols are used
- use their strong sense of inquiry to begin making generalizations about number properties and operations

To have time to build this competence and confidence, math instruction easily takes a minimum of one hour every day. This hour is filled with engaging experiences that focus students' attention on learning important math concepts and conventions. It takes repeated experiences to gain solid understandings of new mathematical concepts, conventions, and skills. These experiences are guided by a teacher's careful use of engaging problems and students' reflections on their solutions. During the hour, the students also have some time to practice their skills and build fluency.

Structuring the Math Hour

Teachers organize math instruction around four components of the math block. The four components are: *problem solving, number work, inspecting equations, and fluency & maintenance.*

- **Problem solving** is the heart of math instruction. Problems may have a context and require more than one operation to solve. Problems also include geometry, measurement, or data analysis and probability.
- Thinking about numbers themselves, without a context, is also an important element of math instruction. This occurs during **number work**.
- Students in third, fourth, and fifth grade develop the ability to think about number relationships through **inspecting equations.** They do this by analyzing the ways in which T/F or open number equations communicate equality relationships.
- When students are able to use their new knowledge and skills independently, they need practice in order to develop fluency. The fourth part of instruction, **fluency & maintenance**, focuses on independent mathematical work.

Table 5.1 shows the recommended time allocation for each of the four parts of the math block. The lines within the diagram are dotted to indicate that the activities in each block can easily cross over into the other blocks. Teachers can organize the blocks in any sequence that best meets student needs and the class schedule.

Instruction	**Problem Solving** (Differentiated small group instruction followed by independent work or whole group investigations) **30-45 minutes/day**	**Number Work** (Whole Class) *or* **Inspecting Equations** (Whole class or small group) **10-15 minutes/day**
Practice	**Fluency & Maintenance** (Differentiated individual practice) **10-15 minutes/day (may be assigned as homework)**	

Advantages of Organizing the Math Hour

Structuring the math hour this way:

- keeps the focus on important mathematical content and processes
- facilitates using appropriate settings to develop students'
  - competency (ability)
  - proficiency (skill)
  - fluency (ease)
- provides time to differentiate for small group or individual needs based on assessment data
- provides opportunities for students to examine the same concepts within a variety of learning contexts

Number Work or Inspecting Equations (about 15 minutes)

In grades 3-5, devoting fifteen minutes daily to thinking about numbers, operations, and concepts represented in equations provides a consistent experience that fosters strong conceptual development in preparation for algebra. Number work and inspecting equations:

- provides students opportunities to analyze the conventions of symbolic representations and to consider the properties of numbers and operations in a whole class or small group setting throughout the year
- builds knowledge of decomposing and composing numbers, operations, base ten, fractions, equality in a number context
- supplements problem solving and fluency work and ensures that number sense continues to build throughout the school year

See Chapter 7: Number Work and Chapter 8: Inspecting Equations for more details.

Problem Solving (30-45 minutes)

In grades 3-5, students are able to spend longer periods of time working on their own with more complex problems chosen at their instructional level. Problem solving is differentiated to meet individual students learning needs. However, it can also include whole class introductions to new topics or for investigations in Geometry, Measurement & Data Analysis. Problem Solving:

- provides time for students to work independently, in partners, or small groups
- provides opportunities for each student to discuss solution strategies in a small group setting with a teacher facilitating the conversation
- provides optimum time for assessment as the teacher observes and listens to students explain their thinking
- provides time for students to learn to represent and communicate their solution strategies to teachers and classmates

See Chapter 6: Problem Solving for more details.

Fluency & Maintenance (about 15 minutes or as homework)

Intermediate students need adequate time to develop fluency with number facts, estimation, and mental computation. Fluency and maintenance activities can provide much needed practice while students meet with the teacher in small groups. Fluency and maintenance includes:

- daily practice with concepts at each student's independent mental computation levels which may be assigned as homework
- number games, fact practice, geometric puzzles and problems

See Chapter 9: Fluency & Maintenance for more detail

Three ways to organize

I.

One teacher begins the math hour by engaging the whole group in *number work* activities or *inspecting equations*. Sometimes students lead the class in number work routines.

Each teacher will organize the math hour in ways that are most effective for the students and their teaching situation.

Then this teacher provides several *story problems* for students to solve independently at their table groups. Each problem has a range of number choices with one set circled by the teacher. This teacher asks students to solve the problem with the circled set of numbers first. Students may try the other number sets if there is time that day.

The teacher meets with small groups of students to guide their development of problem solving strategies for number and algebraic thinking.

Students move to the *fluency & maintenance* independent activities in their math folders or math boxes when they have completed working on the assigned problems.

One day a week this teacher interviews each student on their number facts while the other students solve geometry puzzles and play number games.

When introducing a new topic (such as "nets of an open cube") this teacher uses whole class activities (such a "finding all pentominoes") during problem solving to assess students knowledge and determine what aspects of the topic may need to be highlighted in small groups or individually (such as congruence). Small group problem solving then supports the specific needs of individual students.

II.

This school uses a purchased program which necessitates dividing the students into two classes, one of all fourth graders and one of all fifth graders. This teacher keeps 4th grade students but 5th grade students move to another 4/5 classroom and vise versa.

The teacher turns students' attention to the *fluency & maintenance* activities listed on the board so they know what they can do when they finish their assignment for the day. Then, the teacher facilitates an opening activity related to the lesson for that day, usually *number work* or *inspecting equations*.

Students begin work on the assigned student pages in their student *problem-solving* book. Several students have alternate assignments. One group works on more advanced problems on the same topic. Another group meets with the teacher who modifies the assignment so that the main concept is accessible. The teacher encourages independent thinking, making sense, and sharing strategies as well as how to communicate thinking using mathematical symbolic notation. The teacher meets with two groups each day to discuss their work and to bring a focus to a concept in the lesson.

The teacher collects workbooks from the students not in a math group that day and writes a question or comment on one particular aspect of the students work. The teacher expects students to write a response in their workbook. Students move to fluency and maintenance activities as they finish their work.

III.

This teacher begins each day with a review of the *fluency & maintenance* or *number work* assigned as homework the previous day. The teacher facilitates a discussion of strategies and students check their work for accuracy. Then the teacher assigns the *problems* for the day.

These problems come from a variety of sources including teacher written problems that fit into a yearlong framework that covers the standards for both 4th and 5th grade. For some topics, the entire class works on the same problems. For other topics the teacher pulls small groups (based on assessment) together for guided instruction. Students have a student book of math problems to solve once they have finished their assigned work.

Twice a week, the teacher extends the math hour 15 minutes for a whole class discussion for *inspecting equations*.

One day a week, usually Friday, the teacher assesses individual students while the class plays math games, solves logic or geometry puzzles in small groups or at the computer activities.

For More Information:

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D. & Hanlie, M. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Madison Metropolitan School District. (2006). *MMSD K-5 Grade-level Mathematics Standards*. Madison, WI.

National Council of Teachers of Mathematics. (2000). *Principles & Standards for School Mathematics.* Reston, VA: AN online version retrieved April 2, 2007 from http://standards.nctm.org/

National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points for Pre-Kindergarten through Grade 8 Mathematics*. Reston, VA: AN online version retrieved April 2, 2007 from http://standards.nctm.org/

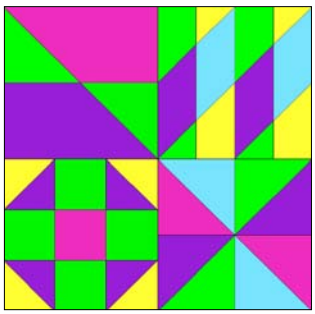
Ronfeldt, S. & Burns, M. 2003. *Third-Grade Math: A Month-To-Month Guide*. Sausalito, CA: Math Solutions Publications.

Wisconsin Department of Public Instruction. *Wisconsin Knowledge and Concepts Examination Assessment Framework*. Madison, WI.

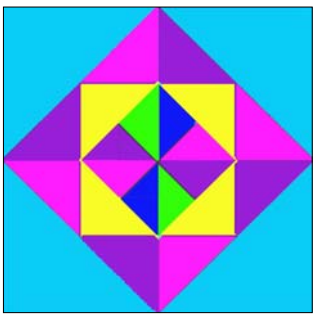
Intermediate Grade Math Reference Guides:

Cavanaugh, M. (2006). *Math to know: Grade levels 3-4*. Wilmington, MA: Great Source Education Group, Houghton Mifflin.

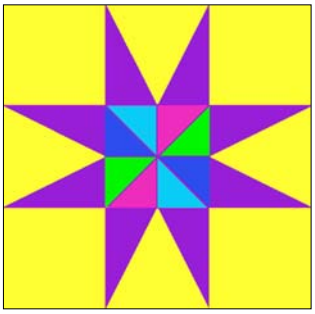
Lilly, M. (2006). *Math at hand: Grade levels 5-6*. Wilmington, MA: Great Source Education Group Houghton Mifflin.



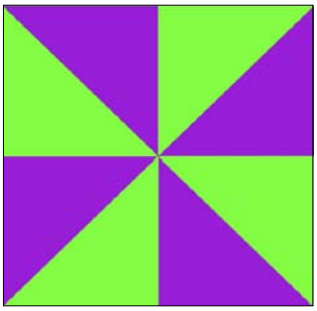
PROBLEM SOLVING



NUMBER WORK



INSPECTING EQUATIONS

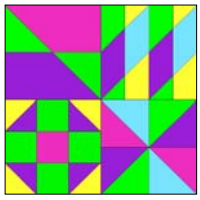


FLUENCY & MAINTENANCE

Chapter 6

Problem Solving

PROBLEM SOLVING



Problem Solving

Problem-solving activities should invite the study of mathematics and provide a context in which concepts and skills are learned. Through problem solving students learn to make effective use of their knowledge which in turn builds competence, a productive belief in their ability to do mathematics, and prepares them for everyday life.

The Problem Solving Block

- ☑ includes solving problems in all content strands: number, operations, and algebraic relationships; geometry; measurement; and data analysis and probability
- ☑ can take place in large and small group settings
- ☑ requires from 30 to 45 minutes of the math hour (depending on group size and purpose)
- ☑ may serve multiple purposes, such as:
    - introducing new concepts
    - challenging students to develop and apply effective strategies
    - providing a context for improving skills
- ☑ utilizes problems that:
    - emerge from the students' environment, a familiar context, a common experience, or from purely mathematical contexts (not a story problem)
    - can be found in the school's curricular materials
    - allow for multiple solution strategies
    - lead to justification and generalization

In mathematics, when one tries to solve an open problem, there is a high likelihood that one does not succeed at all, or only reaches a partial solution, which makes it at times very frustrating to work in mathematical research. On the other hand, often one returns to work on the same problem years later and has gained a new insight by then, which now allows one to "see" a solution one had missed before."

S.Lempp, UW research mathematician, 2007

The Problem Solving Block allows students to build new knowledge within a supportive classroom environment. During problem solving, students:

- ☑ identify and understand the elements of challenging problems and explore new number relationships
- ☑ develop reading comprehension strategies required to understand problems or tasks
- ☑ compare solution strategies to build connections between ideas or concepts
- ☑ understand and use different representations for solutions
- ☑ develop justifications for their solution strategies based on accepted mathematical ideas
- ☑ increase flexibility, efficiency, and accuracy in computation
- ☑ communicate solutions orally and in writing so that classmates and teachers can understand various aspects of a solution
- ☑ learn that perseverance is an important aspect of problem-solving
- ☑ make sense of mathematics and take intellectual risks by asking questions, discovering generalizations, making conjectures, and contributing to mathematical debate

The Problem Solving Block enables teachers to:

- ☑ gain information about each student's conceptual and procedural knowledge.

  Written and oral explanations, drawings, and models can provide evidence of students thinking. Teachers must look beyond the answer and assess student reasoning behind the solution.

- ☑ differentiate instruction by:
  - choosing or adjusting problems or tasks to fit mathematical learning goals
  - flexibly grouping students to meet learning needs

Problem-solving activities challenge students to solve problems in all content strands: number, operations, and algebraic relationships; geometry; measurement; and data analysis and probability. Teachers and students must establish classroom norms such that everyone's ideas are valued and each student gains confidence and self-assurance as a problem solver.

Depending upon the purpose of the problem-solving task (e.g. to foster growth in a content strand, to expand students' understanding of a new set of numbers) a teacher plans to work with a large group, with a small group, or with an individual during problem-solving activities.

This chapter provides detailed discussion of the components of the Problem Solving Block including:

1. STORY PROBLEM TYPES
2. CHOOSING NUMBERS TO BUILD NUMBER SENSE
3. USING STORY PROBLEMS TO BUILD PLACE-VALUE KNOWLEDGE
4. ESTIMATION
5. MENTAL CALCULATIONS
6. SOLUTION STRATEGIES
7. REPRESENTING SOLUTIONS
8. USING STORY PROBLEMS TO BUILD FRACTION KNOWLEDGE
9. GEOMETRY
10. MEASUREMENT
11. DATA ANALYSIS & PROBABILITY
12. CLASSROOM DISCOURSE
13. MATH & LITERACY
14. MATH JOURNALS AND WRITTEN FEEDBACK

Content and Process Standards
BIG IDEAS: Content

Number, Operations, and Algebraic Relationships

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems including unitizing, place-value patterns when computing, proportional reasoning, fractions, decimals, and percents
- Understand meanings of operations and how they relate to one another
- Compute fluently and make reasonable estimates
- Understanding patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships

Geometry

- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Apply transformations and use symmetry to analyze geometric situations
- Use visualization, spatial reasoning, and geometric modeling to solve problems

Measurement

- Understand measurable attributes of objects and the units, systems, and processes of measurement
- Apply appropriate techniques, tools, and formulas to determine measurements

Data Analysis and Probability

- Formulate questions that can be addressed with data and collect, organize, and display the relevant data
- Select and use age-appropriate statistical measures to analyze data
- Develop and evaluate inferences and predictions that are based on data
- Understand and apply basic concepts of probability

Content and Process Standards
BIG IDEAS: Process

*The **Math Processes** for problem solving include:*

Problem Solving

- Build new mathematical knowledge
- Solve problems that arise in everyday situations
- Apply and adapt a variety of appropriate strategies to solve problems (e.g. illustrate, simplify, look for patterns and relationships, test reasonableness of results, generalize)
- Monitor and reflect on the process of problem solving
- Formulate questions for further explorations

Representation

- Create and use representations to organize, model, record, and communicate mathematical ideas
- Select and apply mathematical representations to solve problems

Communication

- Organize and consolidate mathematical thinking
- Communicate mathematical thinking coherently to peers, teachers, and others using mathematical language and representations to express mathematical ideas precisely
- Analyze and compare strategies

Reasoning and Proof

- Develop and analyze problem-solving strategies, mathematical arguments, and justifications
- Make and investigate mathematical conjectures
- Learn that reasoning and justification are fundamental aspects of mathematics

Connections

- Read and understand mathematical texts and other instructional materials and recognize mathematical ideas as they appear in other contexts
- Recognize and use connections among mathematical ideas
- See relationships between problems and actual events
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Apply mathematics in contexts that include personal experiences, interests, and current events

Story Problems in the Problem Solving Block

Teachers consider the following when planning a problem-solving session:

- ☑ mathematical goal of the session
- ☑ composition of student groups
- ☑ selection of the problems
- ☑ MMSD math standards
- ☑ adaptations and extensions that may be needed for specific children

The mathematical goal of the session should connect what the student knows (based on regular informal and formal assessment) with what students need to learn next.

To use problem-solving activities effectively, teachers plan the grouping structure that best fits the learning objectives. Some activities lend themselves to engaging the entire class at one time. However, focused small group sessions provide opportunities for teachers to observe their students as they:

1. develop skills in solving a particular problem type
2. focus on representing a problem and solution strategy
3. work with a particular set of numbers

When planning which problems to pose, teachers choose problem-solving activities from the school's curricular resources and also write problems tailored to the experiences and interests of students using situations and number choices that are familiar to the students.

Most importantly, teachers reflect on what students know and do and choose directions for future instruction that fit with learning goals. Teachers extend or adapt the problem-solving experience by writing problems so a student can read and solve problems independently.

The following pages show how one teacher plans instruction to meet individual student needs through small group instruction.

One 4th/5th grade teacher plans for problem-solving instruction in the following way:

Planning Notes — March 15 — Analyzing last week's work to plan for the next week.

Group 1
Jemiah, Rafael, Amanda, Laura N., Marla, Jake, Zach, Laura B.
They are working with multiplying and dividing fractions in story contexts. Prefer to draw solutions or write lengthy explanations.

Independent work — multi-digit numbers, practice finding factors, work with a partner though fraction story problems in textbook.

Instructional focus — Division story problems. Compare division (MD) with fractions, vs. division with whole numbers. Focus on misconception that division always makes smaller. They have a flexible algorithm for whole numbers. I wonder if they can make it work for fractions. (Perhaps I should suggest a ratio table instead.) Will try to use the same context with all problems-making kites that require 5/6 yd. of fabric vs. 3 yards of fabric (need more of these!) How many can be made with 12 yards of fabric?

Group 2
Naomi & Quintin
They are working with numbers to 100. They use base-ten blocks to solve but still need support to represent their models with drawings.

Independent work — JCU, SCU, C, using numbers to 50, pentomino puzzles, tens facts and decomposing 7, 8, 9 into two parts.

Instructional focus — JCU, SCU, C, M Number work should focus on strengthening strategies to "make a ten." We may use a ten frame to model "across ten facts." Then practice, practice, practice.

Group 3
Lisa, Natalie, Johan, Misty, Tua, Tessa, Zach (Rachel join the group this week)
They are building multi-digit strategies to multiply two-digit x two-digit numbers. They use arrow language to show repeated addition and ratio tables but are shaky with explanations for multiplying by decades. They want to use the "dropping common zeros" trick and can't justify it

Independent work — fact games for x7s and x8s, work with a partner though beginning fraction story problems in textbook after small group session on sharing problems.

Instructional focus — Support an area model for multiplication problems (start with decade numbers only). Focus on justification, where is the 100 in the model. Use a story problem to elicit the model, perhaps a symmetric array problem first. Also, one lesson to introduce equal sharing. Focus on organizing their drawings to support their thinking.

Group 4
John, Sari, Paris, Jesus, Brendan
They are building multi-digit strategies to multiply one-digit x two-digit numbers. They use arrow language sometimes but usually repeated addition. Just beginning to understand base ten and multiplication.

Independent work — fact games for x4s and x6s, work with a partner though beginning fraction story problems in textbook after small group session on sharing problems.

Instructional focus — Continue work on multiplication (start with one-decade numbers). Encourage Use a story problem to elicit the model, perhaps a symmetric array problem first. Also, one lesson to introduce equal sharing. Focus on organizing their drawings to support their thinking.

A Typical Sequence for a Story Problem Session

The teacher:

- ☑ selects problem(s) and numbers based on a student's knowledge of a particular concept
- ☑ checks for comprehension of the story problem(s) (Students may re-tell or ask questions about the problem(s) until they feel confident enough to begin working.)
- ☑ may ask students to pay attention to a particular element of the problem or solution they want students to think about or use as they work on the solution to the problem
- ☑ asks students to work independently, in pairs, or small groups to solve the problem(s) and record their solutions
- ☑ meets with students working on the same problem(s) for discussion and facilitates a conversation about their solutions focusing on one or two teaching points
- ☑ the teacher may introduce a new method or strategy to consider for discussion (being careful not to give it more authority than the student generated solutions)
- ☑ plans the next lesson based on information gained during discussion and from student's written work or assessments

The students:

- ☑ work on the problem(s) and record their solution strategies in ways that others can interpret
- ☑ ask clarifying questions of each other
- ☑ understand and assess each other's solutions to:
    - find connections between solutions
    - make generalizations
    - assess accuracy and efficiency
    - discuss concepts
    - discuss representations and meanings of symbols
- ☑ reflect on strategies and their knowledge of number relationships and math concepts
- ☑ may only use new strategies or representations that they can justify mathematically

The following table indicates what intermediate students should know and be able to do in Number and Operations as a result of instruction in Problem Solving.

Number and Operations in the Problem Solving Block

When planning instruction for each student, teachers use the MMSD K-5 Grade Level Mathematics Standards to guide the selection of problem types and number sizes. Teachers know that every student will take a unique path in becoming proficient with the problem types, solution strategies, and number sizes. The following table summarizes the MMSD Number & Operations standards for grades 3-5. **Boldface** type indicates "new" for the grade-level.

Third grade students: Read, write, and order whole numbers up to **10,000.** Solve all CGI Problem Types Solve multi-step story problems Flexibly uses the following strategies for computation: • **incrementing or compensating** (using landmarks of 10s, 100s, 1,000s), **standard or other algorithm** to find sums of 3-digit or smaller numbers and differences of 2-digit or smaller numbers • **repeated** addition, composing/decomposing, or doubling to multiply **1-digit X 2-digit** numbers • **repeated subtraction, partitioning/sharing, measuring) to divide with a dividend up to 45 and divisor up to 5** • counting by groups of 2, 5, 4, 3, 10 • recall addition facts • **recall subtraction facts** • **recall all multiplication facts (with 2, 5, 4, 3 as multiplier or multiplicand)** Represent solutions by: • modeling with objects or drawings • using the empty number line and arrow language • writing equations Solve problems with money values \$0.01-\$1.00 Investigate fraction concepts through: • **reasoning about basic equivalencies to solve problems** • **relating fractions to benchmarks of 0, whole numbers, 1/2s (order simple fractions)** • **solving joining and separating story problems involving commonly used fractions (with like denominators)** • solving partitioning or equal sharing story problems where the solution has a fractional part (Ex: Four children share 5 cookies equally. How much does each child get?) • solving fractions as 'operators' problems (Ex: How many eggs in $\frac{1}{4}$ of a dozen eggs?) • drawing fractional parts of a set of objects or a single unit (Ex. cookies, rectangles) • naming and using fraction notation for a set of objects or a single unit for $\frac{1}{4}$ and $\frac{1}{2}$ Note: Problems should involve unit fractions ($\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6},\frac{1}{8}$), non-unit fractions (ex. $\frac{5}{6},\frac{3}{8}$), improper fractions (ex. $\frac{3}{2},\frac{6}{4}$), mixed numbers ($2\frac{1}{2},1\frac{2}{3}$).	**Fourth grade** students also: Read, write, order, and compare, whole numbers up to **100,000 and decimals (money context).** **Solve multi-operation story problems.** Flexibly use the following strategies for computation: • find sums of **4-digit** or smaller numbers and differences of **3-digit** or smaller numbers • **efficiently** multiply 1-digit X 2-digit numbers • divide a **2-digit dividend with a single-digit divisor** except zero • recalls **all** multiplication facts • estimation Represent solutions: • **ratio tables** Solve problems with money values \$0.01-**\$10.00** Investigate fraction concepts through: • **relating fractions to benchmarks of 25%, 50%, 75%, 100%** • **exploring the connections between operations with whole numbers and operations with fractions** • **determining the approximate location of fractions on a number line** **Demonstrate an understanding of fraction concepts "investigated" in third grade.** • **names and uses fraction notation for a set of objects or a single unit (Ex. ¼, ½)** • **renames improper fractions** • **compares two fractions by relating them to benchmarks of 0, whole numbers, 1/2s**	**Fifth grade** students also: Read, write, order and compare whole numbers up to 1,000,000 and decimals (money context). Flexibly use the following strategies for computation: • find sums of **5-digit** or smaller numbers and differences of **4-digit** or smaller numbers • efficiently multiply **2-digit X 3-digit** numbers • divide a **4-digit by itself** or a single digit divisor except zero • recalls **all division facts** • **knows the first ten multiples of 2-10 and 25** • estimation Represent solutions: • **ratio tables** Solve problems with money values \$0.01-**\$100.00** Investigate fraction concepts through: • **Generate and justify equivalencies**

Story Problem Types

Teachers use twelve basic problem types for instruction. These problem types cover a range of ways to structure simple math stories that include:

- joining or separating situations
- comparisons
- part-whole situations
- grouping
- two types of partitioning situations

For examples, see the following CGI Problem Type Chart.

During the intermediate grades the basic problem types are useful when paired with numbers that are unfamiliar to students (very large numbers or decimals). However, story problems should also reflect both the growing complexity of language and concepts appropriate for older students. For example, in the primary grades multiplication and two types of division problems involve grouping and partitioning collections of discrete objects that children readily count. Intermediate students should also solve related problems that involve "rates" and "multiplicative comparisons" rather than collections of countable objects. These problems promote proportional rather than additive reasoning.

Intermediate students should have many opportunities to solve the following story problems types discussed in this chapter:

1. CGI (Cognitively Guided Instruction) Story Problem Types
2. Division with Remainders
3. Rate Problems
4. Price Problems
5. Multiplicative Comparisons
6. Symmetric Problems (Array, Area, and Combination Problems)
7. Multi-Step Problems
8. Fraction Problems

CGI Story Problem Types

Teachers should assess how each student solves these problem types with contexts and numbers appropriate for the grade level before moving on to more complex problems described in this chapter. Numbers used in this chart are appropriate for third grade students. *Focus on developing proficiency with the un-shaded problem types for students in the intermediate grades.*

JOIN	**(JRU) JOIN RESULT UNKNOWN** Connie had 43 marbles. Juan gave her 19 more marbles. How many marbles does Connie have all together? $43+19=a$	**(JCU) JOIN CHANGE UNKNOWN** Connie has 43 marbles. How many more marbles does she need to have to have 62 all together? $43+a=62$	**(JSU) JOIN START UNKNOWN** Connie had some marbles. Juan gave her 19 more marbles. Now she has 62 marbles. How many marbles did Connie have to start with? $a+19=62$
SEPARATE	**(SRU) SEPARATE RESULT UNKNOWN** **Connie had 62 marbles. She gave 19 to Juan. How many marbles does she have now?** $62-19=a$	**(SCU) SEPARATE CHANGE UNKNOWN** Connie had 62 marbles. She gave some to Juan. Now she has 19 left. How many did she give to Juan? $62-a=19$	**(SSU) SEPARATE START UNKNOWN** Connie had some marbles. She gave 19 to Juan. Now she has 43 marbles left. How many marbles did she have to start with? $a-19=43$
PART-PART-WHOLE	**(PPW-WU) PART-PART-WHOLE (WHOLE UNKNOWN)** Connie has 43 red marbles and 19 blue marbles. How many marbles does she have? $43+19=a$	**(PPW-PU) PART-PART-WHOLE (PART-UNKNOWN)** Connie has 62 marbles. 43 are red and the rest are blue. How many blue marbles does Connie have? $62-43=a$ or $43+a=62$	
COMPARE	**(CDU) COMPARE DIFFERENCE UNKNOWN** Connie has 62 marbles. Juan has 43 marbles. How many more marbles does Connie have than Juan? $62-43=a$ or $43+a=62$	**(CQU) COMPARE QUANTITY UNKNOWN** Juan has 43 marbles. Connie has 19 more than Juan. How many marbles does Connie have? $43+19=a$	**(CRU) COMPARE REFERENT UNKNOWN** Connie has 62 marbles. She has 43 more marbles than Juan. How many marbles does Juan have? $62-43=a$ or $a+43=62$

(M) MULTIPLICATION	**(MD) MEASUREMENT DIVISION**	**(PD) PARTITIVE DIVISION**
Connie has 3 bags of marbles. There are 15 marbles in each bag. How many marbles does Connie have all together? $3\times15=a$	Connie has 45 marbles. She wants to put 15 marbles in each bag. How many bags can she fill? $a\times15=45$ or $\frac{45}{15}=a$	Connie has 45 marbles. She wants to put the marbles into 3 bags with the same number in each bag. How many marbles are in each bag? $a\times3=45$ or $\frac{45}{3}=a$

Adapted with permission from Carpenter, T.P., Fennema E., Franke, M.L., Levi, L., Empson, S.B. 1999. Children's Mathematical Thinking. Cognitively Guided Instruction. Portsmouth, NH: Heinemann.

+ — × ÷

Story Problem Types

Division with Remainders

In real life, there are many more situations that divide with leftovers than those that do not. Intermediate students should have many experiences with division problems that have "leftovers."

The context of a problem generally indicates how to treat the remainder in answering the question. There are four basic situations:

1. An extra whole unit must be included.

   156 children are going on a field trip to the Science Museum. 47 children can ride in each bus. How many buses are needed to get all of the children to the Museum? How many should ride on each bus?

2. The remainder is left off.

   It takes 4 eggs to make an omelet. How many omelets could you make with 13 eggs?

   Mrs. Carpenter has a bag of 235 nails. She wants to put them equally into 4 bins. How many nails should he put in each bin?

3. The remainder is the answer to the problem.

   A sporting company has 526 tennis balls, which they want to pack in tubes with four tennis balls in each tube. If they fill as many tubes as possible, how many balls will be leftover?

4. The answer includes a fractional or decimal part.

   Ms. Long has 17 blocks of clay for an art project. She wants to put an equal amount of clay at each of the 4 tables. How much will each table get if she puts out all of the clay?

   How many tens in 87?

+ — × ÷

Story Problem Types

Rate Problems

These problems are conceptually different than Grouping and Partitioning (CGI Multiplication, Measurement & Partitive Division) problems because they involve a rate rather than a number of objects. Rate problems do not necessarily have countable objects in them although the quantities could be represented with counters. In each example below, the rate is growth over time (millimeters per day):

A plant grows 3 millimeters each day. How many millimeters will the plant grow in 9 days?

A plant grows 3 millimeters each day. How many days will the plant take to grow 27 millimeters?

A plant grew 27 millimeters in 9 days. If the plant grew the same amount each day, how much does the plant grow in one day?

Other common situations involving rate are:

How many miles does a bicycle travel in 3 hours at an average speed of 12 miles per hour?

A baby sitter earns 6 dollars per hour for baby-sitting. How many hours will he have to baby-sit to earn 18 dollars?

+ — × ÷

Story Problem Types

Price Problems

Price problems are a special kind of rate problem. The rate is a price per item. Students are generally familiar with money and solve price problems with the same strategies as other Grouping and Partitioning problems.

How much would 5 pieces of gum cost if each piece costs 15 cents?

Gum costs 15 cents for each piece. How many pieces of bubble gum can you buy with 60 cents?

If you can buy 5 pieces of bubble gum with 60 cents, how much does each piece cost?

+ — × ÷

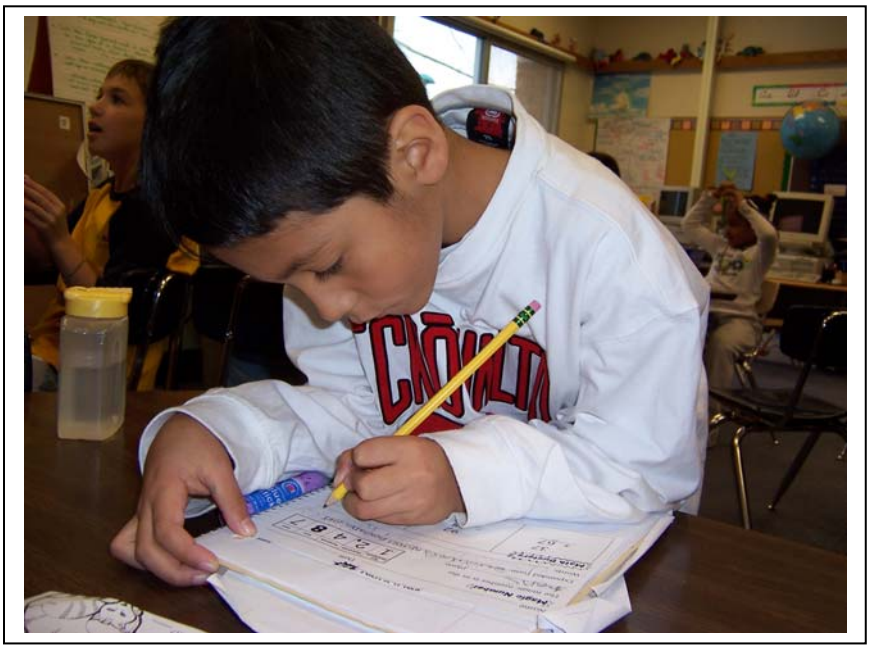
Story Problem Types

Multiplicative Comparisons

These problems involve a comparison of two quantities. The relation between quantities is described in terms of how many times larger one is than the other. In the example below the number "15" quantifies the relationship "times your height."

> *If you could jump like a frog you could jump 15 times your height. How high could you jump?*

The following "Related Problems: Rate, Price, and Multiplicative Comparison" table provides more examples of these story problem types.



Rate, price, and multiplicative comparison problems (also called ratio and proportion problems) involve "*reasoning up and down* in situations in which there exists an invariant (constant) relationship between two quantities that are linked and varying together." (Lamon, 2006) Explorations with ratio and proportion should begin in the intermediate grades. Students who have many experiences solving these types of problems using their own strategies will learn to reason proportionally and use rational numbers flexibly and with understanding in middle school and beyond.

These problems can be very challenging. For this reason, teachers sometimes reserve them as extensions or challenges for students who have strong number sense. However, these problems are appropriate for small group work with all students when the problems are designed with numbers that match the student's number level of fluency.

The following four problems provide an example *problem set.* See the appendix for more examples of *problem sets*.

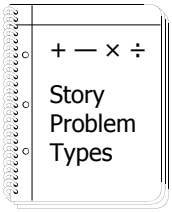
> *Sammy is feeding his fish. The directions on the box tell him that 4 little scoops of food are enough for 12 fish. How many little scoops of food should feed the 24 fish in Sammy's aquarium?*
>
> *Shay has invited his friends for pizza. He estimated that he would need 2-18" pizzas for 4 people. How many pizzas does he need to buy if 26 of his friends are coming?*
>
> *At the Humane Society it is time to feed the cats. If 6 cans of cat food feed 8 cats, how many cans are needed for 36 cats?*
>
> *It is feeding time at a fox farm. Four kg of meat are needed for 5 foxes. How many kg do 22 foxes need?*
>
> *Review these four problems—how are these problems similar or different? Compare your problem-solving strategies for each of them – what strategies did you use? Did you use the same strategies for all of the problems? Were some easier or harder? Why do you think so?*
>
> *.*

Related Problems: Rate, Price, and Multiplicative Comparison

The following chart illustrates the differences between Multiplication, Measurement Division, and Partitive Division problems. Rate, Price, and Multiplicative Comparison problems provide opportunities for students to solve a variety of problems involving different kinds of quantities. These problem types can also include fractions, decimals, in addition to whole numbers. Solving these problem-types lays the groundwork for developing proportional reasoning in middle school and are important to include in intermediate grades instruction.

Problem Type	(M) MULTIPLICATION	(MD) MEASUREMENT DIVISION	(PD) PARTITIVE DIVISION
Grouping/ Partitioning	Shakira has 14 pea plants. There are 8 peas on each plant. How many peas are there in all? $14 \times 8 = p$	Shakira has some pea plants. There are 8 peas on each plant. All together there are 104 peas. How many pea plants does Shakira have? $p \times 8 = 104$ $\frac{104}{8} = p$	Shakira has 14 pea plants. There are the same numbers of peas on each plant. All together there are 104 peas. How many tomatoes are there on each plan? $14 \times p = 104$ $\frac{104}{14} = p$
Rate	Johan walks 3½ miles an hour. How many miles does he walk in 8 hours?	Johan walks 3½ miles an hour. How many hours will it take him to walk 28 miles?	Johan walked 28 miles. It took him 8 hours. If he walked the same speed the whole way, how far did he walk in one hour?
Price	A package of pencils cost $1.98. How much do 12 packages cost?	A package of pencils cost $1.98. How many packages could you buy for $23.76?	Mrs. Martin bought 12 packages of pencils. She spent $23.76. If each package cost the same amount, how much did each package cost?
Multiplicative Comparison	The blue whale is about 22 times as long as the Hector's dolphin. If a Hector's dolphin is 5 ft. long, about how long is the blue whale?	The blue whale is about 110 feet long. The Hector's dolphin is about 5 feet long. The blue whale is how many times longer than the Hector's dolphin.	The blue whale is 110 feet long. If the blue whale is 22 times as long as the Hector's dolphin. How long is the Hector's dolphin?

Adapted with permission from Carpenter, T.P., Fennema E., Franke, M.L., Levi, L., Empson, and S.B. 1999. Children's Mathematical Thinking. Cognitively Guided Instruction. Portsmouth, NH: Heinemann.



Symmetric Problems (Array, Area, and Combination Problems)

Symmetric problems provide a problem situation for students to explore the commutative property.

Typically, the referents (labels for each quantity) in basic Grouping or Partitioning Problem types are not interchangeable. For example:

> *There are 27 cars. Each car has 4 wheels, how many wheels all together?*

In this problem, the referents "cars," "wheels on each car," and "wheels all together" cannot be interchanged.

Students typically solve this problem by modeling or adding 27 groups of 4. However, 4 x 27 also answers the computation but is not easily justifiable with these referents.

Symmetric problems that have the same or interchangeable referent for each quantity in the problem help intermediate students understand the reason why 4 x 27 is the same as 27 x 4. The following area, array, and combination problems are examples of symmetric problems.

Area and Array Problems

Unlike the "27 cars with four wheels" problem, the referents in an area or array problem are interchangeable. For example:

> *A baker has a pan of fudge that measure 12 inches on one side and 9 inches on the other side. If the fudge is cut into square pieces 1 inch on a side, how many pieces of fudge does the pan hold?*

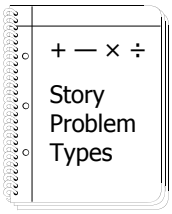
The two factors in this Area problem, 12 and 9, have the same referents. Students can model this problem before they have a formal concept of calculating area. With this type of problem students can begin to justify why adding 12 groups of 9 is the same as 9 groups of 12.

Area problems can also involve division and help students make connections between the inverse operations of multiplication and division. Example:

> *Mrs. Vang wants to plant a rectangular flower garden. She has enough room to make the garden 6 meters along one side. How long does she need to make the adjacent side in order to have 48 square meters of garden?*

Array problems suggest the same concept of multiplication as the area problems. In array problems discrete objects are arranged in rows and columns. "Rows" and "columns" are interchangeable referents.

> *Mr. Wee set up chairs for the school play. He put chairs into 5 rows with 26 chairs in each row. How many chairs were set up for the play?*



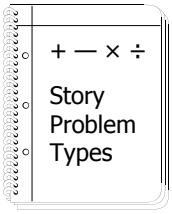
Combination Problems

Combination problems involve making combinations from given sets. This example is symmetric.

> *The Pancake Palace makes 3 types of pancakes. They have 4 different toppings. How many different combinations of pancakes and topping can you get at the Pancake Palace?*

These problems are symmetric because the types of pancakes and the toppings can be interchanged when thinking about this problem. There is no real difference in thinking in terms of the number of pancakes that could go with each topping or the number of toppings that could go with each pancake.

Some students may recognize that it is not necessary to make all of the combinations to solve the problem. Others may need to make an organized model.



Multi-step Problems

Multi-step problems provide opportunities for students to interpret more complex language and use more than one operation or multiple math concepts within one problem. Example:

> *A sandbox measuring 72 inches by 60 inches is filled with 12-inches of sand. By the end of the summer the sandbox was only ¼ full. How much sand must be added to replace the sand that was lost over the summer?*

Multi-step problems can involve a series of related problems that may combine topics of geometry, measurement, and data. For example, the following problems all refer to the same set of data:

> *Students surveyed their classmates to find out approximately how many glasses of milk (8 oz.) they drink per day. In a fifth grade class they found: 0, 1, 3, 1, 2, 1, 3, 4, 0, 1, 1, 2, 2, 3, 3, 3, 1, 0, 1, 4, 3, 2, 5, 1*
>
> *Find the average (mean) number of glasses of milk the class drinks per day. Explain your calculations.*
>
> *What is the range of the glasses of milk the students drink per day?*
>
> *Do you think the mean or the range a good way to describe the amount of milk a fifth grader drinks per day? Why or why not?*
>
> *Make a histogram of the data. What do you notice?*
>
> *If a gallon of milk has 128 oz., how many days would a gallon of milk last for the average fifth grader?*

Multi-step problems may also involve a comparison of two or more situations and require students to make a decision based on several computations. For example:

> *Your neighbor needs some work done around the house and has offered you the job. The neighbor has offered to pay you using one of three plans. You plan to complete the job as quickly as you can and estimate that it will take you at least 6 hours, but not more than 9 hours to do the work. Decide which payment plan is the best one for you and explain why.*
>
> *Plan A You can be paid a flat amount of $55 regardless of how long it takes you to do the job.*
>
> *Plan B You can be paid $40 for the job plus $2 an hour, but not for more than 8 hours of work.*
>
> *Plan C You can take the job and be paid $6.50 an hour.*

+ — × ÷

Story Problem Types

Problems without a Story Context

Numbers alone with a supporting story can be used as a "context" for problem solving. Example:

> *How could you use 55-30 to solve 55-28?*
>
> *How could you use 312 x 10 to solve 312 x 9? How about 300 x 9?*

Solving or comparing a series of equations builds number sense as students look for relationships between the two problems.

Be sure students understand the meaning of the operations before using equations without a story context. One way to assess student knowledge is to ask students to write a story problem that can be solved by a given equation. For example:

> *Write a math story that can be solved by 64/4 (64÷4)?*

See Chapter 4: Assessment for more examples.

Choosing Numbers to Build Number Sense

Teacher observation and assessment determine the particular number domain that students need to work with for problem solving. The MMSD standards provide grade-level expectations for proficient use of numbers in a particular range. Students should have a flexible use of strategies within that number range.

One approach to developing proficiency is to provide the same story problems with several related number choices.

> *Mr. Party has ___ bags of balloons in his store. Each bag has ___ balloons. How many balloons does Mr. Party have at his store?*
>
> *(52, 37) (50, 30) (2, 37) (50, 7)*

Using the number pairs above helps students develop ideas about the distributive property of multiplication. In this case, each student decides which problem to solve first. During discussion students talk about how to use one computation to help solve another similar computation.

Alternatively, teachers may provide a range of number sets appropriate to the learners in the classroom. Teachers either give students the option to choose or direct students to use a particular set of numbers in the problem.

> *There are ____ eggs in a carton. How many eggs are in ___ cartons?*
>
> *(12, 10) (12, 30) (12, 39) (12, 80) (12, 180) (12, 1,390)*
>
> *The chickens on the farm laid_____ eggs. How many dozen-egg cartons could be filled with these eggs?*
>
> *(1,980) (544) (144) (48)*

Teachers then facilitate discussions with students who have solved the problem with the same set of numbers. Students may solve the problems with the other number choices as challenges or for practice once they have solved the assigned or "just right" problem.

Some of the number domains teachers consider when writing, choosing, or modifying existing story problems include:

- Magnitude (.01s, .1s, 10s, 100s, 1,000s, 10,000s, etc.)
- Decades (numbers ending in zero e.g. 50, 520, 5230)
- Hundreds (numbers ending in two zeros e.g. 500, 2500, 12,500)
- Near decades (numbers ending 1, 2, 8, or 9 e.g. 59, 102, 1,348)
- Decimals in money contexts
- Decimals as fractional parts (e.g. .3 of a mile)
- Decimals with fraction equivalents (e.g. .25, .33, .125)
- Decimals close to whole numbers (e.g. 3.9. 45.99, 195.01)
- Negative numbers (in a familiar context such as temperature or below sea level)

An easy adjustment to textbook problems that are too difficult for students is to cross off one or two digits. Students can solve the problem with the smaller numbers first and once they understand the problem attempt the problems with the larger numbers.

Using Story Problems to Build Place-value Knowledge

A student's understanding of the base-ten system is built upon a broad conceptual foundation of number size, number relationships, and flexibility in decomposing and reconfiguring numbers. Place-value understanding is much more than merely the naming of a digit's value according to its position.

Kamii (2004) refers to base-ten understanding as "simultaneous thinking," where a child is capable of seeing the number 42 as 4 tens, two ones, as 42 ones, and potentially, as 3 tens and 12 ones. The student knows the different combinations are equivalent.

Place-value knowledge:

- takes years to develop.

  Children may be inconsistent for a long time. Children who can work with numbers under 100 often struggle with numbers above 100. As the number magnitude changes students readdress how the system works.
- can and should be developed for different components simultaneously. (e.g., number order, grouping in tens, composing/decomposing numbers)
- The understandings for different components reinforce one another and are not learned sequentially.

  See "Components of Place-value Understanding" (Chapter 4).

Problem solving supports developing place-value knowledge because it provides practice for more than one component at a time. Choose problems, number sizes, and activities that help students learn the underlying structure of our place-value system (base-ten).

It is important to consider how different representations (e.g. base-ten blocks, hundreds charts, open number lines) promote different components of place-value understanding.

The following instructional options (Brickwidde, 2002) describe how the strategic use of story problems and the choice of number sizes in them can support place-value development.

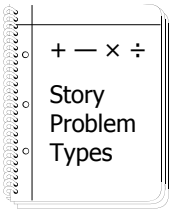
☞ *Blend addition & subtraction instruction together*

Teaching addition and subtraction in an integrated fashion, both with and without regrouping, helps a child build a broad understanding of the number system, placing in check more quickly any misconceptions that might develop.

Textbooks traditionally present addition and subtraction separately, moving from single digit, through multi-digit without regrouping, to multi-digit with regrouping calculations. However, studies of student development patterns have consistently found that children can build misconceptions at one stage that become difficult to readdress later when more complicated calculations are encountered.

Join Change Unknown, Separate Change Unknown, Comparison, and Part-Part-Whole Whole Unknown Problems used interchangeably with strategically chosen number sizes foster growth in understanding the relationships *between numbers* including the inverse relationship between addition and subtraction and base ten. Number choices can include large numbers, zeros strategically located within numbers, numbers close to decades, and decimals for older students.

☞ *Use multiplication to develop base-ten understanding*



Include collections of hundreds, thousands, and decimals (money) in story contexts. For example:

> *The Post Office sells rolls of 100 stamps. If they sell on average 250 rolls of stamps every month for a year, how many stamps would they sell in a year?*
>
> *A card store has 2,530 boxes of cards to sell for a holiday. If each box costs $5.00, how much money will the store get if it sells all of the boxes?*

Repeated opportunities to justify multiplication of numbers ending in zero builds base-ten knowledge. Be sure students have a mathematical justification for their solutions. Multiplying numbers that end in zero(s) can result in generalizing a pattern without mathematical reasoning to support it.

☞ *Use measurement division problems for building base-ten understanding*

+ — × ÷

Story Problem Types

Measurement division problems draw attention to the quantity in a group. The emphasis on groups of ten helps students see how the base-ten system works. For example:

> *A company that makes mechanical pencils wants to sell them in packages of ten. The company produces 12,550 pencils a week. How many packages of pencils would that be in a month?*

☞ *Focus on magnitude of the numbers in the problem and the answer*

Ask students to predict whether the answer after a computation will be in the 10s, 100s, 1000s, etc. Students will begin to think about the recursive 10s within 10s concept of the base-ten system.

☞ *Use base-ten manipulatives in a different way*

Change the unit of base-ten materials so that students think about relationships between quantities rather than a label. Give the ten-stick a value of one. Ask the students how that changes the number assigned to the 100s flat (now ten) or the single original unit cube (now one-tenth).

Students Writing and Solving Their Own Story Problems

Intermediate students should be able to write a story problem that can be solved by a given equation for any of the four operations. This provides teachers and students insight about the relationship between story problems, number size, language, and symbolic notation. Students who can create story problems have an easily accessible approach to help them understand or "visualize" operations. For example: third grade students should be able to write story problems for:

35 + 27	*53 – 29*
5 x 9	*45 ÷ 5*

Consider the following when asking students to write story problems:

- ☑ student's language (can they write in English and first language)
- ☑ number size in the given equation (student's fluent or instructional range)
- ☑ each student's experience solving a range of story problems with a variety of contexts
- ☑ whether a solution to the problem is necessary

When assessing the story problems, reflect on:

- ☑ the types of story problems students write (e.g. partitive or measurement division, take away, part-part-whole, or comparison)
- ☑ the words students use to describe the operations (e.g. "in each group" or "times as many" for multiplication)
- ☑ the setting or story contexts the students typically use
- ☑ whether the student needs to solve the problem before writing the story

Student written math stories can serve as Fluency & Maintenance work. Students can write and edit their own math stories to provide pages for a class book of practice problems. (See appendix, "A Walk for Rocks.")

Estimation

"Estimated answers should not be viewed as "second rate" answers. In most cases, an estimated answer is sufficient to solve a problem, and in some cases, it might be the only answer possible."

-Interactions Curriculum , 1994

"The production of an estimated or approximated calculation is an important life skill, as well as being a significant aspect of number sense. In the everyday world of consumer and worker, estimated answers are more frequently needed than exact answers. Estimation and checking the reasonableness of answers need particular emphasis if children are to make the most effective use of calculators." (Hope, J. & Small, M. 1994)

Research has shown that children can construct estimation approaches that make sense to them. (Sowder, J. & Wheeler, M., 1989)

Intermediate grade students who are developing base-ten knowledge should routinely predict the approximate answer (relative size or magnitude) of a computation. Teachers should establish a classroom norm to make estimation a routine.

Estimation also provides an effective way for students to learn to choose an appropriate strategy for a given set of numbers and know when results make sense. Estimation promotes:

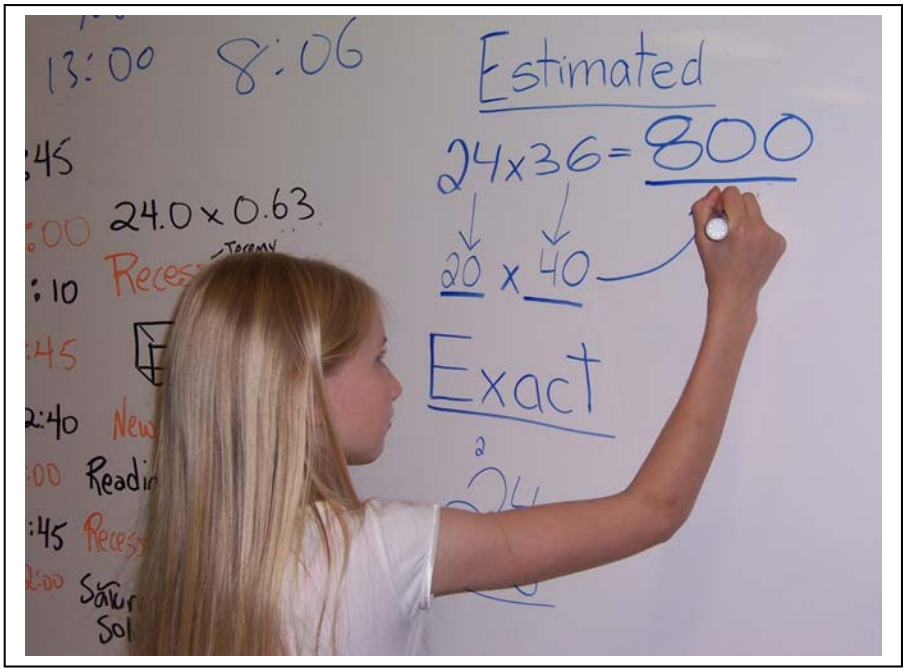
- number sense
- flexible choice of computational method or algorithm
- mental math capacity
- less reliance on calculators for simple computations

Teachers should ask students to use their own methods for estimation and engage them in discussions to compare methods.

Rounding rules do not help students become better at estimation. However students can use common approaches such as compensating or reformulating the problem to estimate. Both use number relationships to refine an estimate. For example 27 + 34 can be thought of as 20 + 30 + 10 (from 7+4) or simply 30 + 30 (since both 27 and 34 differ from 30 by a similar but "opposite" amount).

To scaffold estimation:

- ☑ use numbers in the students' range
- ☑ use operations that students understand
- ☑ ask students to choose from a set of three estimates that you provide
- ☑ ask students to provide an expected range the answer should fall between
- ☑ have discussions about how the operations change numbers (be aware that division doesn't always make a number smaller or multiplication make a number bigger)
- ☑ ask students to estimate "out loud"



Mental Calculations

Research has shown several benefits of mental calculation (Reys, 1984, Hope, 1986). "The crucial issue is not how the calculations are accomplished but rather knowing when and how to use arithmetic to solve problems or answer questions that, in fact, matter in the lives of people. Insisting that all children must be excellent pencil-paper calculators puts the emphasis in the wrong place-on the means, rather than on the ends, of calculation." (Usiskin, 1978) The use of calculators has caused educators to re-examine the purpose of computation in math instruction.

Computational proficiency means that children calculate mentally, estimate, and use written algorithms for smaller numbers and reserve calculators for unwieldy numbers. Mental calculation is:

- a practical life skill
- can improve the efficiency of pencil-and-paper calculations.
- the cornerstone of most estimation procedures
- leads to better understanding of place-value, mathematical operations, and number relationships

Mental calculation should be encouraged. Teachers can help students calculate mentally by:

- ☑ encouraging students to use a method that makes sense to them
- ☑ using pencil and paper as needed for part of a more difficult computation
- ☑ talking with students as they calculate (teachers provide sub-calculations as needed)

Solution Strategies

A solution strategy may consist of two components:

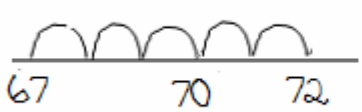
- a student's mental construct (a particular understanding of the mathematics)
- a student's way of representing the construct (when used)

The teacher's analysis of a student's mental construct and the student's representation (when used) provides insight into a student's number sense development and guides instruction.

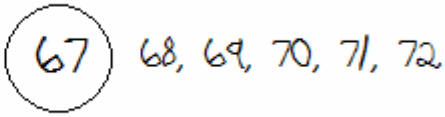
For example, a student who counts up 5 ones to solve 67+5 has a different mental construct than a student who decomposes the five into 3 and 2, then adds 3 to 67 to make 70, and 2 to 70 to make 72. Both students may represent their mental construct with the equation 67+5 = 72.

However, it's important to note that representing the solution with this equation does not show that the first student counts on by ones or that the second student decomposes the number five. Only through observation and discussion will a teacher know the mental constructs that a student has.

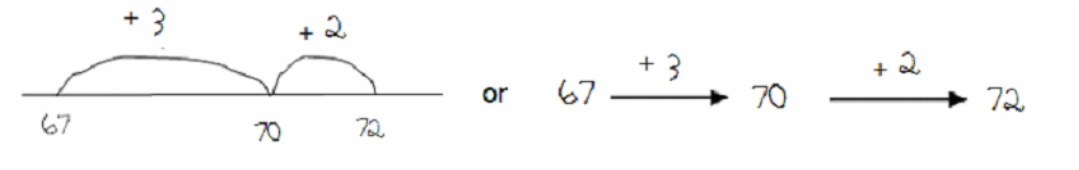
There is often more than one way to represent a particular mental construct. For example, the student who "counts on by ones" could use an empty number line with 5 jumps beginning at 67 and ending at 72. Alternatively, that same student could write the number "67 "and each number that is counted "68, 69, 70, 71, 72" or show five tally marks.



or



The student who decomposed the 5 into 3+2 could use an empty number line or arrow language to show adding 3 and 2 to 67 in increments.



A student's written work provides a valuable resource for discussion and assessing progress. Teachers help students develop routines and habits for recording their solutions.

Furthermore, students who develop their own strategies often provide new mathematical concepts useful for class discussion. For example, to solve 235-89 a student who understands negative numbers may represent their thinking with the following:

```
 235
 -89
 ---
  -4
 -50
 200
 ---
 146
```

Students should develop flexible use of strategies throughout the intermediate grades and continue to grow toward higher levels of abstraction (from using concrete objects to using symbols and from additive thinking to proportional thinking).

Discussions about student work provide excellent opportunities for teachers to understand their students and the effects of their instruction.

More Than One Strategy

Teachers should engage students in many conversations about the differences and similarities between strategies presented by the students. Also consider analyzing each strategy for accuracy, efficiency, or generalization.

One way to develop flexibility, accuracy, and efficiency with computation is to ask students to solve problems in more than one way. When asking students to find a second strategy the numbers in the selected problem should lend themselves to different strategies such as involve decomposing/composing, compensating, or incrementing strategies.

The "second strategy" should begin with the numbers in the problem as if the solution is unknown rather than using the answer and an inverse operation to "check" a solution. The following problem lends itself to a variety of solution strategies:

> *Jamal collected 178 shells at the beach. Mark collected 52 shells. How many shells did the boys collect?*

Some possible strategies include:

- 178 + 2 makes 180 plus 20 makes 200 + 30 makes 230
- think of 52 as 50, add 50 to 178 to get 228 plus 2 more to make 230
- 17 tens and 5 tens are 220, 8 and 2 make one more 10, so 230
- 8 + 2 is ten, add 170 +50 to make 220, then add the ten to make 230

Consider the following when asking students to solve problems a second way:

- ☑ Can the student represent and talk about strategies and make comparisons to other student strategies?
- ☑ What is the purpose for asking the student to solve the problem in more than one way?
- ☑ Does the student have the number sense needed to solve the problem in more than one way?
- ☑ Will your approach help the student build number sense or develop more efficient strategies?
- ☑ Will solving a second way help the student check the accuracy of their first strategy?
- ☑ What will the student do with the second strategy?
    - Will the student compare the second strategy to the first strategy?
    - What are the criteria for comparison? (efficiency, to make connections and build number sense, check for accuracy?)

MMSD standards indicate the minimum level of proficiency for computation at each grade level.

Note: Students who can only model a solution with blocks or drawings will not have a second strategy and will need support to build number sense.

Also, students who can only solve a problem using a memorized algorithm may need support to build number sense to understand or use a second more efficient strategy. For example, students who use the standard algorithm for 301-6 or 301-298 may need support in developing number sense for subtraction.

Representing Solutions

Students represent their thinking steps to solve math problems in a variety of ways. Purposes for recording thinking steps include:

- acquiring new concepts
- modeling a problem
- solving problems that have multiple steps or involve numbers or operations that are not easily computed mentally (ex. large numbers, fractions, etc.)
- communicating solutions to others
- analyzing strategies

Students should devise or choose representations that make sense to them and reflect their understandings. They may need focused attention on developing skills to record their own solutions and support with using new methods.

Each representation is unique in its utility and the conceptual development that it supports. For example, base-ten blocks suggest place-value strategies, hundreds charts stimulate a sequential pattern of counting by tens, and an empty number line develops an important linear organization of numbers.

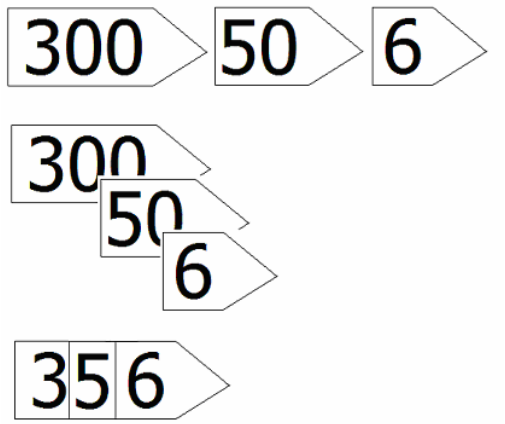
Some students may become dependent on a particular model to solve problems and not develop more mathematical connections. Teachers will want to remind students to use what they already know as they solve problems.

Written calculation proficiency is one of several goals for math instruction. In the intermediate grades, calculators in elementary school can interfere with number sense development and should be reserved for explorations such as revealing number patterns.

The following representations are appropriate to support building mathematical concepts in the intermediate grade students. If the student does not have a useful representation of their strategy, teachers can model a representation.

Place-value Arrows

Place-value arrows help students to see the "hidden zeros" behind a multi-digit number. Paired with base-ten blocks they can help students understand how to write quantities and decompose numbers using expanded notation.

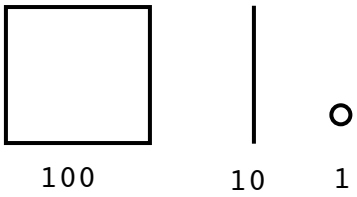


Base-ten Blocks

Base-ten blocks help students see that a group of ten ones is the same as one group of ten. Similarly, students can more easily see that one "flat of 100" is made of 10 tens and 100 ones.

Base-ten blocks should be readily available for every problem-solving session and purposefully used *as needed* to solve or explain solution strategies.

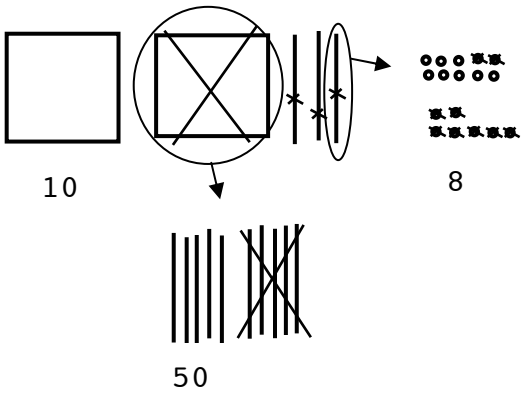
Teach students to represent their use of base-ten blocks as symbols on whiteboards or paper using the following symbols:



There are many ways to use these symbols to reflect a student's solution path. The following method utilizes decomposing a higher value unit of 100 and 10.

To show 237-79:

Students can write the quantity below each group of 1s, 10s, & 100s that make the answer.



Encourage students to show 'ones' and 'tens' in groups of 5.

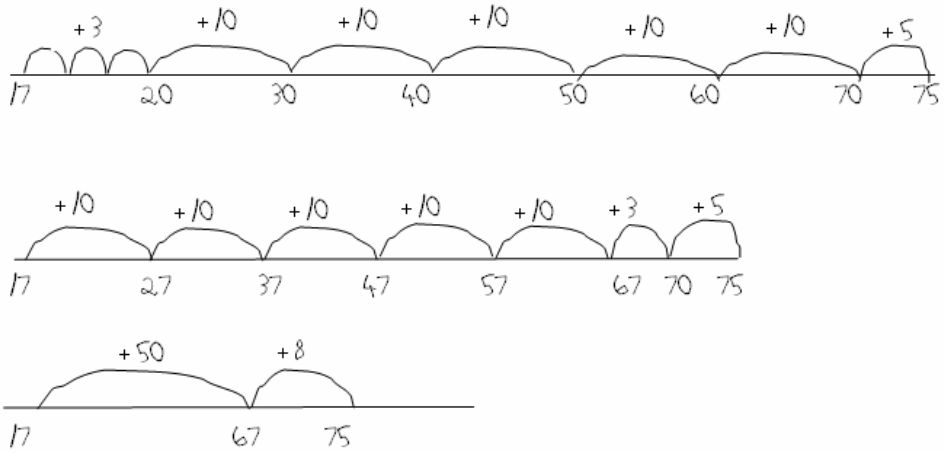
Empty Number line

The 'empty number-line' is a model that reinforces a linear organization of numbers (mental counting line). It can be adapted to meet student needs as their understanding of place-value concepts changes from using objects to using mental computation strategies.

The empty number line is particularly useful for understanding or "seeing" the difference between two numbers. As student fluency with place-value concepts develops, the empty number line provides a useful way to keep track of each step in the process of solving a more complex problem.

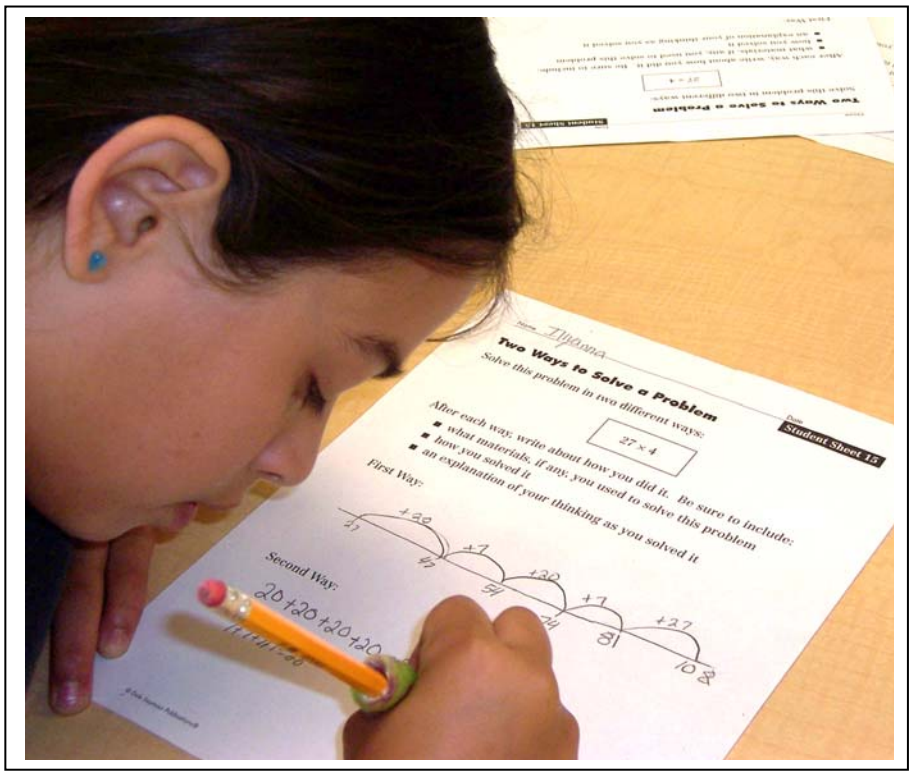
Three possible solutions paths for a Join Change Unknown problem are shown below using an empty number line.

> *Christel has 17 dollars. How many more dollars does she need to buy a puppy that costs 75 dollars?*



Teaching Tips

- SHOWS A STUDENT'S KNOWLEDGE OF NUMBER ORDER AND MAGNITUDE AND USE OF BENCHMARKS (DECADES, HUNDREDS, THOUSANDS)
- ALIGNS CLOSELY TO STUDENTS INTUITIVE MENTAL STRATEGIES AND SHOULD REFLECT THEIR THINKING RATHER THAN BE TAUGHT AS A PROCEDURE TO BE USED WITH ALL COMPUTATIONS
- TRACKS ERRORS AND ALLOWS STUDENTS TO THINK ABOUT WHAT TO DO NEXT WHEN A PARTICULAR COMPUTATION IS TOO DEMANDING TO DO "MENTALLY"
- NUMBERS SHOULD ALWAYS INCREASE IN SIZE LEFT TO RIGHT
- "JUMPS" DO NOT NEED TO BE PROPORTIONAL TO THE NUMBER SIZE
- RECORD ONLY THE RELEVANT NUMBERS ON THE LINE (E.G. LABEL "LANDING" POINTS UNLESS COUNTING BY 1S AND LABEL "JUMPS" GREATER THAN ONE WITH A NUMBER (OPERATION IS OPTIONAL)
- USEFUL FOR SHOWING "ADDING ON," INCREMENTING, OR COMPENSATING STRATEGIES AND FOR SHOWING SOLUTIONS TO JOINING, SEPARATING, COMPARING PROBLEMS, AND REPEATED ADDITION FOR MULTIPLICATION
- START AT ZERO FOR MULTIPLICATION



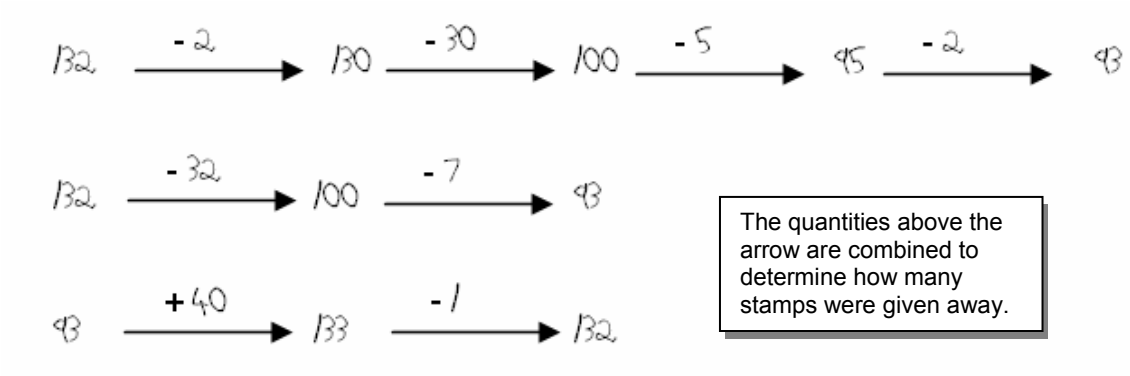
Arrow Language

Arrow language provides another easy-to-use representation of a student's thought processes. Teachers sometimes refer to arrow language as a "thinking train."

To teach the use of arrow language, begin with a simple computation that the student can do "in their head." Then build the "thinking train" that represents each step of the student's thought process. Show several solutions to the same problem. Then, compare and contrast for accuracy and efficiency.

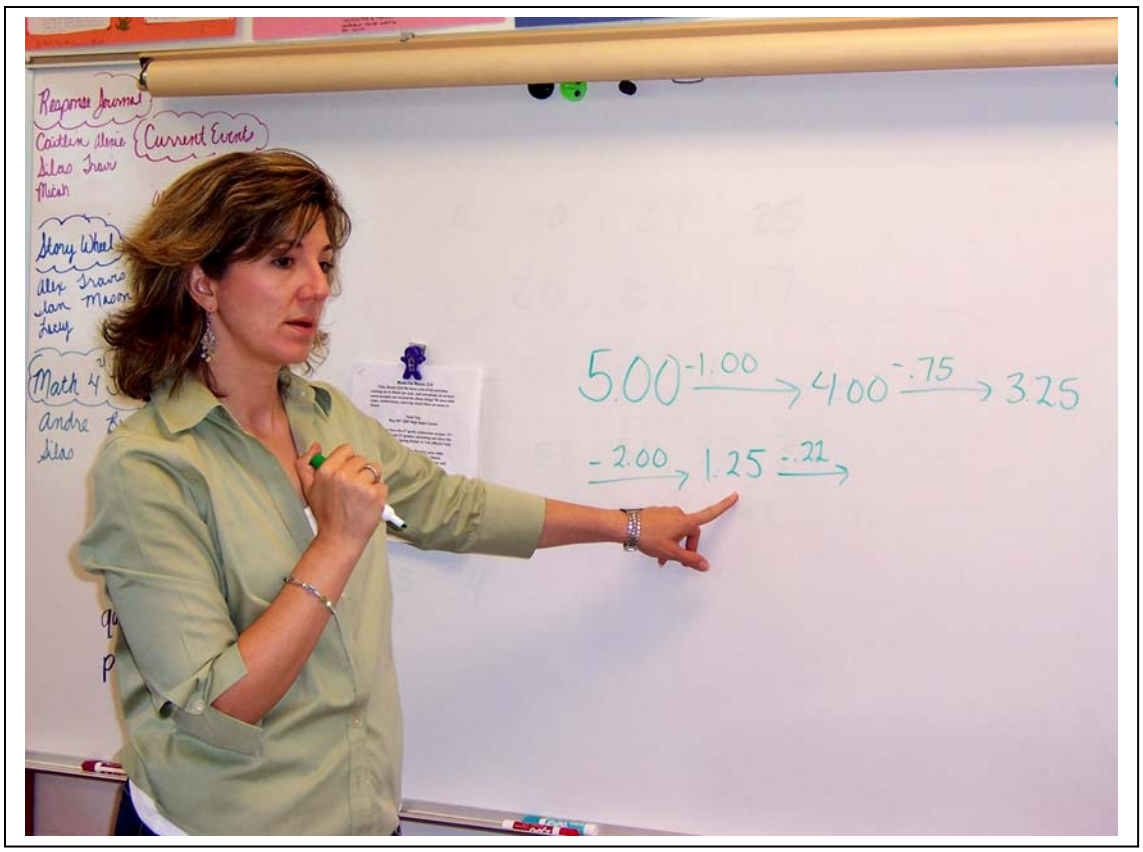
Three possible solutions for the following Separate Change Unknown problem are shown below using arrow language.

> *Madelyn has 132 stamps in her stamp collection. She gave some of her stamps to Angie. Now she has 93 stamps left. How many stamps did she give to Angie?*



Teaching Tips

- CAN REVEAL A STUDENT'S KNOWLEDGE OF NUMBER ORDER AND MAGNITUDE
- ALIGNS CLOSELY TO STUDENTS INTUITIVE MENTAL STRATEGIES AND SHOULD REFLECT THEIR THINKING RATHER THAN BE TAUGHT AS A PROCEDURE TO BE USED WITH ALL COMPUTATIONS
- TRACKS ERRORS AND ALLOWS STUDENTS TO THINK ABOUT WHAT TO DO NEXT WHEN A PARTICULAR COMPUTATION IS TOO DEMANDING TO DO "MENTALLY"
- WRITE ARROWS FROM LEFT TO RIGHT ONLY
- CAN BE USED FOR ALL OPERATIONS AND MORE THAN ONE OPERATION IN A SERIES
- LENGTH OF THE "THINKING TRAIN" WILL SHORTEN AS STUDENT'S MENTAL COMPUTATIONS IMPROVE FOR A GIVEN SET OF NUMBERS AND OPERATION
- DO NOT END THE TRAIN WITH AN EQUAL SIGN



Array Model (for Multiplication)

This model may help student see a connection between repeated addition and multiplication as well as develop an understanding of the commutative property for multiplication.

Students model the array using Unifix cubes or with pencil and paper.

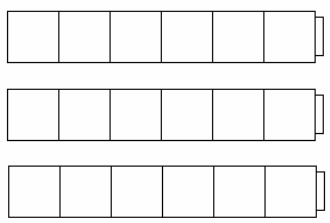
For the problem:

> *The music teacher wants to arrange the chairs in 3 rows with 6 chairs in each row. How many chairs will the teacher need?*

Using symbols:

X X X X X X
X X X X X X 3
X X X X X X
6

Using Unifix cubes:



Using an area model (discussed on the next page):

Area Model (for Multiplication)

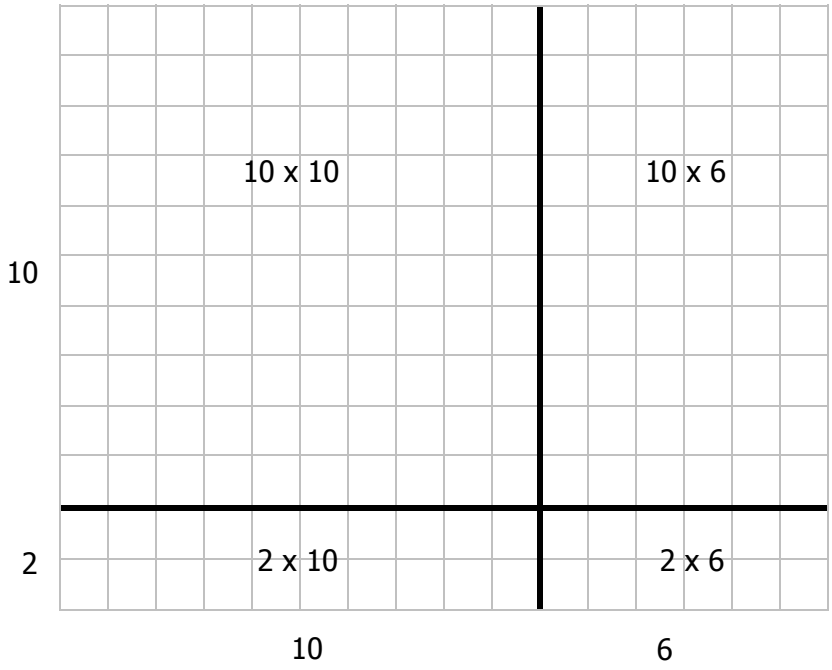
An area model helps students keep track of the partial products of a multi-digit computation and understand the distributive property.

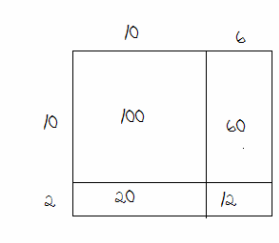
Introduce the area model using graph paper first. Later students can sketch the model without showing all of the units.

For example students could model 16 x 12 on graph paper as:

10 x 10 = **100**
10 x 6 = **60**
2 x 10 = **20**
2 x 6 = **12**

100 + **60** + **20** + **12** = **192**





A sequence of models:

3 x 4
10x4
30x4
30x40
30x46
35x46

Teaching Tips

- STUDENTS BEGIN BY MODELING SINGLE-DIGIT MULTIPLICATION THEN 10 X SINGLE-DIGIT, DECADE X SINGLE DIGIT, DECADE X DECADE, DECADE X DOUBLE-DIGIT (NON-DECADE), NON-DECADE DOUBLE-DIGIT X NON-DECADE DOUBLE-DIGIT,
- STUDENTS SHOULD KNOW BASIC MULTIPLICATION FACTS, 10X10, AND MENTAL COMPUTATION STRATEGIES BEFORE USING THIS MODEL
- MARK GROUPS OF TEN ON THE GRAPH AND LABEL THE SIDES OF GRAPHED AREA
- ASK STUDENTS TO RECORD THE PARTIAL PRODUCTS IN A LIST
- COMPARE THIS STRATEGY TO OTHER ALGORITHMS TO FIND CONNECTIONS
- EVENTUALLY STUDENTS MAY SIMPLY SKETCH AN OUTLINE AND USE THE MODEL AS A GRAPHIC ORGANIZER FOR THEIR THINKING.

Ratio Table

The ratio table helps students keep track of multiplicative relationships (proportions). It is useful for solving grouping and partitioning & measurement division problems.

Introduce the ratio table with a story problem that involves an obvious grouping. For example:

A car has four wheels. How many wheels will there be on 48 cars?

You may see students add one or two groups at a time, double groups, use 10s or a combination of these strategies as seen below for the following measurement division problem:

How many wheels would 48 cars have?

cars	1	2	4	8	16	32	48
wheels	4	8	16	32	64	128	192

cars	2	4	6	8	10	20	30	40	48
wheels	8	16	24	32	40	80	120	160	192

cars	10	20	40	8	48
wheels	40	80	160	32	192

cars	10	40	8	48
wheels	40	160	32	192

Teaching Tips

- USEFUL FOR ALL GROUPING & PARTITIONING PROBLEMS (MULTIPLICATION, MEASUREMENT DIVISION, PARTITIVE DIVISION)
- STUDENTS DECIDE HOW THE RATIO "GROWS" AND "SHRINKS"
- OBSERVE HOW STUDENTS REASON USING THE TABLE
- RATIO TABLES WILL SHORTEN AS MENTAL COMPUTATIONS IMPROVE BASED ON IMPROVED NUMBER SENSE – BE SURE STUDENTS JUSTIFY THEIR CHOICE OF NUMBER

Equations

A sequence of equations can be used to keep track of a computation. For example, students could use a series of equations to represent the solution to:

> *A baker could make 6 dozen cookies at one time (in a batch). He can make 8 batches in one hour, how many cookies can he make in 4 hours?*

6 x 10 = 60	70 x 8 = 560	500 x 4 = 2000
6 x 2 = 12	2 x 8 = 16	70 x 4 = 280
60 x 12 = 72	560 x 16 = 576	6 x 4 = 24
		2000 + 280 + 24 = 2304

Teaching Tips

- A SERIES OF EQUATIONS CAN BE USED TO SHOW COMPUTATIONS FOR ALL OPERATIONS
- STUDENTS DECIDE WHICH EQUATIONS BEST REPRESENT THEIR THINKING
- THE TEACHER CAN ACT AS A SCRIBE TO MODEL HOW EQUATIONS CAN SHOW THINKING STEPS.
- ASK STUDENTS TO LOOK FOR CONNECTIONS BETWEEN EQUATIONS AND OTHER REPRESENTATIONS

Algorithms

An algorithm is "a finite step-by-step procedure for accomplishing a task that we wish to complete." (Usiskin 1998, p. 7) More specifically, "an algorithm consists of a precisely specified sequence of steps that will lead to a complete solution for a certain class of computational problems." (Bass, TCM, 2003 p.323)

Before teaching any algorithm teachers consider:

- What understandings/skills does a child need to understand the algorithm?
- What understandings/skills does a child need to do the algorithm?

There has been much confusion over whether children should learn "standard" or more traditional algorithms. Research has shown that teaching standard algorithms too early acutely interferes with student's growing number knowledge (Kamii, 2004). The question of whether to teach the standard algorithms is really a question about whether the student understands the math behind them.

The standard four algorithms we learn are quick to memorize but error prone for the same reason. While efficient, they hide the mathematical concepts needed to build number sense. Students can generally justify their use only as "tricks" or something "my teacher taught me." Learning algorithms without concurrent sense-making results in ungrounded "competence" with mathematics.

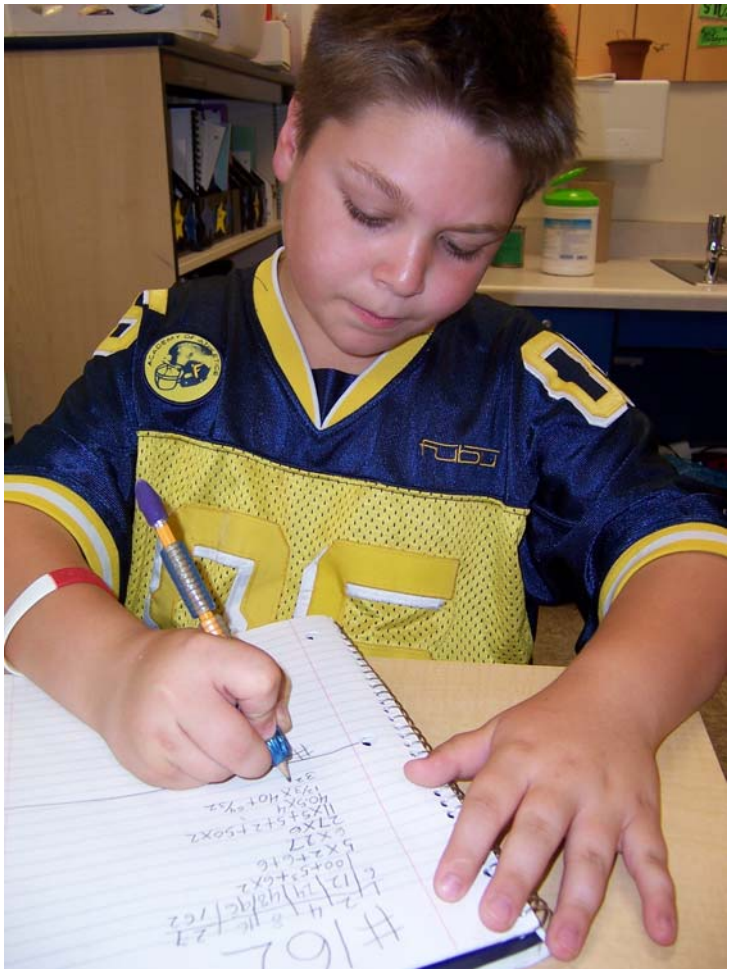
Algorithms do have practical uses and theoretical importance in mathematics. They should be judged on their reliability, generality, efficiency, ease of accurate use, and transparency. The primary goal of teaching algorithms should focus on the analysis of why they work and comparison between algorithms. Comparing two or more algorithms for a given computation can lead to a deeper understanding of mathematics and expand a student's flexibility to choose an appropriate method for a given set of numbers.

Students become more discriminating about which algorithms or methods to use when teachers encourage students to explore and discuss alternative approaches. Students should always be able to explain how the methods or algorithms they choose work and answer questions about the mathematical concepts that may be "hidden" in any approach they use.

Writing equations horizontally can prompt student to use number sense and mental strategies by encouraging students to pause and consider the numbers before they solve.

Teachers should frequently assess whether students have become overly reliant on a few algorithms rather than using number sense to determine the best approach for a particular set of numbers.

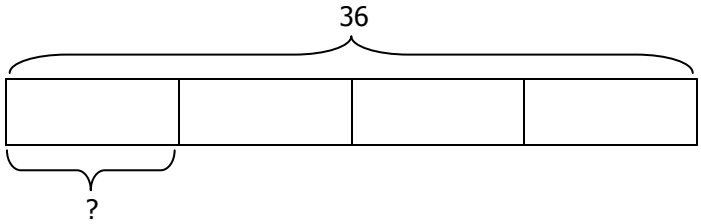
For example, students with a good sense of number order won't use an algorithm to solve 1000 — 9. However, students who rely too heavily on the standard subtraction algorithm will "borrow" from the tens, hundreds, and thousands and often make errors in the process of this simple calculation.



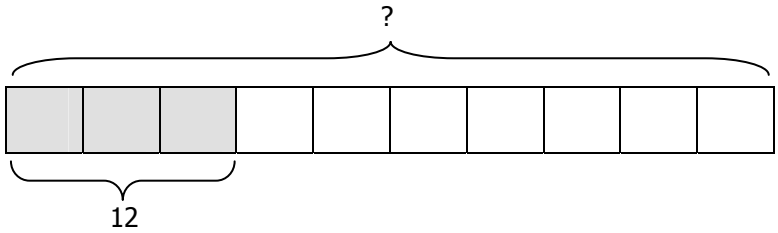
Bar Model

This "linear" model represents the various relationships in a story problem. Students may have to represent the computation needed to arrive at the answer depending on the numbers in the problem.

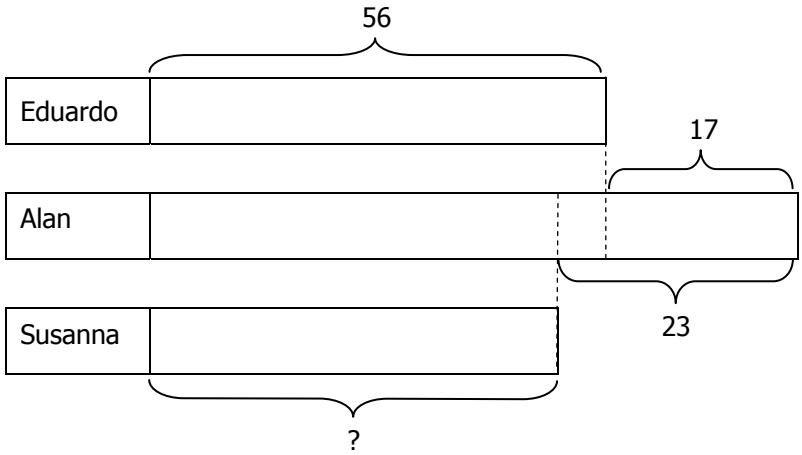
Pamela has 36 books. If she puts them 4 shelves equally, how many does she put on each shelf?



Hillary spent 3/10 of her money on a CD. If the CD cost $12 how much money did she have left?



Eduardo has 56 trading cards. Alan has 17 more trading cards than Eduardo. Susanna has 23 fewer than Alan. How many cards does Susanna have?



Using Story Problems to Build Fraction Knowledge

Understanding fractions is critically important for the future study of algebra and more advanced mathematics. The goal of fraction instruction is "to discuss and extend concepts that emerge while students solve problems." (Saxe et al, 1999)

Just as in other domains of teaching number, teaching fractions begins with solving story problems. Using story problems helps avoid misconceptions. Common limited conceptions intermediate students have about fractions include:

- Fractions are pieces
- Fractions are always smaller than a whole
- Fractions values are determined by counting parts
- Fractions are two numbers

To develop fraction concepts, teachers:

- ☑ choose or design problems based on assessment and the fraction concepts they want to address. See Table 6.4 for problem types and sample problems
- ☑ select numbers that result in answers with specific fractional parts to address learning goals
- ☑ analyze the strategies students use to solve the problems
- ☑ help children coordinate fraction concepts and fraction symbols

"Equal Sharing" problems provide a good starting point for fraction study especially when designed with contexts familiar to students. These problems often result in a variety of solutions that lead to discussions about equivalence.

"Equal Groups" and "Division" problems provide a logical "next step" that involves combining like fractional units and provides discussion about the need for a common unit.

Teachers should introduce symbols, number sentences, and use mathematical language to go with student's strategies as they solve problems.

- ☑ ask students to draw or use physical materials to create models of fractional amounts (draw, fold, cut, or shade)
- ☑ use fraction words such as "two-thirds of a candy bar" or "a third plus a third," before writing fraction symbols
- ☑ relate unknown fractions to well-known fractions such as ½ or ¼ ("It's more than a fourth but less than a half." or " It's smaller than one-fifth.")
- ☑ use language that emphasizes the relationship of fractional quantity to the unit instead of the number of pieces ("How many of this piece would fit into the whole candy bar?" instead of "How many pieces is the candy bar cut into?")
- ☑ use "How much?" rather than "How many?" to signal a fraction answer is needed.

The following problem types build a solid foundation in understanding fractions.

Comparison

Division (total divided by the number of groups)

Joining/Separating (combining like and unlike units)

Equal Groups

Division (total divided by the size of the group)

Operator

Equivalence

Fraction Problem Types

Problem Type	SAMPLE PROBLEMS
Comparison	• Who will get more pizza: a child at a table where 5 children are sharing a medium pizza or a table where 6 children are sharing a medium pizza? • Susan and Jeremiah each order the same sized pizza. Susan ate 3/4 of her pizza. Jeremiah ate 5/6 of his pizza. Who ate more pizza?
Division (total divided by the number of groups)	Equal Sharing with an answer > 1: • Two children want to share 5 cookies equally. How much can each child have? • Four children want to share 10 candy bars so that each one gets the same amount. How much can each child have?
	Equal Sharing with an answer < 1: • There is 1 brownie for 4 children to share equally. How much brownie can each child have? • Three children want to share 2 candy bars equally. How much will each child get? • At a birthday party, 2/3 of a watermelon is left on the table. There are 4 children at the party who want to share this leftover watermelon. They all want the same amount and they want to finish it off. How much can each child have?
Joining/Separating	Combining like units: Janie has 4/5 of a gallon of blue paint left over from painting her room. John has 3/5 of a gallon of the same blue paint left over from painting a table. How much blue paint do they have?
	Combining unlike units: • Janie has 3/4 of a gallon of blue paint left over from painting her room. John has 3/8 of a gallon of the same blue paint left over from painting a table. How much blue paint do they have? • Jason ate 2/3 of an ice-cream sandwich. He let the rest melt. But he was still hungry, so he ate 5/6 of another ice-cream sandwich. He let the rest melt. Jason's brother told him he had eaten a lot. Jason didn't think so. Did Jason eat more or less than 1 whole ice cream sandwich altogether? How much did Jason eat?
Equal Groups	Eric and his mom are making cupcakes. Each cupcake gets 1/4 of a cup of frosting. They are making 20 cupcakes. How much frosting do they need?
	Backwards sharing context: Six friends shared some cookies. Each person got 2 2/3 cookies. How many cookies did they have altogether?
Division (total divided by the size of the group)	• Ollie has a snow cone machine. It takes 2/3 of a cup of ice to make a snow cone. How many snow cones can Ollie make with four cups of ice? • You can make 12 peanut butter sandwiches with a jar of peanut butter. How much peanut butter do you need to make 15 sandwiches?
Operator	Myra brought 18 pencils to school today. One-third of the pencils have stripes. The rest were plain. How many pencils had stripes? How many were plain? If the soccer team sold 300 raffle tickets, it would have enough money to pay for new team shirts. So far the players have sold 2/3 of the tickets. How many more tickets do they need to sell?
Equivalence	In art class, the teacher is handing out sheets of construction paper to groups of children for a project. There are 2 children at one table. The teacher gives them 3 pieces of construction paper to share equally between themselves. At another table there are 6 children. How many pieces of construction paper should the teacher give them so that each child gets as much paper as a child at the first table?

Geometry

These problems challenge students to develop spatial reasoning or spatial sense by working with figures and shapes to discover relationships, see shapes within shapes, sort by attributes (including symmetry), and examine the results of flipping, sliding and turning shapes. Typical geometry problems in the intermediate grades include "filling space" with pentominoes or tangrams, decomposing and composing shapes, understanding point of view and "footprints" of 3-D shapes.

Intermediate students also develop definitions of classes of shapes and begin to use geometric vocabulary. Students expand their knowledge of the properties of figures. They apply properties to entire classes of figures and shapes rather than to individual models and determine new properties. For example, they find ways to sort triangles into groups and define the different types.

During the intermediate grades teachers ask questions that lead to making and testing hypothesis. Questions such as "Does that work all of the time?" or "Is that true for all triangles or just equilateral triangles?" causes students to expand their understanding of shape.

Geometry in the intermediate grades includes:

- understanding what makes shape different and alike based on geometric properties
- describing shapes in terms of their location in a plane or space (coordinates)
- learning how shapes can be moved in a plane (translations)
- "seeing" shapes from different perspectives and understanding relationships between 2-D and 3-D figures
- transforming, decomposing and composing shapes mentally

Teachers can use the school district's curricular resources for geometry problems. Additional resources for geometry problems are listed in *For more information* at the end of this chapter.

Geometry Activities with Manipulatives

The following activities introduce 2- & 3-dimensional shapes and provide opportunities for students to develop and use geometric language. Teachers or students should chart the vocabulary they use during these opening investigations and re-address the charted vocabulary often.

- ☑ EXPLORE - Students become familiar with shape through building and talking about their constructions.
- ☑ SORT AND CLASSIFY – Sort a set of geometric manipulatives and talk about the sort. Make a list of the properties for the group. Alternatively, organize the different manipulatives so that each one is related to the next one in some way. (e.g. line up the pentominoes so that each pentomino has only one square in a different position than the one next to it.)
- ☑ ATTRIBUTES - Students need time to learn the attributes of the manipulatives such as size, number of sides, symmetry, corners. Activities such as "Guess My Shape" (one student builds a shape, the other student must make the same shape by listening or reading directions) or "Guess My Rule" (Venn diagram) provide excellent opportunities for students to build their geometric vocabulary.
- ☑ FRACTIONAL PARTS **-** Ask students to find combinations of shapes that make other shapes. Then ask students to identify the fractional relationships between the shapes and explain their reasoning.
- ☑ TRANSFORMATIONS – Ask students to determine which shapes become new shapes through a transformation (slide, flip, turn) and which shapes stay the same.
- ☑ SIMILARITY/CONGRUENCE – Ask students to determine which shapes, if any, are similar or congruent to other shapes.
- ☑ DRAW – Have students draw both two and three-dimensional shapes. Asymmetric dot paper allows students to draw three-dimensional shapes and see relationships between sides.

Geometry in the Problem Solving Block

Geometry problems challenge students to think about spatial relationships, see shapes within shapes, sort by attributes (including symmetry), and examine the results of transformations such as flipping, sliding and turning shapes. When planning Geometry instruction for each student, teachers use the MMSD K-5 Grade Level Mathematics Standards as a guide. The following table summarizes the Geometry standards for grades 3-5. **Boldface** type indicates "new" for the grade-level. See the appendix for introductory ideas for geometry problems.

Third grade students:

Investigate circles, polygons (**octagon,** hexagon, rhombus, trapezoid, parallelogram, square, rectangle, and triangle), polyhedrons (pyramids, cube, **hexagonal-, octagonal-,** triangular-, square- & rectangular prism) and other solids (**hemisphere,** sphere, cylinder, cone). Child:

- **compares attributes of a classification**
- **identifies properties of shapes**
- names shapes and **uses geometric language (Ex. side, face, vertex, edge)**
- builds with geometric shapes or computer models (Ex: **Geo-logo, tetrominoes,** Geoboards, pentominoes, square tiles, Geoblocks, pattern blocks)
- predicts the results of putting together and taking apart shapes
- sorts and classifies shapes according to attributes
- matches geometric models to shapes in the environment

Draw:

- the geometric shapes of objects in the environment
- a two-dimensional figure (rectangle, triangle, square, circle)
- front or top view of a three-dimensional object

Investigate the symmetry of two-dimensional shapes by:

- determining which movements leave plain (un-patterned) shapes unchanged (Ex. sliding, **rotating half-turn, quarter-turn, up-down flip, sideways flip**)
- folding paper to make a shape with mirror symmetry
- using a mirror to **identify all lines of symmetry for a given object**

Determine multiple nets(flat patterns)of a cube and square pyramids by:

- constructing and deconstructing shapes
- drawing nets

Specify locations, spatial relationships, and movement. Child:

- locates points on maps and simple coordinate grids with letters and numbers
- represents points and simple figures on maps and simple coordinates grids with letters and numbers
- solves problems involving shape, movement, and space
- uses words such as **½ turn, full turn, parallel, perpendicular, intersection, adjacent to, interior of,** forward, back, right, left, near, far, over, under, next to, and between

Use geometric models to solve problems in other areas of mathematics such as number and measurement. (Ex. area model of multiplication or fractional parts, filling an open box to determine volume)

Fourth grade students also:

- **identify and describe 3-dimensional shapes from multiple perspectives**
- **determine the number of faces, edges, and vertices (corners) given an illustration of a 3-dimensional figure**
- compare attributes of a classification
  - **parallel sides**
  - **number of sides (two-dimensional shapes)**
- name shapes and use geometric language (Ex. **lines**, **line segment, parallel, perpendicular, right angle**)
- build with geometric shapes or computer models (Ex: **Geoboards or dot paper**)

Demonstrate an understanding of symmetry of two-dimensional shapes.

- determine which movements leave plain (un-patterned) shapes unchanged (Ex. **180 degree turn**, **90 degrees**)

Determine multiple nets (flat patterns) of **geometric solids.**

Specify locations, describe spatial relationships, and movements.

- **state the coordinates of points, objects and simple figures on maps or one-quadrant coordinate grids**
- locate **and plots points on maps and one-quadrant coordinate grids**
- use words such as **congruent and similar**

Fifth grade students also:

Investigate understanding of **irregular** polygons (3-, 4-, 5-, 6-, 8-sides):

- **classify plane figures by characteristics of angles**
- name shapes and uses geometric language (Ex. **ray, line, line segment, acute-, obtuse-)**

Demonstrate an understanding of symmetry of two-dimensional shapes.

- **design shapes that have at least one-line of symmetry**
- **determine congruency and similarity of shapes**

Specify locations, describe spatial relationships, and movements.

- **locate the fourth coordinate pair when given three vertices of a quadrilateral on a one-quadrant coordinate grid**

Measurement

Measurement is the most common use of number in everyday experience. The topic of measurement brings together number and geometry. When students figure out how many inch-cubes fill a container they use number to communicate about three-dimensional space.

The use of number to represent measurement requires abstract thinking. As children move from counting concrete objects to counting space they often do not realize that the numbers refer to the entire space they are counting. For example, they count the marks on a ruler or on a cup as if they are discrete objects rather than continuous quantities.

Zero point is another concept that children often miss. Providing "broken" rulers (part of a ruler without a zero mark) provides practice and a quick assessment of students' understanding of zero point (starting point of a measurement). Also, asking students to measure a crooked line serves as an easy assessment of the fundamental concepts of measurement.

Manipulating units to measure length, weight, capacity, and volume and learning how to use tools to measure these attributes serves as the focus for problem solving. Measurement can be a topic for story problems.

For instance, students collect measurements in a science experiment to construct a table of data. Teachers then use the measurement data to write a series of story problems.

Measurement conversions provide a natural grouping and partitioning situation for problem solving. Groups of 16 ounces become one pound, inches become feet and so on. Many experiences with measurement conversions develop proportional reasoning.

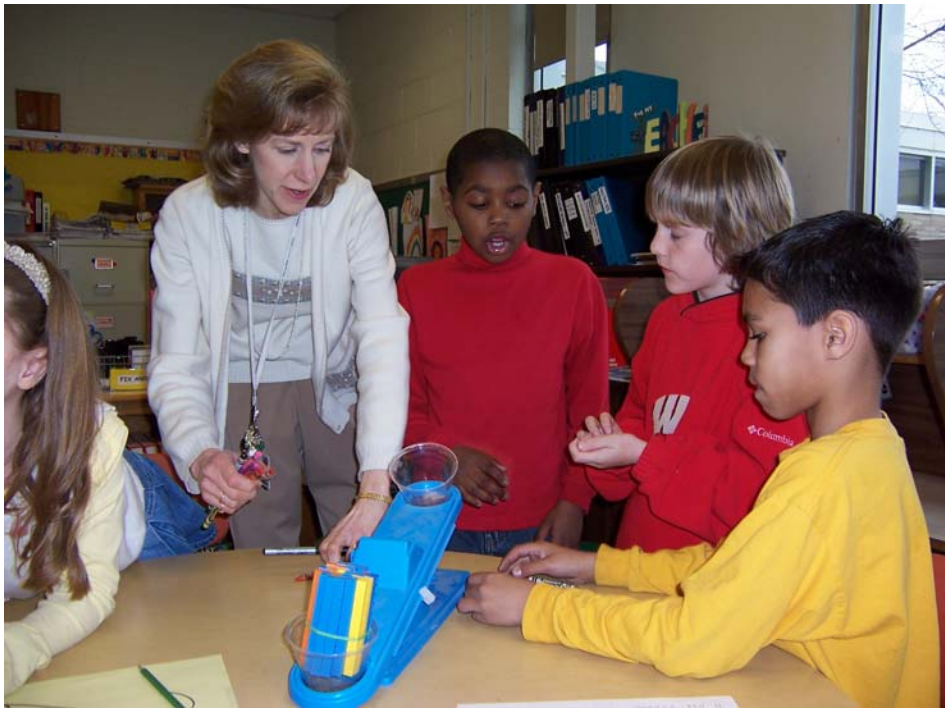
Linear measure provides an excellent context to develop understanding of fractional parts. Both naming fractions and combining or separating fractional parts emerges from work with linear measure.

Area measurement provides a context to develop understanding of multi-digit multiplication and supports algebraic concepts such as the distributive property.

Measurement in the intermediate grades includes:

- understanding attributes and comparing objects with the same attribute
- becoming familiar with common units used for measuring
- estimating measurements from personal benchmarks
- understanding measurement tools
- exploring area and volume formulas
- understanding how area, perimeter, and volume are related

The school district's curricular materials for both math and science are sources for measurement problems.



Measurement in the Problem Solving Block

Measurement problems involve relationships between number and shape, understanding measure attributes of objects and the units, systems, and processes of measurement. Students develop appropriate techniques, tools, and formulas to determine measurements. Measurement problems can emerge from work in science. When planning to teach concepts related to Measurement, teachers use the MMSD K-5 Grade Level Mathematics Standards as a guide. The following table summarizes the Measurement standards for grades 3-5. **Boldface** type indicates "new" for the grade-level.

Third grade students:

Name, discuss, compare, and order objects according to attributes of, weight, capacity, area, length (perimeter), and temperature through observation or actual measurement.

Demonstrate an understanding of measurement concepts including**:**

- choosing an appropriate tool and unit (Ex. inches, centimeters, miles, feet, yards, millimeters, cups, quarts, gallons, liters, pounds, ounces, grams, degrees F/C)
- apply estimation techniques using non-standard measure
- **zero point (any point can act as the starting point of a measurement)**
- **iteration (repeatedly laying one unit next to an object to measure its length)**
- **subdividing units to increase the precision of a measurement**
- **the relationship between the size of the unit and the number of units needed to make a measurement**
- **the necessity for identical units**
- **conventions for communicating measurements by identifying the quantity and the name of the unit (Ex. 12 strips of paper)**

Measure length (perimeter)**, area, capacity, mass, weight, and temperature. Child:**

- **solves problems involving measurement**
- **selects appropriate measurement tools and units (standard and non-standard)**
- **measures with accuracy to the nearest ½", cm** *(WKCE in 3rd grade)*
- **measures area by iteration (Ex: square tiles covering a surface)** *(WKCE in 3rd grade)*
- **reads a thermometer to the nearest 5 degrees F/C** *(WKCE in 3rd grade)*

Estimate measurements using:

- non-standards units (Ex: estimation jars, paper clips, square tiles)

Tell time to the nearest minute using analog and digital clocks. Child:

- translates time between analog and digital clocks
- **records time**

Identify increments of time:

- **seconds, minutes, days, months, years**
- **minutes grouped by fives**
- **benchmarks of 15, 30, 45**
- **twelve numbers indicate 12 hours**

Fourth grade students also:

Name, discuss, compare, and order objects according to attributes of, weight, **volume and liquid** capacity, area **(regular and irregular)**

Demonstrate an understanding of measurement concepts including:

- converting measurement units (inches/feet/yards, cups/pints/quarts/gallons)
- **choosing an appropriate unit (**miles)

Estimate measurements using:

- proportional contexts (Ex. using map scales)

Tell time to the nearest minute using analog and digital clocks:

- compare elapsed time in problem-solving situations (across two adjacent hours in quarter-hour increments)

Convert units (minutes/hours/days/months/years)

Fifth grade students also:

Demonstrate an understanding of measurement concepts including:

- additivity (the measurement of the whole is equal to the sum of the measure of the parts)
- knowing that all measurements are approximations
- knowing how differences in unit size affects precision
- converting measurement units (millimeters/centimeters/meters, grams/kilograms)

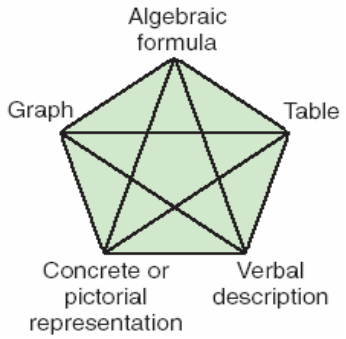
Measure angles, **length (**perimeter of regular and irregular shapes)**, area** (rectangles and irregular shapes on a grid) **and** volume.

- **measures length and perimeter to the nearest** mm (WKCE in 5th grade)
- **reads a thermometer to the nearest** 1 degree **F/C** (WKCE in 5th grade)

Estimate measurements using:

- common benchmarks (Ex. a paperclip has a mass of about one gram)
- U.S. customary measurements

Data Analysis & Probability



Data analysis is about gathering data to answer questions. When students formulate their own questions, data analysis becomes more meaningful.

During the intermediate grades, students expand their knowledge about which data to gather, how to organize the data, and how to interpret the results.

Data analysis also provides an excellent site for the study of multiple representations such as objects, pictorial, verbal descriptions, tables, different kinds of graphs, and possibly formulas. Teachers should engage students in conversations about the usefulness of each representation as well as connections between the various representations. Questions about sets of data should include questions that promote additive thinking, "How many more hot than cold lunches?" and multiplicative thinking, "About how many times more hot than cold lunches?"

During the intermediate grades, students explore the "shape" of data including the spread (range, variance) and the measures of center (mean, median, mode). They begin to predict how these "characteristics" might change when the conditions of the data collection changes.

Data concepts include:

- data is gathered and organized to explore questions
- a sample (collection of data) can provide insight but requires inference
- data can be analyzed in various ways
- data displayed in different ways conveys different information
- data usually raises more questions

During the intermediate grades, teachers depend on everyday events to provide plenty of opportunities to explore probability. They use the language of probability to talk about future events and engage students in simple experiments.

Data Analysis & Probability in the Problem Solving Block

When planning to teach concepts related to Data & Probability, teachers use the MMSD K-5 Grade Level Mathematics Standards as a guide. The following table summarizes the Data & Probability standards for grades 3-5. **Boldface** type indicates "new" for the grade-level.

Third grade students:

Design investigations to address questions that will lead to data collection and analysis.

- **determine what data to collect, when and how to collect it**
- predict possible results and their implications

Collect and organize data from:

- **observations**
- **surveys**
- experiments

Create appropriate representations of data such as:

- **tables and charts**
- **bar graphs**

Describe the important features of a set of data including:

- **shape**
- **high and low values (minimum and maximum)**
- **difference between the high and low values (range)**
- **most frequent value (mode)**

Discuss possible conclusions and **implications** based on the data**.**

Use data presented in Venn diagrams, tables, charts, and graphs (picture and bar) to answer questions.

Describe familiar events as impossible or certain (more, less, or equally likely) to occur.

- **describe the likely outcome of a simple event (e.g. one toss of a coin, one role of a number cube)**
- **design fair and unfair spinners**

Fourth grade students also:

Design investigations to address questions that will lead to data collection and analysis.

Determine the important features of a set of data (7 items or fewer) including **middle value (median)**

Use data presented in line plots

Describe familiar events as impossible or certain (more, less, or equally likely) to occur.

- **test predictions using data from a variety of sources**
- **use words to express probability**

Fifth grade students also:

*Determine the important features of a set of data (10 or fewer items) including **average value (mean**)*

*Create appropriate representations of data such as **line plots***

Predict outcomes or trends from graphs and tables

Determine and describe the possible combinations of three items

Describe familiar events as impossible or certain (more, less, or equally likely) to occur.

- **test predictions using data from a variety of sources**
- use words, **percents, and fractions** to express probability

Classroom Discourse

"Supporting productive discourse is easier when the mathematical tasks allow for multiple strategies from core mathematical ideas that are of interest to students."

Franke. M., et.al 2007

Teachers and students need to develop a learning community where the discourse about mathematics supports learning by all participants. The primary role of this community is to understand and extend each member's thinking.

Providing a problem-solving block that includes small group explorations and discussion allows both teachers and students to learn how to engage in productive discourse targeted at specific learning goals for individual students. All classroom conversations whether between individuals, small, or large groups of students should focus on understanding mathematical reasoning.

Teachers notice the diversity of thinking and utilize it as the basis for lessons or extensions of the mathematics. They allow discussion to flow as needed for change in conceptual understanding. They notice in what ways students participate and invite all students to engage in conversation. This means that teachers encourage students to develop a "need to know" attitude for themselves. This is an environment where students have confidence that all ideas are valued and are comfortable contributing ideas.

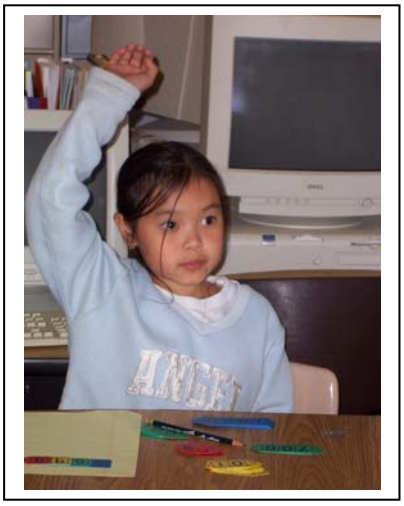
In this setting, students will respectfully interject their own ideas into conversations and ask questions that necessitate justification. They will question to satisfy their own need to understand.

"…..no matter how lucidly and patiently teachers explain to their students, they cannot understand for their students."

Schifter and Fosnot, 1993.

Elements of classroom conversations include (Huffer, K. Ackles, et.al., JRME, 2004):

- questioning
- explaining
- discovering mathematical ideas
- accepting responsibility for learning



Use think, pair, share strategies to support language development between students.

During classroom conversations teachers coach and assist students to:

- ☑ explain their thinking
- ☑ initiate and solicit questions
- ☑ make the details of their explanations more complete
- ☑ defend or justify their answers and ideas with mathematical reasoning
- ☑ compare and contrast ideas
- ☑ be responsible for listening and co-evaluating strategies
- ☑ help each other sort out misconceptions and correct errors

It takes time for a classroom community to develop where students take a leading role in classroom conversations about math. Teachers play a central role in developing this learning community. Step-by-step changes such as the following can affect classroom conversations that benefit the learning community:

- ☑ establish classroom norms for sharing, including explicitly telling students why they are sharing strategies
- ☑ model questioning that focuses on strategies rather than answers
- ☑ limit questions to things that are not immediately obvious
- ☑ assist students to clarify their thoughts and become more explicit in their communication
- ☑ coach students to ask genuine questions rather than mimic questioning techniques
- ☑ provide productive rather than punitive feedback
- ☑ help students feel confident that their mistakes are sites for learning

Math education researchers (Franke, M. et al, 2007) have identified that five practices make student thinking the focus of classroom discussions:

1. Anticipate likely student responses to mathematical tasks (problems).
2. Monitor student responses to the tasks as they work on them.
3. Select particular student responses to present for sharing.
4. Purposefully sequence the student responses that will be displayed and highlighted.
5. Help students make mathematical connections between different student responses.

When students expect to go beyond simply reporting how they solved a problem to asking questions and justifying each other's thinking they learn to put mathematical ideas together, identify and explain errors in their thinking, and develop mathematical arguments and reasoning.

Math & Literacy

The language used in school for academic subjects provides a connection between math and literacy for all learners. All students are academic language learners. All students benefit from strategies used to help English Language Learners acquire a second language.

English Language Learners

Language acquisition occurs in four domains: listening, speaking, reading and writing. When organizing a classroom, it's important to consider that some students may have varying levels of English language proficiency and background knowledge in these four domains. In order for students to become proficient in English, it is important to give them practice in each domain throughout the day and during the math block.

Students generally acquire listening skills first. While in the first stage of language acquisition, a student may participate only in their native language, use gestures, or model using math manipulatives.

Students need ready access to manipulatives at all times to support language acquisition.

As students increase in their listening abilities, they speak with increasing confidence. Reading and writing are the last domains to solidify.

A student who is fully proficient in reading and writing in their home language will be more capable and confident in English.

When teaching students who are acquiring a second language, teachers:

- ☑ use accessible language that includes:
  - new words used in many contexts
  - slowing down the rate of speech
  - enunciating
  - reducing use of complex speech, idioms, and jargon
- ☑ develop an awareness of each student's level of proficiency in listening, speaking, reading, and writing to build on what students already do (see Appendix for "Can Do Descriptors")
- ☑ build on the mathematics students already know based on assessment
- ☑ encourage use of models (manipulatives) to explain mathematical thinking
- ☑ support growth in content knowledge with the use of home language in the classroom. Support includes:
  - clustering students with like languages for small group problem solving and follow-up discussion
  - providing the support of a bilingual resource teacher or bilingual resource specialist
  - providing written translation when possible
  - allowing students to write in their home language

Note: The teacher report generated from the ACCESS for ELL's assessment gives information for where students fall in each of the four domains as well as where they are in oral proficiency (listening and speaking combined) and literacy (reading and writing combined).

Math Vocabulary

It is important for students to know and become comfortable using math vocabulary. However, math vocabulary is not simply naming an object or procedure. For example, a rectangle is a particular class of shape that has the specific attributes of a four-sided polygon with two sets of parallel sides and four right angles. This means that a square is a rectangle. Ways to develop math vocabulary are similar to building vocabulary in any content area. For example:

- ☑ use new words in many contexts
- ☑ promote intentional conversation about math language in all of the contexts in which the students encounter the words
- ☑ develop meaning as part of defining
- ☑ add new vocabulary to a word web or Venn diagram rather than a simple list or word wall
- ☑ use appropriate labels clearly and consistently
- ☑ incorporate lessons on content vocabulary such as Latin measurement pre-fixes (centi-, milli-, deci-) as needed
- ☑ avoid teaching "key words"

The Problem with Teaching "Key Words"

Teachers sometimes emphasize "key words" to help children solve story problems. For example, "all together" suggests addition, "left" suggests subtraction. The intent is good: Have students think about the situation. But, unfortunately children often misuse key words – they skim, looking just for the key words, or they trust them too much, taking what is intended as a rough guide as a dictum.

In the following examples, the underlined key words potentially mislead a child who does not read the whole problem or who does not think about the situation.

1. *Dale <u>spent</u> $1.24. Then Dale had 55 cents. How much did Dale have at the start?*
2. *Each classroom at one school has 32 children. The school has 12 classrooms. How many children are at the school <u>all together</u>?*
3. *Ben <u>divided</u> up his pieces of candy evenly with Jose and Cleveland. Each boy got 15 pieces of candy. How many pieces did Ben start with?*
4. *Flo has 3 <u>times</u> as much money as Lacy does. Flo has 84 cents. How much does Lacy have?*
5. *Manny's mother bought some things at the grocery store. She gave the clerk $10 and got $1.27 in change. <u>In all</u>, how much did she spend at the store?*

Ineffective Strategies

Students may use the following strategies when they have little experience solving story problems or typically rely on others to tell them how to solve problems. (Sowder, San Diego State Univ.)

Desperation Strategies

- Find the numbers and add or guess at the operation to be used

Computation-Driven Strategies

- Look at the numbers. They will "tell" you what operation to use.

- Try all operations and choose the answer that is most reasonable.

Limited Strategies using some degree of meaning

- Look for isolated "key words" to tell what operation to use. (e.g., "all together" would mean add, "left" would mean subtract, "of" would mean multiply)

- Decide whether the answer should be larger or smaller than the given numbers. If larger, try both + and x, and choose the more reasonable answer. If smaller, try both – and ÷ and choose the more reasonable answer.

Desired Concept-Driven Strategy

- Choose a solution strategy that fits the story

Math Journals and Written Feedback

The simplest prescriptions for improving education must be 'dollops' of feedback."

Marzano, 2003

Math journals or notebooks that provide plenty of space for students to work on problems allow students and teachers to keep a permanent record of progress which models the work of a mathematician.

A math journal has both the components of a work log and a workspace. Teachers help students build habits and routines to keep track of their work in their notebooks. Teachers use student journals as a resource to provide feedback. (See the appendix for examples for math journal pages)

"In skills teaching, the teacher tells the learner what to do and then "corrects" or "marks" the response.

In strategy teaching, the teacher induces the learner to behave in an appropriate way and encourages the learner to confirm or correct his own responses – the teacher does not usurp the control which is crucial to mastering a strategy."

Don Holdaway
Foundations of Literacy. 1979.

One of the most effective ways to engage children in reflective practice is for teachers to provide written feedback in the form of questions or comments rather than simply correcting work with checkmarks and grades.

Students learn to read and respond to the questions in the math journal or on workbook pages and correct their own work. Questions can ask students to look again at a solution, clarify thinking, compare a solution to a previous one, request that a student try a strategy that they are familiar with but haven't used in awhile, practice a few quick calculations, or look for a short cut.

Intermediate students enjoy these written exchanges that maximize learning.

- MARK PAGES IN THE JOURNAL THAT NEED STUDENT RESPONSE WITH A SMALL POST IT OR SYMBOL
- PROVIDE WRITTEN FEEDBACK FOR FOUR OR FIVE STUDENTS EACH DAY (MARK THE DAY FOR EACH STUDENT ON THE OUTSIDE OF THE JOURNAL)
- EXPECT STUDENTS TO RESPOND TO FEEDBACK AS PART OF THEIR INDEPENDENT WORK
- USE GENUINE QUESTIONS AND POSITIVE, GUIDING COMMENTS

For More Information:

Activities to Generate Discussion

Erickson, T. (1989). *Get it Together: Math Problems for Groups (Grades 4-12)*. Berkeley, CA: EQUALS / Lawrence Hall of Science.

Fulton, B. & Lombard, B. (2003). *The Language of Math: Helping Students Speak, Write, and Think Mathematically*. Millville, CA: Teacher to Teacher Press

Lilburn, P. & Rawson, P. (1995). *Let's Talk Math: Encouraging Children to Explore Ideas*. Portsmouth, NH: Heinemann.

Engaging in Discussion

Bickmore-Brand, J. (Ed.). (1990). *Language in Mathematics*. Portsmouth, NH: Heinemann.

Bushman, L. (2007). *Making Sense of Mathematics: Children Sharing and Comparing Solutions to Challenging Problems*. Reston, VA: National Council of Teachers of Mathematics.

Bushman, L. (2003). *Share & Compare: A Teacher's Story about Helping Children Become Problem Solvers in Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Chapin, S. H., O'Connor, C. & Anderson, N. C. (2003). *Classroom Discussions: Using Math Talk to Help Students Learn Grades 1-6*. Portsmouth, NH: Heinemann.

Corwin, R .B. & Trafton, P. (1996). *Talking Mathematics: Supporting Children's Voices*. Portsmouth, NH: Heinemann.

O'Connell, S. (2005). *Now I Get It: Strategies for Building Confident and Competent Mathematicians, K-6*. Sausalito, CA: Math Solutions Publications.

Spiegel, D. L. (2005). *Classroom Discussions: Strategies for Engaging All Students, Building Higher-Level Thinking Skills, and Strengthening Reading and Writing Across the Curriculum*. New York: Scholastic.

Fractions

Lamon, S. J. (2006). *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

Litwiller, B. & Bright, G. (Eds.). (2002). *Making sense of fractions, ratios, and proportions: 2002 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Geometry

Bachman, V. (2003). *Third-grade math: A month-to-month guide*. Sausalito, CA: Math Solutions Publications.

Confer, C. (2002). *Math by all means: Geometry grades 3-5*. Sausalito, CA: Math Solutions Publications.

Dana, M. *Geometry: Experimenting with shapes and spatial relationships. Grade 3*. Grand Rapids, MI: Instructional Fair, Inc. (Out of print, see an elementary math resource teacher) Also available for grades 4 and 5.

Findell, C. R., Small, M., Cavanagh, M., Dacey, L., Greenes, C. R. & Sheffield, L. J. (2001). *Navigating through geometry in grade 3 – grade 5*. Reston, VA: National Council of Teachers of Mathematics.

Bright, G. W. & Clements, D. H. (2003). *Classroom activities for learning and teaching measurement: 2003 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Dacey, L., Cavanagh, M., Findell, C. R., Greenes, C. E., Sheffield, L. J. & Small, M. (2003). *Navigating through measurement in grade 3 through grade 5*. Reston, VA: National Council of Teachers of Mathematics.

NCTM Navigations –best online source for lessons and lesson plans.

Usiskin, Z. (1998). Paper and pencil calculations in Calculator—and-Computer Age. *The Teaching and Learning of Algorithms in School Mathematics*. 1998 Yearbook of the National Council of Mathematics.

Number, Operations and Algebraic Relationships

Brickewidde, J. (2002). *Children's Development of Place Value Understanding*. Project for Elementary Mathematics. Unpublished manuscript, Hamline University, St. Paul, MN.

Carpenter, T. P., Fennema, E., Loef Franke, M., Levi, L. & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Coggins, D. (Ed.). (1999). *A mathematics sourcebook for elementary teachers: Key concepts, teaching tips, and learning pitfalls*. Novato, CA: Arena Press.

Cuoco, John A. (1987). A Case Study of a Highly Skilled Mental Calculator. *Journal for Research in Mathematics Education*, *18*(5), 331-342.

Kamii, C. (2004). *Young children continue to reinvent arithmetic: 2nd grade implications of Piaget's theory*. New York: Teacher's College Press.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Litwiller, B. & Bright, G. (Eds.). (2002). *Making sense of fractions, ratios, and proportions: 2002 NCTM Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Madison Metropolitan School District. (2004). *MMSD mathematics Grade Level K-5 standards*. Madison, WI: Retrieved April 29, 2009 from http://mathweb.madison.k12.wi.us/node/268

Pilar, R. (2004). *Spanish-English language issues in the mathematics classroom*. In Ortiz-Franco, L., Hernandez, N. G. & De La Crus, Y. *Changing the faces of mathematics perspectives on Latinos* (pp. 23-33). Reston, VA: National Council of Teachers of Mathematics.

Reys, R. E. (1984). Mental computation and estimation: past, present, and future [Special Issue: Mathematics Education]. *The Elementary School Journal*, *84*(5), 546-557.

Small, M., Sheffield, L. S., Cavanagh, M., Dacey, L., Findell, C. R. & Greenes, C. E. (2004). *Navigating through problem solving and reasoning in grades 3-5*. Reston, VA: National Council of Teachers of Mathematics.

Sowder, J. & Schapelle, B. (Eds.). (2002). *Lessons learned from research*. Reston, VA: National Council of Teachers of Mathematics.

Sowder, J. & Wheeler, M. (1989). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education, 20*, 130-146.

Usiskin, Z. (1978). Are Calculators a Crutch? *Mathematics Teacher 71*, 412-413.

von Rotz, L. & Burns, M. (2002). *Lessons for algebraic thinking*. Sausalito, CA: Math Solutions Publications.

Questioning

Schuster, L. & Anderson, N.C. (2005). *Good Questions for Math Teaching: Why Ask Them and What to Ask (Grades 5-8)*. Sausalito, CA: Math Solutions Publications

Sullivan, P. & Lilburn, P. (2002). *Good Questions for Math Teaching: Why Ask Them and What to Ask (Grades K-6).* Sausalito, CA: Math Solutions Publications

Reading

Kenney, J.M. (2005). *Literacy strategies for improving mathematics instruction.* Alexandria, VA: Association for Supervision and Curriculum Development

Hyde, A. (2006). *Comprehending math: Adapting reading strategies to teach mathematics, K-6*. Portsmouth, NH: Heinemann

Barton, M.L. & Heidema, C. (2002). *Teaching reading in mathematics*, 2nd ed: A supplement to teaching reading in the content areas teacher's manual. Aurora, CO: McREL Eisenhower Regional Consortium for Mathematics and Science

Representation

Cuoco A.A., Curcio, F.R. (2001). *Representation in School Mathematics.* 2001 Yearbook. Reston, VA: National Council of Teachers of Mathematics

Teaching

Chapin, S.H. & Johnson, A. (2006). *Math Matters: Understanding the Math You Teach Grades K-6.* Sausalito, CA: Math Solutions Publications.

Lampert, M.L. (2001). *Teaching problems and the problems of teaching.* New Haven: Yale University Press

Marzano, R.J. (2006) *Classroom Assessment & Grading that Work.* Association for Supervision & Curriculum Development

Sakshaug, L.E., Olson, M., & Olson, J. (2002). *Children are mathematical problem solvers.* Reston, VA: National Council of Teachers of Mathematics

Van de Walle, J.A. & Lovin, L.H. (2006). *Teaching student-centered mathematics: Grades 3-5.* Boston, MA: Pearson Education, Inc

Vocabulary

Block, C.C. & Mangieri J.N. (2006). *The vocabulary-enriched classroom: Practices for improving the reading performance of all students in grades 3 and up*. New York, NY: Scholastic Inc.

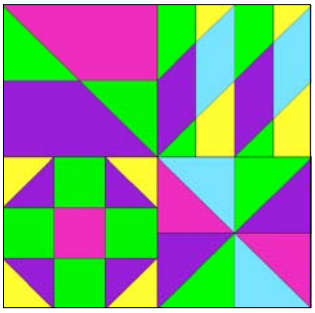
Graves, M. (2006). *The vocabulary book: learning & instruction.* New York, NY: Teacher's College Press

Murray, M. (2004). *Teaching mathematics vocabulary in context: Windows, doors, and secret passageways*. Portsmouth, NH: Heinemann

Writing

Newmann, V. (1994). *Math Journals: Tools for Authentic Assessment.* San Diego, CA: Teaching Resource Center

Whitin, P. & Whitin, D.J. (2000). *Math is Language Too: Talking and Writing in the Mathematics* Classroom. Reston, VA: National Council of Teachers of English



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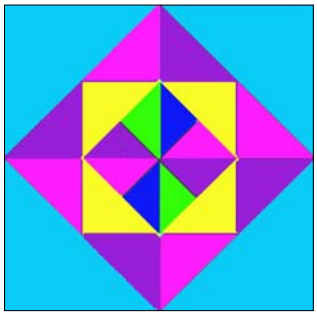
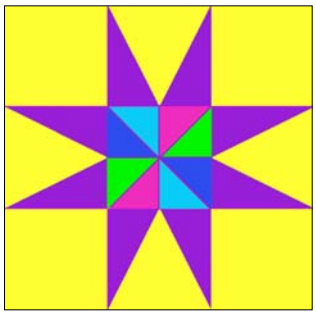
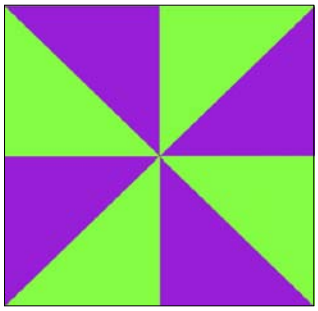


NUMBER WORK



INSPECTING EQUATIONS

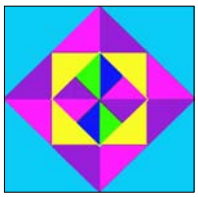


FLUENCY & MAINTENANCE

CHAPTER 7

Number Work

NUMBER WORK



Number Work

Number work activities challenge students to develop a sense of number including how large numbers are in comparison to other numbers, the base-ten system, mental computation, and problem solving without a story context.

Number work can include such activities as: find different ways to compose or decompose a number, determine a function relating two sets of numbers, numeric patterns, patterns on number charts, estimate before computation, or investigate effective strategies for computations of specific types of numbers (e.g. add numbers ending in 8 or 9 or the many ways to decompose a fraction)

Number work is most productive when it becomes routine. Because number work often involves activities with multiple entry points and a search for a diverse set of responses, teachers frequently use it as a whole class activity. Repeating the same activities with different number choices builds fluency.

The Number Work Block

- ☑ focuses on developing number sense without a story context
- ☑ can be posed as one problem or a series of problems
- ☑ uses activities that often have multiple entry points or more than one answer
- ☑ accommodates students with a range of skills and abilities
- ☑ promotes communication and making connections
- ☑ supports algebraic reasoning
- ☑ supports learning in a heterogeneous large group
- ☑ requires from 10 - 15 minutes of a math hour
- ☑ can be interchanged with *Inspecting Equations*

Content and Process Standards
BIG IDEAS: Content

*The **Math Content** big ideas for number work activities include:*

Number, Operations, & Algebraic Reasoning

- Learn that numbers have many representations
- Develop flexible use of computational methods
- Develop knowledge of number and operation relationships
- Learn and apply place value concepts
- Recognize and generalize patterns when counting, computing, and solving problems
- Solve money problems

Geometry

- Determine how many of one shape can make another
- Divide geometric shapes into fractional parts
- Understand classes of geometric shapes
- Nets
- Transformations

Measurement

- Understand measurement is a comparison process
- Choose a useful unit
- Count the number of units and partial units used to measure (using both fractions and decimals)
- Investigate the relationship between units (e.g. 12 inches is the same as 1 foot or 1/4 pound is the same as 4 ounces)
- Investigate the measurement of angles

Data Analysis & Probability

- Gather and graph data
- Analyze data
- Formulate questions

Content and Process Standards
BIG IDEAS: Process

*The **Math Process** big ideas for number work activities include:*

Problem Solving

- Build new mathematical knowledge
- Engage in finding many ways to solve problems in a variety of "number only" contexts
- Reflect on one's own understanding of number relationships

Reasoning and Proof

- Explain the reasoning one uses to draw a conclusion. Describe patterns (e.g. patterns on a multiplication table or multiples on a hundreds chart)
- Make and investigate mathematical conjectures (e.g. when you multiply two odd numbers the answer is always odd)

Representation

- Create or select and apply representations to organize, record, communicate, and compare ideas
- Learn to use conventional forms of representing mathematical ideas (e.g. expressions, equations, and operations)

Communication

- Share solution strategies using drawings, models, and numbers and mathematical language
- Analyze the mathematical thinking of others

Connections

- Discuss how new ideas are alike and different from ideas seen before
- Discuss how two number representations are alike and different

For specific grade level information on math processes and content, see the MMSD K-5 Grade Level Mathematics Standards.

Teaching Activities for the Number Work Block

"Understanding number is a precursor to calculating effectively and flexibly with the base ten number system."

A Mathematics Sourcebook for Elementary and Middle School Teachers, Arena Press, 1999

Number work activities are most effective when they become daily routines. If students are familiar with the format of the activities, they can better focus their attention on learning about the number relationships and concepts involved in the activities rather than on how to do the tasks.

Teachers select number work activities that best meet the needs of their students. When beginning to teach a particular number work activity, the teacher may guide the class through the activity for a period of several days. As the students participate with greater independence, the teacher begins to ask more probing questions in order to change the focus from learning the activity itself to learning about the number relationships and mathematical concepts.

There are a wide variety of activities that provide excellent number work experiences. They include games and practice with mental computations. Repeated use of the same type of activity can build fluency and confidence as well as expand student's knowledge. Curricular guides and purchased programs also have ideas for number work.

Number Work activities tend to be divergent in nature and allow for multiple responses, lending themselves to whole class activities. However, teachers may use these activities with small groups or individual students depending upon the number size and the students' levels of proficiency or use them for fluency and maintenance.

Teachers will find ideas for Number Work activities in the school's curricular resources. This chapter includes a few suggestions for easy to learn Number Work activities to get started:

- Add (or Subtract) My Number
- Coins for the Day!
- Compare
- Compare Arrow Language Representations
- Compare Empty Number Line Representations
- Count and Compare
- Counting By ____ Toss
- Difference Of
- Estimation
- Fluency & Maintenance Games
- Game of 24
- Guess the Sort
- Hundred Chart Patterns
- Nickname, Real name
- Numeric Patterns
- Number of the Day or "Ways to Make"
- Number Squeeze
- Reading and Interpreting Graphs
- Reading Math Literature
- What Do You Know About ____?
- What Do You Notice?
- What's the Rule? (Function Machine)
- Which Number Does Not Belong?
- Writing Math Stories

GRADE LEVEL
3

Add (or Subtract) My Number
Purpose: mentally add and subtract single digits from a multi-digit number

What to do:

- ☑ Designate an order for taking turns with this activity.
- ☑ The teacher writes a starting number (39) and a student picks "my" number to add or subtract and announces it to the class (9) and says, "Add (or subtract) my number."
- ☑ The next student adds (or subtracts) the number to the number on the board (39+9=48).
- ☑ Continue around the class adding (or subtracting) the same number (48+9, 57+9, 66+9….)
- ☑ Continue until all students have had a turn.
- ☑ One person may record the count on the board to use for discussing the patterns observed.

Variation: Use fractions or decimals.

GRADE LEVEL
3

Coins for the Day!
Purpose: know and add values of coins

What to do:

- ☑ Students work together to find as many ways as they can to create the "number of the day" with coins.
- ☑ Students share one or two from their list and discuss their approach.
- ☑ *Variation:* Find the selection that uses the least number of coins, the most number of nickels, etc.

GRADE LEVEL
3-5

Compare
Purpose: comparisons

What to do:

- ☑ Write two numbers on the board and ask students to write as many relationships as they can between the two numbers (e.g. for compare 10 to 50 they might write 10 is 40 less than 50, both are even, both are multiples of 10, 10 is 1/5 of 50).
- ☑ Repeat the activity several times with different numbers.

GRADE LEVEL
3

Compare Empty Number Line Representations
Purpose: represent the operations of addition or subtraction with equations and empty number line

What to do:

- ☑ Write two empty number line representations large enough for the entire group to see.
- ☑ Ask students to tell the ways the two empty number line representations are the same and list their responses.
- ☑ Ask students to tell the ways the two empty number line representations are different and list their responses.
- ☑ Ask students to write a series of equations that represent the empty number line representation.

GRADE LEVEL
3-4

Compare Arrow Language Representations
Purpose: represent computation with equations and arrow language

What to do:

- ☑ Write two arrow language representations large enough for the entire group to see.
- ☑ Ask students to tell the ways the two arrow language representations are the same and list their responses.
- ☑ Ask students to tell the ways the two arrow language representations are different and list their responses.
- ☑ Ask students to write a series of equations that represent the arrow language representation.

GRADE LEVEL
3-4

Counting and Compare
Purpose: place value

What to do:

- ☑ Hand out plastic bags filled with assorted collections of 15 place-value blocks.
- ☑ Ask students to count the number of blocks in the bag and indicate the value of the bag (how many unit cubes represented) E.g. A bag with 3 hundreds, 10 tens, and 2 ones has 15 blocks and a value of 402. A bag with 11 tens and 4 ones has 15 blocks and a value of 114.
- ☑ Discuss the reasons for the different values.

Variation: Supply bags filed with collections of any number of blocks. Students total two bags using pencil and paper. Then confirm the count with the place-value blocks or compare the values of two bags to find the difference.

GRADE LEVEL
3-5

Counting By ____ Toss
Purpose: fluency with multiples

What to do:

- ☑ Small group stands in a circle.
- ☑ Roll a prepared number cube to determine the quantity to count by.
- ☑ Student tosses a beanbag or ball to a classmate who states the next number.
- ☑ Play continues until all students have had a turn adding that number.

GRADE LEVEL
3-5

Difference Of
Purpose: differences

What to do:

- ☑ Ask students to write pairs of numbers with a given difference (e.g. difference of 7) for a predetermined number of minutes.
- ☑ Share responses and repeat several times. (e.g. 32/39, 54/61, 1009, 1016, etc.)

Variation: Use decimals and fractional differences

GRADE LEVEL
3

Estimation
Purpose: visualize quantities

What to do:

- ☑ Find two jars that are transparent and exactly the same.
- ☑ Have a collection of items in one jar and an appropriate small quantity (2, 3, 5, and 10) in another jar to use as a reference. Tell the students how many objects are in the reference jar.
- ☑ Ask students to think about how much space the reference quantity fills. They should use that understanding to help them think about the quantity in the estimation jar.
- ☑ After students share their thinking, count the quantity in the estimation jar.
- ☑ Students determine the closest and farthest estimate and share computation strategies.

GRADE LEVEL
3-5

Factor Patterns
Purpose: factors for numbers to 100

What to do:

- ☑ Ask students to shade "count bys" for one number (2-9) on a 100s chart (using a highlighter.)
- ☑ Students discuss or list the patterns they notice.
- ☑ Ask students to shade in another "count by" using a different color on the same chart. Discuss.
- ☑ Repeat with a third "count by."

GRADE LEVEL
3-5

Fluency & Maintenance Games
Purpose: number relationships and fact strategies

What to do:

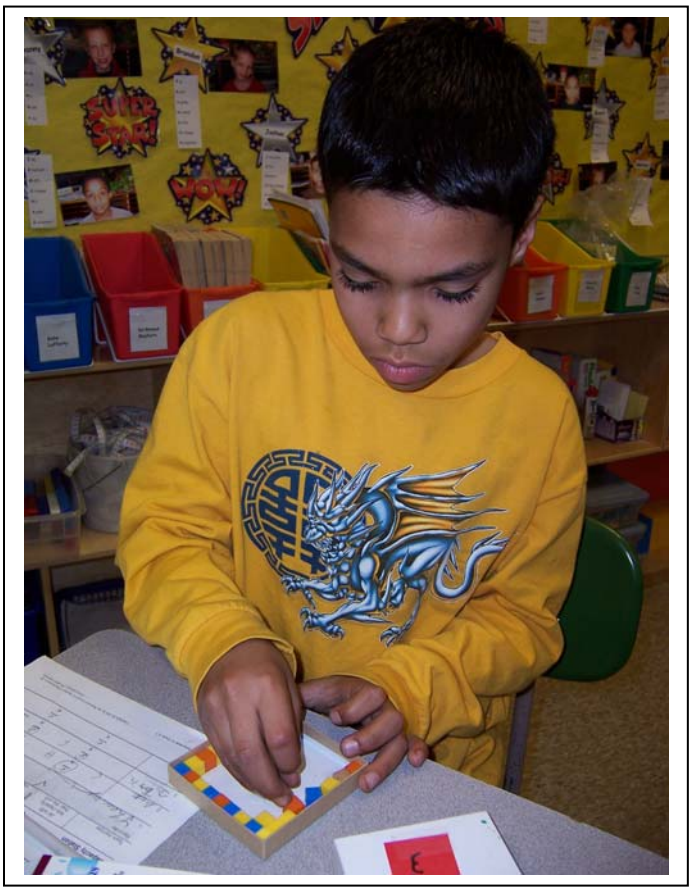
- ☑ Teach the rules for the games in the Fluency & Maintenance Block.
- ☑ See Chapter 9, Fluency & Maintenance Block for game rules.

GRADE LEVEL
4-5

The Game of 24
Purpose: build number sense

What to do:

- ☑ This is a purchased game that provides a great mental math challenge.



GRADE LEVEL
3-5

Guess the Sort
Purpose: number classification

What to do:

- ☑ Put a Venn diagram on the board with hidden rules for each circle. Rules can be things such as odd numbers & divisible by 5, or shapes with certain characteristics such as "closed" & "one set of parallel lines.
- ☑ Students take turns guessing a number or drawing shape.
- ☑ Teacher puts the number or shape in the correct place on the Venn diagram which includes the universal set outside of the circles so that every guessed number or object has a place.
- ☑ As the set of numbers grows in each area of the Venn diagram, students may first make a guess and then conjecture about the label for the circle after guessing a new number or shape.
- ☑ Continue playing until the labels for each circle can be determined.
- ☑ Check to make sure that each number or shape in the diagram meets the sorting criterion.

Variation: Use composite, prime, multiples, or factors

GRADE LEVEL
3-5

Hundred Chart Patterns
Purpose: patterns

What to do:

- ☑ Ask students to look for patterns on a hundred chart. (See appendix for different number charts to use for a variety of grade-levels.)
- ☑ Keep a chart describing the patterns.
- ☑ Try even/odd numbers, multiples, factors

GRADE LEVEL
3

Nickname, Real name
Purpose: naming numbers (best for small group number work)

What to do:

- ☑ Make a chart with three columns—the counting numbers beginning at 1, number names ("nicknames"), and real names (quantity and value for each place).
- ☑ Fill in the chart as the students provide the information. E.g. 17, seventeen, one ten and seven. (Note: The numbers 1-10 only have real names.)
- ☑ Challenge students to give a nickname when given the real name and visa versa.

GRADE LEVEL
3-5

Numeric Patterns
Purpose: repeating and growing patterns

What to do:

- ☑ Write a numeric pattern large enough for the entire group to see it. For example: 1, 5, 9, 13, …
- ☑ Ask students to think about how the numbers change from one number to the next. (In the 1, 5, 9, 13 . . . sequence, each number is 4 more than the previous number.)
- ☑ As students share their ideas, check them out to see if they work consistently from one number to the next.
- ☑ Students identify the rule and keep the pattern going. (In the 1, 5, 9, 13 sequence, the rule is +4.)

Variations: Use fractions or decimals in the pattern for more advanced students

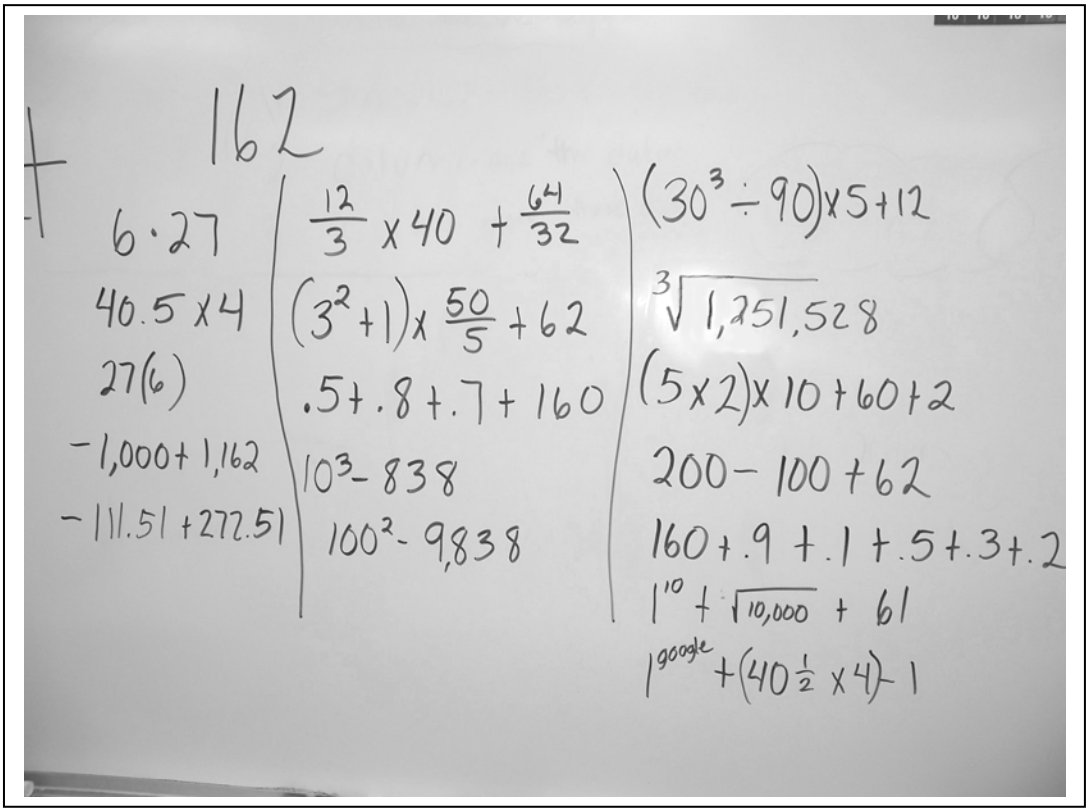
Extension: Once students identify the change pattern, ask if a particular number will be in the pattern if you keep the pattern going.

GRADE LEVEL
3-4

Number of the Day or "Ways to Make"
Purpose: compose/decompose numbers and relationships and relationships between operations

What to do:

- ☑ Write a target number on the board
- ☑ Students write as many different mathematical expressions for the target number they can think of in 3-5 minutes
- ☑ The teacher records a few examples the board (or students show their expressions written on a whiteboard)
- ☑ The teacher highlights a few new ideas (e.g. fractions. negative numbers, or square numbers) observed
- ☑ Students re-do the activity with the same or a different number with the goal of extending their domain of numbers, operations, or representations.



GRADE LEVEL
3

Number Squeeze
Purpose: number order language

What to do:

- ☑ One student chooses a number between 1 and 100 and keeps it hidden. (Hang a number line or hundreds chart in plain view)
- ☑ The class takes turns asking questions to narrow the possibilities and find the number using order vocabulary such as "before," "after," in-between, greater than, less than.
- ☑ Tally the questions needed to determine the number. No random guessing is allowed.
- ☑ Repeat several times.

Variation: Provide students with their own number charts or number lines to eliminate numbers.

GRADE LEVEL
3-5

Reading and Interpreting Graphs
Purpose: count, represent, and compare quantities

What to do:

- ☑ Use the endless opportunities you have to create graphs such as: graphing hot lunch counts, types of shoes, favorite authors, etc.
- ☑ Pose questions about the data.

GRADE LEVEL
3-5

Reading Math Literature
Purpose: number sense

What to do:

- ☑ Have a variety of books with math concepts available.
- ☑ Students complete a response page. (e.g. make a new page for this book, or what three number ideas do you remember from this book?)

GRADE LEVEL
3-5

What Do You Know About ____?
Purpose: number relationships

What to do:

- ☑ Teacher asks the students to share what they know about a given number and writes the responses on a chart.
- ☑ Elicit responses from all students.
- ☑ Responses can be a personal reference, such as my brother is 10.
- ☑ Encourage statements that indicate a relationship, such as 10 is 2 more than 8.
- ☑ Bring out charts created earlier in the year to compare responses and see growth.

GRADE LEVEL
3-4

What Do You Notice?
Purpose: model numbers on a 10x10 grid and equivalent expressions

What to do:

- ☑ Shade in a portion of a 10x10 grid (See example in the Appendix)
- ☑ Ask students to write expressions that represent what they "see" on the chart which can include the non-shaded area. Share examples.
- ☑ Repeat the activity several times.

GRADE LEVEL
3-5

What's the Rule? Function Machine
Purpose: patterns

What to do:

- ☑ Input any single or two-digit number, output a change in quantity based on a function appropriate to the grade level such as n/2 or 3n.
- ☑ Have students offer numbers to input.
- ☑ Students observe the relationships and decide the rule the machine is following.
- ☑ Student who figured out the relationship decides the next function and provides the outputs.

GRADE LEVEL
3-5

Which Number Does Not Belong?
Purpose: number relationships

What to do:

- ☑ Write a set of numbers on the board like this one:

6	15
10	12

- ☑ Ask students "Which number does not belong with the others?

- ☑ Students explain their choice such as:
  - 10, because it is not divisible by 3
  - 15, because it is not even
  - 6 because is less than 10

- ☑ These kinds of problems are very complex. They are fun to do every now and then. Working on them builds perseverance. These problems challenge students to think from more than one perspective.

For More Information:

Bresser, R. & Holtzman, C. (2006). *Minilessons for Math Practice: Grades 3-5*. Sausalito, CA: Math Solutions Publications.

Dana, M. *Inside-out math problems: Investigate number relationships & operations*. Grand Rapids, MI: Instructional Fair.

Hope, J. A., Reys, B. J. & Reys, R. E. (1987). *Mental Math in the Middle Grades: Blackline Masters*. Parsippany, NJ: Dale Seymour Publications.

Kamii, C. (2004). *Young children continue to reinvent arithmetic* (2nd ed.). New York: Teachers College Press.

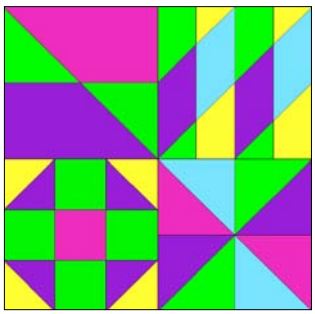
Richardson, K. (1999). *Developing number concepts: Place value, multiplication, and division*. Parsippany, NJ: Pearson Education.

Sheffied, S. (2002). *Lessons for introducing place value*. Sausalito, CA: Math Solutions Publications.

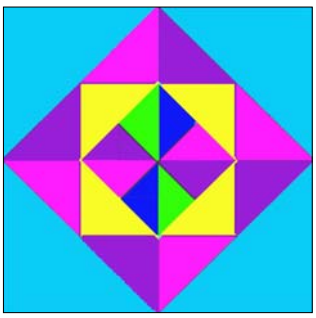
Trafton, P. R. & Thiessen, D. (1999). *Learning through problems: Number sense and computational strategies*. Portsmouth, NH: Heinemann.

Van de Walle, J. A. & Lovin, L. H. (2006). *Teaching student-centered mathematics: Grades 3-5*. Boston: Allyn and Bacon Pearson Education.

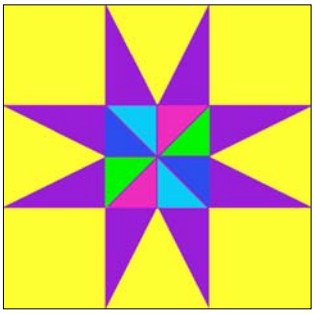
Wheatley, G. H. & Reynolds, A. M. (1999). *Coming to know number: A mathematics activity resource for elementary school teachers*. Tallahassee, FL: Mathematics Learning.



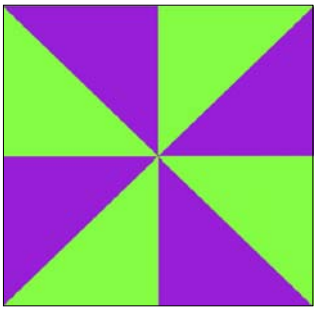
PROBLEM SOLVING



NUMBER WORK



INSPECTING EQUATIONS

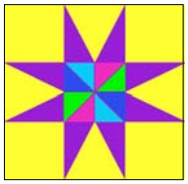


FLUENCY & MAINTENANCE

CHAPTER 8

Inspecting Equations

INSPECTING EQUATIONS



Inspecting Equations

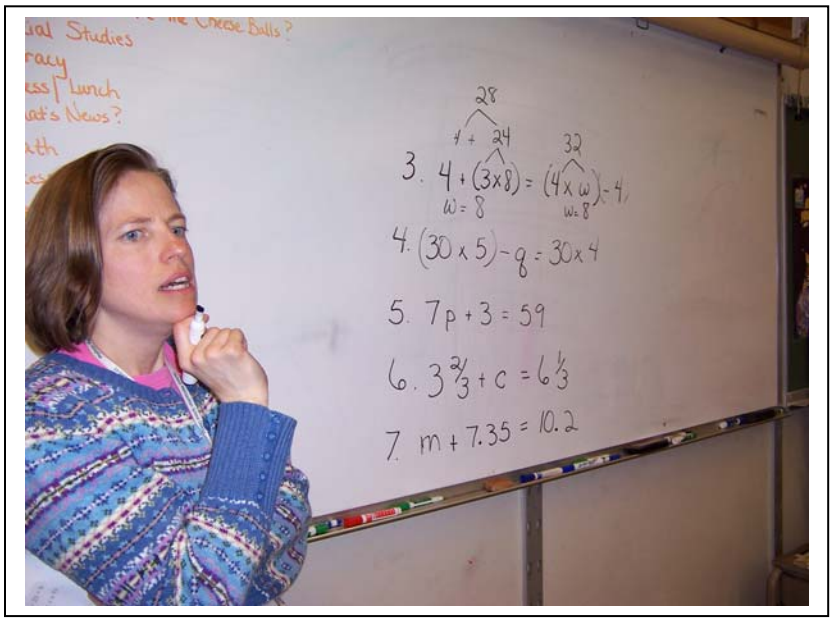
Inspecting equations activities focus on learning about how the equal sign expresses equality relationships. Students:

- discuss true/false or open number sentences (equations)
- use number relationships to reason about equality relationships
- recognize patterns and make conjectures about number properties
- justify their thinking mathematically

Inspecting equations activities can provide a whole class experience because of many entry points.

The Inspecting Equations Block

- ☑ focuses on concepts of equality and the meaning of the equal sign
- ☑ involves students analyzing conventional symbols of number, operations, and relationships
- ☑ encourages students to look for and use number relationships as well as computation to confirm equality relationships
- ☑ can be used interchangeably with the Number Work block
- ☑ utilizes about 15 minutes of a math hour
- ☑ occurs as a heterogeneous large or small group activity



Inspecting equations provides another setting for students to "learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers." (Carpenter, et.al 2003) These skills are essential for success in mathematics and other math related study. Through inspecting equations activities students also begin to recognize reasoning and proof as fundamental to mathematics.

Inspecting equations provides a means for teachers to engage students in learning the big ideas of mathematics. Students learn to make connections between the arithmetic they have learned throughout elementary school to the algebra they will encounter in middle school and beyond.

Short focused discussions with the whole class can work well for inspecting equations. However, small group sessions may provide a more productive setting when the objective of the lesson is to:

- ☑ help a few students understand the meaning of the equal sign
- ☑ develop particular concepts
- ☑ work with a particular set of numbers
- ☑ provide more opportunity to develop language and communication skills such as listening, questioning, or contributing to discussions

Small group discussion provide more opportunity for individual students to share their own thinking about the equations, ask clarifying questions, and reflect on each other's thinking to build number sense.

Content and Process Standards
BIG IDEAS: Content

*The **Math Content** ideas for Inspecting Equations block activities include:*

Number, Operations and Algebraic Relationships

- Analyze equations to understand the equal sign as a mathematical symbol of equality rather than a signal to compute
- Understand and use conventional symbolic notation
- Represent the idea of a variable as an "unknown quantity" using a letter or a symbol
- Develop and use number relationships
- Seeing an entire equation across the equal sign before responding
- Justify conjectures about number properties
- Understand and use the commutative, associative, and distributive properties
- Use number relationships to compute more fluently

Content and Process Standards
BIG IDEAS: Process

*The **Math Process** big ideas for inspecting equations block activities include:*

Problem solving

- Reflect on one's own strategies to continue making sense of the problem-solving experience
- Develop the habit of rethinking the solution to assure accuracy

Representation

- Learn conventional forms of representing operations and number relationships
  $8+5=a+4$

Communication

- Share solution strategies using words and symbolic language
- Learn mathematical vocabulary

Reasoning and Proof

- Explain the reasoning one uses to draw a conclusion
- Look for patterns and explain them mathematically
- Generalize from examples to make conjectures about the properties of number
- Justify generalizations

Connections

- Understand connections between operations
- Understand base ten and the place-value system of numbers
- Understand computational approaches

Learning About the Equal Sign

As early as kindergarten, teachers introduce the equal sign. Students learn that the equal sign means "the same as." Teachers encourage students to say, "5 is the same as 3+2" instead of "5 makes 3+2." Teachers are careful to write expressions and equations only after students understand the numbers in them as quantities and the operation symbols as actions. They write equations in more than one form such as 5=3+2 and 3+2=5.

In first grade, students develop their understandings of how to represent statements of equality. They inspect equations such as:

$7 = 7$
$8 = 3 + 5$ **denotes a false equation*
$8 + 2 = 2 + 7$*
$4 + 2 = a + 2$

Second graders expand their understanding of equations. They begin to use relationships as well as computation to reason about the truth of equality statements. They inspect equations such as:

$8 + 4 = 7 + 5$
$8 + 6 = 7 + 7$
$7 + 8 = 7 + 7 + 1$
$7 + 8 = 8 + 7$

Students often learn that the equal sign indicates a signal to compute or that "the answer comes next," rather than a relationship between two quantities. For this reason, it is important to write equations with numbers and operations to the right of the equal sign.

During grades 3-5, students refine and expand their understanding of base-ten concepts, fractions, and operations. They use their knowledge of facts, basic properties, and number relationships rather than computation to reason about equations such as:

$25 + 47 = a + 26$
$67 + 28 - 29 = 66$
$g + g + 4 = 16$
$75 \times 45 \times 0 = y$

The following benchmarks indicate levels of understanding about the meaning of the equal sign. Not all children follow the same path, however, these benchmarks serve as indicators of a change in a student's conceptions of the equal sign. Students may also have different conceptions depending on the numbers, operations, and placement of the unknowns in a given equation.

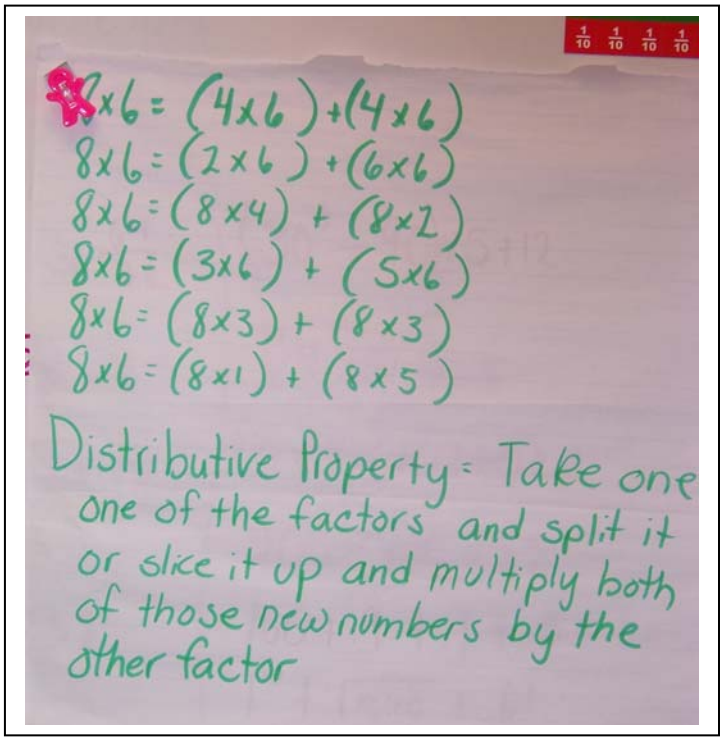
Table 8.1 **Benchmarks for understanding the equal sign**

Talking about meaning	Students communicate own ideas about the meaning of the equal sign. At this benchmark, students may not have correct conceptions but comfortably express their thinking. Common misconceptions about the meaning of the equal sign seen in answers for the equation $8+4=a+5$ include: "the answer comes next" (answer given is 12), "all of the numbers go together to make "a" (answer given is 17), "equation is incomplete" (answer given is 12 and 17).
Accepting new forms	Students accept as true number sentences that are *not* of the form $a+b=c$. For example: $12=8+4$ $\quad 12=12$ $\quad 8+4=12+0$ $\quad 8+4=4+8$.
True meaning	Students recognize that the equal sign represents a relation between two equal numbers. They compare expressions on both sides of the equal sign and carry out calculations on each side of the equal sign. For example, for the equation $8+4=a+5$, a student at this benchmark adds 8 and 4 to get 12, then takes 5 away from 12 to get the answer.
Using the "equals" relationship	Students can compare expressions on both sides of the equal sign without calculations. For example, for the equation $8+4=a+5$, a student at this benchmark compares the 4 to the 5 and knows that the answer must be one less than 8.

Adapted with permission from Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heineman.

Intermediate students also make and discuss conjectures about basic number properties (e.g. zero property, commutative, base ten) that emerge from discussions about T/F or open number sentences. They advance their ideas about justification and proof as they explain why the conjectures they make are always true.

For specific grade level information on math content, see the MMSD K-5 Grade Level Mathematics Standards.



Conventions for Using the Equal Sign

The equal sign is a convention. By working with students on the meaning of this convention, teachers help students learn to use the equal sign consistently and become proficient in thinking relationally.

However, the equal sign is often used for purposes that are not related to its mathematical meaning. Avoid using the equal sign to:

- ☑ list ages or some other numerical characteristic of people or things such as January=31, February=28
- ☑ designate the number of objects in a collection such as pennies=5, dimes=12
- ☑ represent a string of calculations such as:

  $30+30=60+15=75$, or

  $30+30 \rightarrow 60+15=75$

  Instead, use arrows as in:

  $30+30 \rightarrow 60+15 \rightarrow 75$, or

  $30 \xrightarrow{+30} 60 \xrightarrow{+15} 75$

- ☑ show equality using pictures or drawings (reserve use of the equal sign for numbers only)

  This: $3=3$ not this: ♥ ♥ ♥ = 3

Note**:** Also avoid using a balance as a substitute or metaphor for the equal sign. This commonly used metaphor both misrepresents the scientific meaning of "balance" and the mathematical meaning of the equal sign. Students will learn the correct mathematical meaning of the equal sign and learn to use mathematical symbols best when used in equations.

Table 8.2 **Assessing for conceptual understanding**

Use the following sets of equations to capture student conceptions of equality. For each equation ask students, "What does the letter (or box) have to be to make this a true number sentence? These provide a start to Inspecting Equations sessions. A written assessment is included in this chapter.

Purpose	**Grade 3**	**Grade 4**	**Grade 5**
Do students view the equal sign as expressing a relationship between two quantities in the equation forms: $a+b=c+d$ and $a=b+c$	$3+4=y+5$ $5+y=6+2$ $y+4=5+3$ $y=2+3$ $7=3+y$	$36+24=y+25$ $25+y=26+32$ $y+24=25+23$ $y=32+27$ $59=27+y$	$256+289=y+290$ $290+y=291+355$ $y+289=290+254$ $y=283+217$ $500=217+y$
Do students automatically compute from left to right or do they consider the helpful relationships within an equation before computing an equation of the form: $a+b-b=a$	$25+59-59=y$ $67+28-y=67$	$889+118-118=y$ $2\frac{5}{6}+\frac{7}{8}-\frac{7}{8}=y$ $7\frac{1}{2}+\frac{3}{4}-\frac{3}{4}=y$	$10.5+.3-.3=y$ $35.6+.89-.89=y$ $.3+.99-.99=y$
Do students automatically compute from left to right or do they consider the helpful relationships within an equation before computing an equation of the form: $a+b-(b-1)=c$ and $a+b-(b+1)=c$	$54+37-36=y$ $43+28-29=y$	$345+76-75=y$ $436+27-28=y$	$3.5+1.9-1.8=y$ $89.58+.26-.27=y$
Do students take advantage of familiar number relationships?	$25+47+75=y$ $98+65+2=y$	$25+47+75=y$ $98+69+2=y$ $7\times45\times0=y$	$2.50+47+.50=y$ $67.98+2+.02=y$ $32\times58\times0=y$
Can students accurately solve equations with repeated variables?	$g+g+16$ $h+h-3=11$ $a+a+a=15$	$g+g+4=16$ $h+h-3=11$ $a+a+a=15$ $f+f+f+5=23$ $p+p+p+4=p+16$	$2g+4=16$ $2h-3=11$ $3a=15$ $3f+5=23$ $3p+4=p+16$

Adapted with permission from Jacobs, V. R. et al. *Professional Development Focused on Children's Algebraic Reasoning in Elementary School*, JRME: Volume 38, Issue 3, pp. 258 - 288

Table 8.2 continued **Assessing for conceptual understanding**

Use the following multiple-choice tasks with 5th grade students to assess their understanding of algebraic expressions and levels of justification. They can be used to assess individual students, but work best when followed up with whole group discussion.

Can students represent word problems with appropriate number sentences for additive relationships?	1. There are 5 more cars than trucks in the school parking lot. If "c" is the number of cars and "t" is the number of trucks, which number sentence shows the relationship between the number of cars and the number of trucks? $a)\ c+5=t \quad b)\ t+5=c \quad c)\ t+c=5 \quad d)\ 5\times c=t$ 2. John has 5 more marbles than Pete. If "p" stands for Pete's marbles and "j" stands for John's marbles, which number sentence shows the relationship between John's marbles and Pete's marbles? $a)\ p+5=j \quad b)\ j+5=p \quad c)\ j+p=5 \quad d)\ j=5$
Can students represent word problems with appropriate number sentences for multiplicative relationships?	1. There are 6 times as many students as teachers at Linwood School. If "s" is the number of students and "t" is the number of teachers. Which number sentence shows the relationship between the number of students and the number of teachers? $a)\ 6+t=s \quad b)\ 6\times t=s \quad c)\ t\times s=6 \quad d)\ t+s=6$ 2. Stephanie has 4 times as many pieces of candy as Isabel. If "i" stands for Isabel's candies and "s" standards for Stephanie's candies, which number sentence shows the relationship between the Stephanie's candy and Isabel's candy? $a)\ i+4=s \quad b)\ i\times 4=s \quad c)\ s+4=i \quad d)\ s\times 4=i$
How do students generate their own justification for a conjecture?	*When you multiply two numbers, you get the same result even if you change the order of the numbers when you multiply. For example, 2 x 3 is the same thing as 3 x 2. This idea will work for any set of numbers.* How might you convince others that this idea is always true?
What category of justification do students choose?	*The following responses illustrate examples of justification for why the statement is always true. Who do you think did the best job of proving this statement and why?* **Tom:** When you multiply one number by another number, you get the same thing as when you multiply the second number by the first number – you can just flip the numbers around. The order doesn't matter when you multiply, and this should work for all numbers. **Art:** My teacher last year taught me to do that. She said I could use it when I solve problems because sometimes it makes things easier. She also said it should work for all numbers. **Grant:** Let's say I build 5 rows with 4 blocks in each row – that's 5 times 4. But I can just turn the set of blocks and then I have 4 rows with 5 blocks in each row – that's 4 times 5. I didn't add or take away any blocks so it's the same number both ways. This idea should work for all numbers. **Edd:** I spent a lot of time proving this one. I did this type of problem with 50 different sets of numbers. Like I did 3 times 5 is 15 and 5 times 3 is also 15. And I did 45 x 53 and I got the same thing as 53 x 45. Each time I did it, it worked. Since I tried it so many times, I think it should work for all numbers. $3\times 5=15 \quad 5\times 3=15 \quad yes$ $45\times 53=2{,}385 \quad 53\times 45=2{,}385 \quad yes$ $2\times 36=72 \quad 36\times 2=72 \quad yes$ $9\times 10=90 \quad 10\times 9=90 \quad yes$

Adapted with permission from Jacobs, V. R. et al. *Professional Development Focused on Children's Algebraic Reasoning in Elementary School*, JRME: Volume 38, Issue 3, pp. 258 – 288

Name_____ Date_____

Inspecting Equations - Assessment

1. 7 + 5 = 5 + 7 true false

   Explain your thinking:

2. 29 + 17 -17 = 29 true false

   Explain your thinking:

3. 15 + 25 = 29 + 14 true false

   Explain your thinking:

4. 89 = 89 true false

   Explain your thinking:

5. Put a number in the box to make the number sentence true. 8 + 5 = $\boxed{}$ + 7

   Explain your thinking:

Teaching Activities in the Inspecting Equations Block

Inspecting equations activities primarily involve discussion about the meaning of the symbols used in the study of mathematics. The need for student responses that include a range of interpretations makes this block most beneficial as a whole class or small group activity. The interchange of ideas within a group setting supports learners as they question each other about the reasoning behind their responses.

Getting Started with True/False or Open Number Sentences

One of the easiest ways to learn what students know about the equal sign is to ask them to respond to several true/false or open number sentences and explain their reasoning to a partner or in writing before discussion.

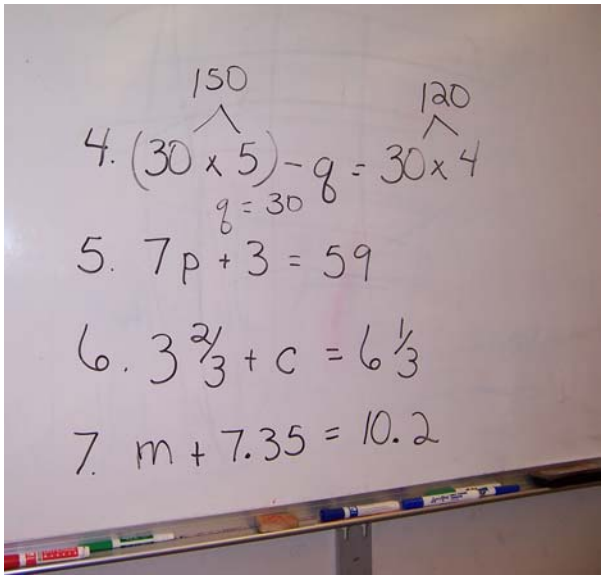
Inspecting Equations sessions may take longer initially. As norms for discussion develop, session length changes based on what students know and the goal of the lesson.

The following set of equations set the stage for further Inspecting Equations discussions. They specifically reveal student concepts about the equal sign. Use part or all of the set for one Inspecting Equations session.

T/*F number sentences		Open number sentences
$3+5=10$	$7+8=15+2$*	$a=3+5$
$8=3+5$	$9+5=14$	$8=a+5$
$37=37$	$9+5=14+0$	$37=a$
$5+6=5+6$	$9+5=0+14$	$5+a=5+6$
$7+8=8+7$	$9+5=14+1$	$7+8=a+5$
$24+47=47+23$*		$a+47=47+24$

Teaching Tips

- Use True or False equations first. Equations without variables are often easier to understand.
- Remember to include false equations as well as true equations as they are often easier for students to justify given an obvious disparity in equality. $100 = 70 + 3$
- Begin with assessing what students know about the meaning of the equal sign. The equations on page 16 will elicit conversation about the meaning of the equal sign.
- You may want students to commit to their idea by writing T or F and a brief explanation on a response page or ask students to talk to a partner.
- Ask several students to explain or read their explanation. If students disagree, allow some time for discussion but do not explain for them. Help them see that they must agree on mathematical terms. It is acceptable to move on when there is not consensus.
- Use the "Open Number" sentences as a follow-up.
- Switch to using lower case letters after using a box as a symbol for the variable. Students may want to choose which letters to use and should avoid those that aren't confused with other mathematical symbols such as x or 0.



When planning a True/False or Open number discussion:

- ☑ consider the benchmarks for development of relational thinking and work toward understanding and using equivalent relationships (see table 8.1)
- ☑ establish expectations or norms for responding to number sentences (E.g. think time, pair-share, write first)
- ☑ design or choose specific equations or sequences of equations in response to student reasoning about the equal sign, number relationships, or operations (see table 8.3)
- ☑ know what to avoid when using the equal sign (see Conventions, page 11)
- ☑ consider keeping a visual record (on chart paper) of equations that have been discussed so that students can reflect on past work and see progress over time

Table 8.3 **True/False and Open Number Equation Sets** ***denotes a false equation**

Purpose	Sample Equation Set A	Sample Equation Set B	Sample Equation Set C
Understanding the meaning of the equal sign as equality rather than a command to compute	$8+7=15$ $15=8+7$ $15=15$ $8+3=8+3$ $7+5=5+7$ $7+5=12+1$* $6+9=15+6$*	$9+5=14$ $14=9+5$ $14=14$ $11+6=11+6$ $8+5=5+8$ $8+5=13+1$* $8+5=13+8$*	$16+4=20$ $20=16+4$ $20=20$ $18+3=18+5$* $23+7=7+23$ $23+7=30+1$* $23+7=30+23$*
Number fact development	$9+5=9+1+a$ $7+8=7+7+a$ $7+5=7+3+a$	$4+1+1+1=4+3$ $6+4=5+a$ $5+7=5+5+2$	$2\times 9=a+9$ $4\times 7=14+a$ $5\times 6=(2\times 6)+(2\times 6)+a$
Noticing additive inverse and identity	$42+9-9=43$* $53+8-z=54$ $192-32+32=a$	$9+\frac{1}{4}-\frac{1}{4}=9$ $2\frac{1}{2}=2\frac{1}{2}+\frac{1}{4}-\frac{1}{4}$ $a=\frac{5}{6}+\frac{1}{3}-\frac{1}{3}$	$7.8+.20-.20=78$* $45.6+2.67-2.67=j$ $34.2-.68+a=34.2$
Noticing numbers with a difference of 1 or 2	$23+48=t+22$ $99+101=100+100$ $53+68=67+52+s$	$39+73=74+38-b$ $376+84=85+c$ $692+45=44+691$*	$4\frac{1}{2}+3\frac{2}{3}=3\frac{2}{3}+4$* $5.2+6.3=a+5.2$ $36.25+15.16=14.16+37.25$
Noticing place value	$45+5+50=90+10$ $56+67+23=50+60+10+6$* $340+306=600+10$*	$358+130+2=400+80+9$* $456+298+302=156+900$* $567+89=560+k$	$35.60+44.40=35+44+p$ $67.98+67.98=120+14+1.00+r$ $18.5+199.25+1.25=s+18+200$

Table 8.3: Continued

Purpose	Sample Equation Set A	Sample Equation Set B	Sample Equation Set C
Noticing distributivity	$6\times3=(3\times3)+(3\times3)$ $4\times15=(4\times10)+(4\times5)$ $23\times9=(20\times9)+(3\times9)$	$12\times3=(10\times3)+(2\times3)$ $6\times8=(12\times8)+(12\times8)$* $4\times9=(4\times3)+(4\times3)+(4\times3)$	$26\times13=(20\times10)+(20\times3)+(6\times a)$ $15\times67=(10\times60)+(5\times7)$* $35\times72=(30\times70)+(5\times70)+(30\times2)+b$
Relating addition to multiplication	$8+8+8+8=y\times8$ $12+12+12=z\times6$ $15+15+15=v\times5$	$14+14=4\times7$ $3\times7=7+7+7$ $7\times4=7+7+7+7$	$3\times6+3=3\times7$ $6\times4=(4\times2)+(2\times2)$* $3\times6=2\times6+a$
Working with variables	$13=h+h+3$ $35=i+i+i+5$ $j+j+j+j=16$ $15+k+k=25$	$11=b+b$ $k+k+k=18$ $m+m+m+12=m+24$ $75-5+j=70+j+j$	$36=3+u+u+3$ $c+c+c+4=16$ $q+q+8=16$ $32+b+b=42+b$
Working with multiplication facts	$10+10+10=(3\times9)-3$* $c\times5=40+5$ $4\times10-4=4\times8$* $4\times9=(4\times10)-4$	$6\times4=5\times4+a$ $8+b=3\times6$ $7\times9=7\times10-a$ $8\times5+a=8\times6$	$2\times9=w+w-2$ $6\times8=5\times8+y$ $7\times8=7\times7+z$
More place-value ideas Note: Problems with two or more variables have many solutions.	$347=(g\times100)+47$ $(b\times10)+8=408$ $(h\times1000)+(4\times10)=2040$	$(b\times10)+t+(r+100)=4758$ $576.8=(m\times10)+(p\times.1)$ $a\times10+b=87$	$576.25=(b\times100)+c+(p\times.01)$ $3.5=(m\times10)+g$ $(t\times.1)+s=8.6$
Using more than one operation	$\frac{3+p}{4}=8$ $6\times j-3=21$ $3\times m-5=10$ $15+y=2\times y-3$ $25+3\times g=5\times g-7$	$5-d=d-3$ $6\times m-2\times m+4=12$ $a=\frac{5\times6}{3}$ $23+r=27+33$ $6\times w+43=10+9\times w$	$\frac{z+7}{2}=9$ $t+45-15=32+28$ $2\times s+5\times s=15+13$ $16=4-t+3\times t$ $3\times r=20-r$
Working with more than one variable	$t+2=v+3$	$k-t=7$	$2\times d+n=14$

Equations with More than One Variable

What to do:

- ☑ Provide a story problem or an equation with more than one variable. For example:

  Michelle bought nine fish. She wants to put them in two ponds in her backyard. What are the different ways she could put the fish in the ponds?

- ☑ Ask students to list all the ways to partition the nine fish between the two ponds.
- ☑ Ask students to suggest an equation that would represent the problem.
- ☑ Challenge students to solve the same problem with larger numbers (e.g. 15, 23) and look for a generalization.
- ☑ Discuss and solve equations (see table above) with more than one variable.
- ☑ Discuss the idea that two different variables can have the same value (e.g. $a+b=6$, both a and b can be 3) but a repeated variable must have the same value (e.g., $a+a=11$, both a's must be $5\frac{1}{2}$).
- ☑ Use examples from Table 8.2 for further discussion.

Teaching Order of Operations

When a number sentence includes more than one operation, carry out the operations in the following order:

1. operations inside parentheses (in the order given by the next two rules)
2. multiplication and division from left to right
3. addition and subtraction from left to right

Teaching "order of operations" works best when a real mathematical need arises. The order-of-operation conventions apply in cases where the order of calculations matters. Without conventions, $4+6\times 5$ can be interpreted in two ways:

$$(4+6)\times 5=10\times 5=50 \qquad 4+(6\times 5)=4+30=34$$

Students most often solve right to left even after learning the conventional order-of-operations rule. Intermediate students should learn how to use parentheses to indicate which part of an equation to calculate first.

Fifth grade students can begin to use order of operations in longer equations such as:

$$8+25\div 3\times 2-5=a$$
$$2\times(6+2\times 4+4)\div 3=b$$
$$15-5+3\times 4=c$$

Order of operations only applies to equations where it makes a difference what order calculations are completed. There are many situations where order does not matter. *For example:*

$$8\times 4\div 2$$

Teachers can provide situations T/F equations to study this situation as well. For example is the following equation true or false?

$$(6\times 5)\div 2=6\times(5\div 2)$$

Always follow-up with a discussion about the reasoning behind the answer and models to help explain if needed.

Students should have experiences that suggest a need for conventions. For example, students may have different ideas about the answer to:

$$3 \times 5 + 2 = s$$

Students could determine where to put parenthesis in the following equation to make it a false statement

$$3 \times 5 + 2 = 3 \times 5 + 2$$

Teaching Tips

- Students should be used to working with simple number sentences involving multiplication and division before learning the three "order of operations" conventions.
- Students should have good number sense before learning the order of operations conventions.
- Parentheses are easiest to learn and use. Teach the use of parentheses first. Generally, students in grades 3-5 will only need to know this convention.
- Engage students in discussion about the need for parentheses through use of the true/ false number sentences involving multiple operations.
- Ask students to write their own true/false equations and include parentheses.
- Introduce "order of operations" conventions as the need emerges from their work with true/false or open number sentences.

Developing Conjectures

What to do:

- ☑ Follow the protocol for true/false and open number sentences using equations that represent basic properties of number operations or classes of numbers such as even/odd numbers.
- ☑ Elicit a conversation about each equation in a related set of equations (See table 8.4.)
- ☑ Pose a question about the entire set. For example, "What do you notice about this group of equations?" or "Could you make a conjecture about what happens when you add a zero to a number?" or "Is this always true?" Students suggest an open number sentence to represent the conjecture.
- ☑ Post the conjecture for further discussion about its validity by providing new equations that use a new domain of numbers such as fractions, decimals, or very large numbers.
- ☑ Ask students to explore operations with even and odd numbers resulting in conjectures about how a class of numbers works. For example, "What happens when you add two even numbers together? Two odd numbers? One even and one odd number?

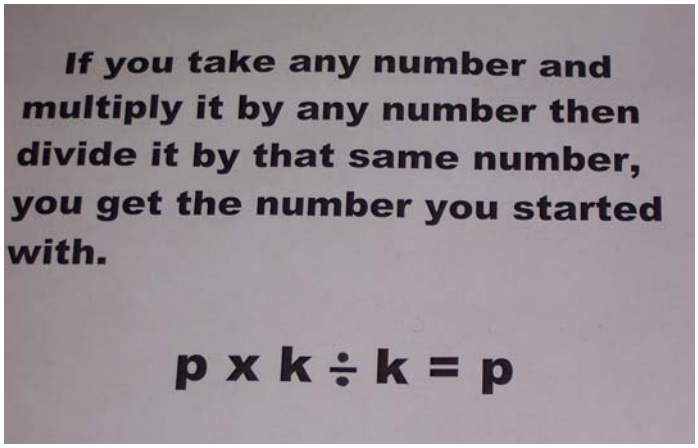


Table 8.4 **Equation Sets to Prompt Conjectures**

Equation Set * Denotes a false number sentence	**Possible student conjecture and number sentence:**	**Number property to discuss:**
$799+0=800*$ $48+2=48*$ $50+0=500*$ $39+c=39$	When you add zero to a number, you get the number you started with. $a+0=a$	Addition involving 0 (Additive Identity)
$536-0=536$ $48-9=48*$ $570-0=57*$ $c-0=654$	When you subtract zero from a number, you get the number you started with. $a-0=a$	Subtraction involving 0
$5,467\times1=5,467$ $48\times1=49*$ $8.4\times1=8.5*$ $c\times1=76$	When you multiply a number times 1, you get the number you started with $a\times1=a$	Multiplication involving 1 (Multiplicative Identity)
$345\times0=0$ $28\times0=28*$ $c\times0=c$ $t\times d=0$	When you a multiply a number times 0, you get zero. $a\times0=0$	Multiplication involving 0 (Zero Product Property)
$45+67=67+45$ $156+78=c+156$	When you add two numbers, you can change the order of the numbers you add. $a+b=b+a$	Commutative property for addition
$17\times28=28\times17$ $39\times46=c\times39$	When you multiply two numbers, you can change the order of the numbers you multiply and get the same number. $a\times b=b\times a$	Commutative property for multiplication

Justification

What to do:

- ☑ The teacher facilitates discussion about one conjecture by asking, "Is this always true?" or "Is this true for all numbers?"
- ☑ Students work together to develop a rationale for their answer and explain their reasoning to the class.
- ☑ The teacher supports discussion and probes students to develop an explanation that develops a higher level of justification.

Levels of Justification include:

1. appeal to authority
2. justification by example
3. generalizable arguments

Examples of student justifications for the classroom conjecture:

> *"When you multiply two numbers it doesn't matter the order that you multiply them."*

1. "We learned that in third grade." (*appeal to authority*)
2. "I tried it a lot of times and it always works." (*justification by example*)
3. "I made a model with cubes and see you don't even have to count them. If you turn the model one way it's 3x5 and the other way its 5x3 and you know everything is there, so they are the same." (*generalization*)

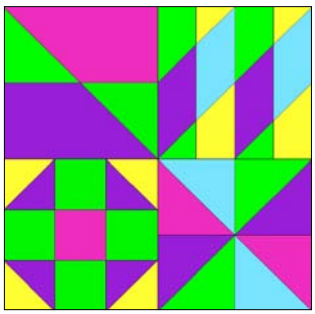
Family of Facts

What to do:

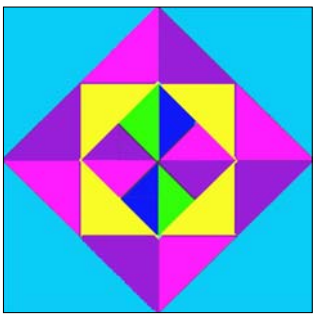
- ☑ The teacher writes three numbers on the board large enough for the entire group to see.
- ☑ Ask students to use the three numbers with an operation symbol and an equal sign to express a true equality relationship, for instance, 8, 7 and 56 or 12, 5, and 60.
- ☑ After several seconds of thinking time, ask students to share their thoughts, listing as many true number sentences as they can think of.
- ☑ Students share reasons for their decisions, working with the others to determine that the number sentences (equations) they have created make sense.
- ☑ Continue the discussion for all eight equations.
- ☑ The teacher writes a new set of three numbers, deciding upon what number relationship or number property relationship to highlight based on the points made during the student discussion.

For more information:

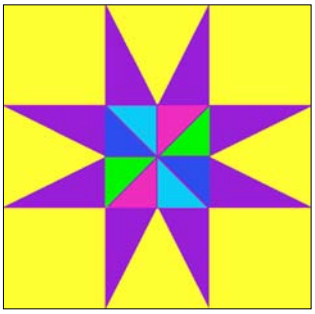
Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heineman.



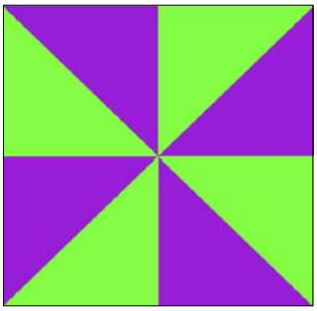
PROBLEM SOLVING



NUMBER WORK



INSPECTING EQUATIONS

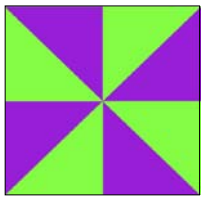


FLUENCY & MAINTENANCE

Chapter 9

Fluency and Maintenance

Fluency & Maintenance



Fluency & Maintenance

Fluency and maintenance work should always be at a student's independent level, determined by teacher observations of daily work, informal teacher created assessments, fact interviews*(See Chapter 4)*, problem-solving interviews, and post-assessments. Fluency and maintenance:

- strengthens concepts and skills
- builds efficiency and accuracy
- reinforces vocabulary

Activities provide experiences to review knowledge, concepts, and skills from all the content strands: number, operations, and algebraic relationships; geometry; measurement; and data analysis and probability.

The Fluency & Maintenance Block

The activities must be at each student's independent level.

- ☑ uses activities that have number sizes within a student's independent mental computation level (See Fact Interviews)
- ☑ requires 15 minutes of a math hour or is assigned as homework
- ☑ may occur while a teacher meets with a small group during problem solving
- ☑ supports learning in a small group or as independent work

When assigning fluency and maintenance as homework it is important to consider factors that make homework effective. Homework "can enhance achievement by extending learning beyond the school day." (Marzano, R., & Pickering, J., 2007) However, fluency and maintenance homework:

- ☑ must be at a student's independent level
- ☑ may foster improved achievement as well as positive attitudes and habits only when appropriately assigned
- ☑ should require about 15 minutes
- ☑ permit appropriate parental involvement (asking questions and clarifying or summarizing)

Fluency and maintenance homework assignments should involve practicing a skill or process that students can do independently but not fluently.

Teachers should carefully monitor the amount of homework assigned and help students learn to monitor its completion. Students need clear guidelines about what to do when they don't understand or can't complete the assignment.

Feedback about the homework is important and helps students reflect on their progress.

Teaching Activities for the Fluency & Maintenance Block

Teachers will find ideas for activities that can build fluency or maintain proficiency in the school's curricular resources. This chapter includes a few suggestions for easy to learn fluency games involving number facts.

Other suggestions for activities in the Fluency & Maintenance block include:

- card games (use number cards instead of playing cards)
- shape puzzles (Ex. tangrams, pentominoes)
- logic puzzles
- daily mental math practice
- problem solving practice pages (at student's independent level)
- gathering data for graphing assignments
- fact games or geometry puzzles on web sites

The following games and activities described in this chapter can easily be adapted for a range of learners. They provide examples of appropriate leveled practice for the Fluency & Maintenance block.

- Salute
- Bing, Bang, Buzz
- Close to 100
- Digits
- Differences
- Winning Touch
- Four-In-A Row
- Products
- Rio
- Juniper Green

Salute

Topic	Identifying a missing factor
Players	3
Materials	Handmade numeral cards (6 each of numbers 1-9) or playing cards (10-K: removed)
Object	Be the first player to state the missing addend or factor
Play	One player serves as the referee. Deal out all cards into two face-down piles in front of the two players. When the referee says, "Go" the players each take their top card and without looking at it, place it on their forehead facing out for the other player to see. The referee states the total and each player tries to be the first to determine what his or her card is based on what number is on the card that the other player is showing.
Variation	Adjust for the facts that students need to practice. For example: Sums to 7 or 15, multiples of 5.

Bing, Bang, Buzz

Topic	Multiples and base ten
Players	Small or large group
Materials	None
Object	Keep the game going as long as possible without error
Play	The group sits in a circle or around a table. The first person says the number 1. The person on the left says 2 and so on. Each time somebody gets to a number that has a 7 in it, or is a multiple of 7, they must say, "Buzz" instead of say the number (e.g. 7, 14, 17, etc.) Once the group has the concept down and is doing well, add Bang. "Bang" must be said for any number that has a 10 in it or is a multiple of ten. Some numbers will be both a multiple of 10 and a multiple of 7 (e.g. 70) – in this case the person must say "Bang, Buzz." Anytime someone says the wrong word, you start back at one.
Variation	To really challenge the group, add Bing, which is said any time there is a five in the number or the number is a multiple of 5. There will be many numbers that are both multiples of 5 and 10, in which case the player must say, "Bing, Bang." If the number is a multiple of 5, 10, and 7, the player should say, "Bing, Bang, Buzz."

Close to 100 (or 1000)

Topic	Composing & comparing numbers
Players	2-3
Materials	Handmade numeral cards (6 each of numbers 1-9) and Close to 100 Score Sheet
Object	Get as close to 100 (or 1000) as possible
Play	Deal six cards for Close to 100 (eight for Close to 1000) to each player face up. Players take turns making two 2-digit numbers (two 3-digit numbers) that when added total as close to 100 (1000) as possible. Write the numbers and the total on the Close to 100 (1000) Score Sheet. Each player calculates their own score which is the difference between the total they created and 100 (1000). Discard the used cards and deal out new cards to replace them. After five rounds, total scores. The player with the lowest score is the winner.
Teaching Tip	Close to 100 is very challenging for students just beginning to understand base ten concepts. Students must be able to easily decompose numbers and keep their thinking organized. Close to 100 builds mental computation capacity. Be sure to observe how the students are using their fact knowledge to play the game. Ask them to explain their strategies to each other using tens and ones language.

Close to 100 Score Sheet

Date_____

Name_____

PLAYER 1 Round Score

Round 1: _____ + _____ = _____ _____

Round 2: _____ + _____ = _____ _____

Round 3: _____ + _____ = _____ _____

Round 4: _____ + _____ = _____ _____

Round 5: _____ + _____ = _____ _____

Total Score: _____

Name_____

PLAYER 2 Round Score

Round 1: _____ + _____ = _____ _____

Round 2: _____ + _____ = _____ _____

Round 3: _____ + _____ = _____ _____

Round 4: _____ + _____ = _____ _____

Round 5: _____ + _____ = _____ _____

Total Score: _____

Close to 1000 Score Sheet **Date_____**

Name_____

PLAYER 1 Round Score

Round 1: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 2: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 3: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 4: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 5: ___ ___ ___ + ___ ___ ___ = _____ _____

Total Score: _____

Name_____

PLAYER 2 Round Score

Round 1: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 2: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 3: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 4: ___ ___ ___ + ___ ___ ___ = _____ _____

Round 5: ___ ___ ___ + ___ ___ ___ = _____ _____

Total Score: _____

Digits

Topic	Composing, comparing, and calculating differences
Players	2 or 3
Materials	Numeral cards (6 each of numbers 1-9) and Digits Score Sheets
Object	Find differences between a target number (100 or 1000) and a 2- or 3-digit number
Play	Decide the target (100 or 1000). Deal out one more card than the digits in the target to each player. Players use the numerals to make a number as close as possible to the target. Record and find the difference between the target and the number. Total scores after three rounds. Lowest score wins

Digits Score Sheet (100)

Date_____

PLAYER _____

Game 1

		Difference
Round 1:	100 – ____ ____ = _____	_____
Round 2:	100 – ____ ____ = _____	_____
Round 3:	100 – ____ ____ = _____	_____
	Total Score:	_____

Game 2

		Difference
Round 1:	100 – ____ ____ = _____	_____
Round 2:	100 – ____ ____ = _____	_____
Round 3:	100 – ____ ____ = _____	_____
	Total Score:	_____

Game 3

		Difference
Round 1:	100 – ____ ____ = _____	_____
Round 2:	100 – ____ ____ = _____	_____
Round 3:	100 – ____ ____ = _____	_____
	Total Score:	_____

Digits Score Sheet (1000) **Date_____**

PLAYER _____

Game 1

		Difference
Round 1:	1000 – ____ ____ ____ = _____	_____
Round 2:	1000 – ____ ____ ____ = _____	_____
Round 3:	1000 – ____ ____ ____ = _____	_____
	Total Score:	_____

Game 2

		Difference
Round 1:	1000 – ____ ____ ____ = _____	_____
Round 2:	1000 – ____ ____ ____ = _____	_____
Round 3:	1000 – ____ ____ ____ = _____	_____
	Total Score:	_____

Game 3

		Difference
Round 1:	1000 – ____ ____ ____ = _____	_____
Round 2:	1000 – ____ ____ ____ = _____	_____
Round 3:	1000 – ____ ____ ____ = _____	_____
	Total Score:	_____

Differences

Topic	Composing, comparing, and calculating differences
Players	2
Materials	Numeral cards (6 each of numbers 1-9) and Differences Score Sheet
Object	Find differences between two 2-digit numbers (or two 3-digit numbers)
Play	Deal four cards face up to each player. Make two 2-digit numbers as close as possible to each other using the four cards and record on the Score Sheet. Find the difference between the two numbers mentally and record. After three rounds players total their differences. The smallest total wins.

Differences Score Sheet **Date**_____

(2-digit numbers)

PLAYER _____

Game 1

		Difference
Round 1:	____ ____ – ____ ____ = _____	_____
Round 2:	____ ____ – ____ ____ = _____	_____
Round 3:	____ ____ – ____ ____ = _____	_____
	Total Score:	_____

Game 2

		Difference
Round 1:	____ ____ – ____ ____ = _____	_____
Round 2:	____ ____ – ____ ____ = _____	_____
Round 3:	____ ____ – ____ ____ = _____	_____
	Total Score:	_____

Game 3

		Difference
Round 1:	____ ____ – ____ ____ = _____	_____
Round 2:	____ ____ – ____ ____ = _____	_____
Round 3:	____ ____ – ____ ____ = _____	_____
	Total Score:	_____

Differences Score Sheet **Date**_____

(3-digit numbers)

PLAYER _____

Game 1

Difference

Round 1: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 2: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 3: ___ ___ ___ – ___ ___ ___ = _____ _____

Total Score: _____

Game 2

Difference

Round 1: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 2: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 3: ___ ___ ___ – ___ ___ ___ = _____ _____

Total Score: _____

Game 3

Difference

Round 1: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 2: ___ ___ ___ – ___ ___ ___ = _____ _____

Round 3: ___ ___ ___ – ___ ___ ___ = _____ _____

Total Score: _____

Winning Touch

Topic	Multiplication
Players	2 or 3
Materials	*Winning Touch* game board, 16 labeled tiles
Object	To have the fewest tiles when the play ends
Play	All sixteen tiles are placed face down and mixed well, and each player takes two tiles to begin the game. The first player chooses a tile and places it in the square corresponding to its two factors. For example, 25 must be placed in the column labeled '5' that intersects the row labeled '5.' The first player then takes one tile from the facedown pile to have two tiles again. The players take turns placing one tile at a time on the board. To be played, a tile must share a side with a tile that is already on the board. In this example, the second tile played must have 5 as a factor. Touching a corner is not enough. If a player does not have a tile that can be played the player misses a turn and takes a tile from the facedown pile keeping it in their collection. They cannot play it during this turn. If a player puts a tile on an inappropriate square, the person who catches the error plays next and the person who made the error takes their tile back. Play ends when no more tiles can be played (must touch another tile).
Teaching Tip	Children should use their own thinking and discussion with others to figure out the products. Do not provide a multiplication table. Introduce new boards as the students become proficient with the easier boards.

Winning Touch to 6

	3	4	5	6
3				
4				
5				
6				

Tiles for Winning Touch to 6

9	12	15	18
12	16	20	24
15	20	25	30
18	24	30	36

Winning Touch to 7

	3	4	5	6	7
3					
4					
5					
6					
7					

Tiles for Winning Touch to 7

9	12	15	18	21
12	16	20	24	28
15	20	25	30	35
18	24	30	36	42
21	28	35	42	49

Winning Touch to 8

	3	4	5	6	7	8
3						
4						
5						
6						
7						
8						

Tiles for Winning Touch to 8

9	12	15	18	21	24
12	16	20	24	28	32
15	20	25	30	35	40
18	24	30	36	42	48
21	28	35	42	49	56
24	32	40	48	56	64

Winning Touch to 9

	3	4	5	6	7	8	9
3							
4							
5							
6							
7							
8							
9							

Tiles for Winning Touch to 9

9	12	15	18	21	24	27
12	16	20	24	28	32	36
15	20	25	30	35	40	45
18	24	30	36	42	48	54
21	28	35	42	48	56	63
24	32	40	48	56	64	72
27	36	45	54	63	72	82

Four-in-a-Row

Topic	Multiplication Facts
Players	2
Materials	*Four-in-a-Row* game board, 36 transparent chips in two colors (18 of each), two paper clips
Object	Place four same color chips in a row vertically, horizontally, or diagonally
Play	Each player takes eighteen chips of the same color. To begin the game, the first player takes the two paper clips and places them on any two of the numbers listed below the square, such as the 4 and 5. The same player then multiplies these numbers and puts one of their eighteen chips on any 20 (4×5). The second player moves one of the two paper clips that are now on the 4 and the 5. If the second player moves one of them from 4 to 3, this person can place one of their eighteen chips on any 15 (3×5). On every subsequent turn, a player must move one of the two paper clips to a different number. Two paper clips can be placed on the same number, to make 5×5, for example. The person who is first to make a line of four chips of their color, vertically, horizontally, or diagonally, is the winner.
Teaching Tip:	The pages included are examples and can be used to focus children's efforts on a few combinations at the correct level of difficulty. When a board becomes too easy, or if it is too hard, make a new page to introduce an appropriate new set of factors and products to match the student's knowledge.

Four-in-a-Row (factors 2-5)

6	9	20	15	8	10
12	8	25	4	6	16
4	15	9	10	20	4
16	4	8	25	12	30
12	20	25	15	6	4
6	16	8	9	25	10

Factors:

2 3 4 5

Four-in-a-Row (factors 3-6)

24	9	20	15	30	18
12	30	25	36	24	16
36	15	9	18	20	36
16	36	30	25	12	30
12	20	25	15	24	36
24	16	30	9	25	18

Factors:

3 4 5 6

Four-in-a-Row (factors 4-7)

24	49	20	35	30	42
28	30	25	36	24	16
36	35	49	42	20	36
16	36	30	25	28	30
28	20	25	35	24	36
24	16	30	49	25	42

Factors:

4 5 6 7

Four-in-a-Row (factors 5-8)

48	49	56	35	30	42
40	30	25	36	48	64
36	35	49	42	56	36
64	36	30	25	40	30
40	56	25	35	48	36
42	64	30	49	25	42

Factors:

5 6 7 8

Four-in-a-Row (factors 6-9)

48	49	56	63	54	42
54	72	81	36	48	64
36	63	49	42	56	36
64	36	72	81	54	56
54	56	42	63	48	36
72	64	72	49	81	42

Factors:

6 7 8 9

Products

Topic	Multiplication Facts
Players	2
Materials	*Product* game board, 18 transparent chips in two colors, two markers or paper clips
Object	Place four chips in a row, vertically, horizontally, diagonally, or until all squares have been covered.
Play	To begin the game, Player 1 moves a marker (paper clip) to a number in the factor list at the bottom of the board Player 2 then moves the other marker to any number in the factor list (including the number marked by Player 1). The product of the two marked numbers is determined and covered by Player 2. Player 1 moves *either* marker to another number and covers the new product and with a transparent chip. Play continues until one player covers fours squares in a row – vertically, horizontally, diagonally, or until all squares are covered. Adapted from the Web applet retrieved April 3, 2007: http://illuminations.nctm.org/ActivityDetail.aspx?ID=29

Products (0-36)

0	1	2	3	5
6	8	9	10	12
14	15	Free Space	16	18
20	21	24	25	27
28	30	32	35	36

0	1	2	3	4	5	6

Products (1-81)

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

1	2	3	4	5	6	7	8	9

Rio

Topic	Multiples
Players	3
Materials	Tiles for the ten multiples of 2-9 (see next pages), fifteen transparent chips or counters (5 each of three different colors), one 10-sided number cube
Object	To be the first player to run out of chips
Play	Choose a multiple to practice (ten tiles). Scatter the tiles in the middle of the table. Each player takes five chips. Example: "Let's practice the fours." The first player rolls the number cube, and if a 5 comes up the player puts a chip on the tile marked '20' (5×4). The second player then rolls the number cube, and if an 8 comes up the player puts a chip on '32' (8×4). If the third player rolls a 5, the tile marked '20' already has a chip on it, so the player must take it. The third player now has six chips and the first player has four. Play continues until one person has no chips.
Teaching Tip:	This is a good introductory game. Most third graders begin by using repeated addition rather than multiplication. As they continue to play Rio, finding products when multiplying by 2 and 10 becomes easy. Then, they master the multiples of 5, etc. Increase the challenge as students become proficient with each new level. Keep accurate records about what group of facts each child knows so that you can match students to the game at their independent practice level.

Tiles for *Rio*

2	4	6	8	10
12	14	16	18	20

3	6	9	12	15
18	21	24	27	30

4	8	12	16	20
24	28	32	36	40

5	10	15	20	25
30	35	40	45	50

6	12	18	24	30
36	42	48	54	60

7	14	21	28	35
42	49	56	63	70

8	16	24	32	40
48	56	64	72	80

9	18	27	36	45
54	63	72	81	90

Juniper Green 50 (100)

Topic	Factors and multiples
Players	2
Materials	Juniper Green game boards (one for each game), highlighter or pencil
Object	Use strategy to force opponent out of play
Play	The first player begins by highlighting an even number. Players take turns highlighting any remaining number that is a *factor or a multiple* of the previous number selected by the opponent. Play continues until no *factor or multiple* can be highlighted.

Juniper GREEN 50

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Juniper GREEN 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

For More Information:

Marzano, R. J. & Pickering, D. J. (2007). The case for and against homework: Responding to changing demographics. *Educational Leadership*, *64* (60), 74-79.

Hope, J. A., Reys, B. J. & Reys, R. E. (1987). *Mental math in the middle grades: Blackline Masters*. Parsippany, NJ: Dale Seymour Publications.

Kamii, C. & Anderson, C. (2003). Multiplication Games: How We Made and Used Them. In *Teaching Children Mathematics*, 135-141.

Kamii, C. (2004). *Young children continue to reinvent arithmetic: 2nd grade implications of Piaget's theory*. New York: Teachers College Press.

Number Development Assessments

Developmental Guidelines

Strategies for Effective Intervention

Intervention Activities

Planning and Progress Record

REKENREK

CHAPTER 10

Intervention Instruction

This work was made possible through the generous support of:

The Madison Community Foundation that understood the importance of all children exiting elementary school as proficient math students and provided the initial funds to secure supplies, the services of a consultant, and the release time for the initial work directed at first grade teachers.

DIME, Diversity in Mathematics Education Project, a National Science Foundation Center for Teaching and Learning Project, that provided the funds to release a team of intermediate grade teachers from their classrooms to develop this intervention project.

Angela Andrews, our consultant from the National Louis University whose expertise, ideas, and knowledge guided us as we extended the primary intervention initiative to the intermediate grades.

And most of all,

The 16 elementary teachers who left their classrooms to join us throughout the school year as well as the many more who make up the instructional teams at every elementary school. Your dedication to children and willingness to try new ideas on behalf of all your mathematics students is truly what makes the Madison Metropolitan School District a great place to be.

CHAPTER 10 TABLE OF CONTENTS

Intervention Instruction

The Goal

The goal of this intervention initiative is to provide teachers with instructional strategies to help intermediate grade students become proficient in math and benefit from grade-level instruction.

To address specific student learning needs, teachers:

- ☑ find out what each child already knows
- ☑ determine what each child needs to know
- ☑ plan an appropriate sequence of learning activities
- ☑ monitor the child's progress

This chapter and accompanying professional development provides teachers with assessment and corresponding learning activities to address number knowledge.

Assessment

A student who scores "minimal" or "basic" on the WKCE or receives a "1" on two or more sections of the report card may need intervention. Two types of assessments are included in this chapter. "Probes" which are given orally provide data for MMSD's Student Information Monitoring System. "Interviews" which are also given orally provide more detailed information about a child's number knowledge. Additional assessments can be found in this binder and in Chapter 4 of *Learning Mathematics in the Primary Grades (LMIP)*.

For the purpose of learning about a student's…	*The teacher collects data from…*
Number Knowledge	Fact Interviews A-E, Chap. 4, pp. 5-36 Number Development Assessment (Interview), Chap. 10, pp. 9-13 Number Development Probe Levels A1, A2 & B, Chap. 10, p. 15
Problem Solving	Problem Solving Probe A or B, Chap. 10, pp. 17-20 CGI Problem-type Assessment, Chap. 4, pp. 44-48 Problem Solving Interviews, *LMPG* - Chap. 4, pp. 75-85
Place Value Knowledge	Problem Solving Probes A or B, Chap. 10, pp. 17-20 Early Base Ten Interview, Chap. 4, p. 68
Fraction Knowledge	Problem Solving Probe A or B, Chap. 10, pp. 17-20 Beginning Fraction Knowledge Interview Chap. 10, p. 21

Developmental Guidelines

Teachers use the information from these assessments and others to guide instruction. The **Developmental Guidelines** on pages 33-38 provide an approximate sequence for number knowledge development. Each table lists the components of number development at a given level.

Directions for Administering Assessments

Number Development Assessment (Interview)

This assessment is administered individually. It is not necessary to administer the entire assessment in one session. An interview script precedes each section. While it is not critical to say exactly what is indicated on the script, variations that provide extra scaffolding for the student (e.g. emphasizing vocabulary, inserting adjectives as clues or gesturing to materials) should be noted as they may provide clues regarding future instructional decision making.

Materials included in this section pp. 23-27:

Numeral Identification Strips
Number Sequencing Cards
Dot Cards

Additional materials needed:

Two colors of counters
Cloth or paper to cover counters
Base-ten blocks and/or ten frames

Number Development and Problem Solving Probes:

The *Number Development Assessment (Interview)* assesses beginning number concepts. Use the *Number Development Probes for Levels A1, A2, and B* pp. 17-20 to assess the same number concepts with larger numbers. The *Problem Solving Probe* assesses student's use of number knowledge to solve story problems given orally.

Beginning Fraction Knowledge (Interview):

The *Beginning Fraction Knowledge (Interview)* p. 21-22 can help identify foundational fraction concepts students need for success in grades 4 & 5.

Number Development Assessment (Interview)

Name: _____*Score*_____

School: _____ *Teacher:* _____

Date of Interview: _____ *Interviewer:* _____

Points for each task apply to progress monitoring.

1. Forward Number Sequence

Say: *"Start counting from ___and I'll tell you when to stop."*

Level 1:	(a) 1 (to 32) _____	(b) 8 (to 17) _____
	(c) 22 (to 30) _____	
Level 2:	(a) 47 (to 53) _____	(b) 77 (to 83) _____
Level 3:	(a) 96 (to 112) _____	

Level 1: Max. of 3 points (1 pt. each for a, b and c) *Score*_____
Level 2: Max. of 2 points (1 pt. each for a and b)
Level 3: Max. of 1 point for this level

2. Number After

Say: *"Tell me the number that comes right after _____. For example, if I say 1, you would say _____?"*

Level 1:	2	5	9	12	19		
Level 2:	49	29	50	80	59	79	39
Level 3:	109						

Levels 1-3: Maximum of 3 points (1 pt. for each level that all are correct) ***Score***_____

3. Backward Number Sequence

Say: *"Count backward starting at 10…"*

Level 1:	(a) 10 (down to 1) _____	(b) 15 (down to 10) _____
Level 2:	(a) 22 (down to 16) _____	(b) 33 (down to 26) _____
	(c) 62 (down to 56) _____	(d) 85 (down to 77) _____
Level 3:	(a) 112 (down to 99) _____	

Level 1: Max. of 2 points (1 pt. each for a and b)
Level 2: Max. of 3 points (1 pt. each for a, b, c and d)
Level 3: Max. of 1 point for this level

*Score*_____

4. Number Before

Say: *"Tell me the number that comes just before _____. For instance, if I say 2, you would say _____?"*

Level 1:	3	5	9	14	20
Level 2:	41	89	60	69	100
Level 3	110				

Levels 1-3: Max. of 3 points (1 pt. for each level that all are correct)

Score_____

5. Numeral Identification

Say: *"What number is this?"*

Level 1:	8	3	5	7	9	2	4	6	1	10
Level 2:	24	29								
Level 3:	12	20	83	14	81	13	21	15		
Level 4:	340	213	850	620	380					

Levels 1-4: Max. of 4 points (1 pt. for each level that all are correct)

Score_____

6. Sequencing Numbers Say: *"Please put these numbers in order from smallest to largest, starting here."* Point to left of workspace. Ask child to identify the numerals in order after sequencing. Record what student says.

Level 1: 1-10 _____

Level 2: 8-17 _____

Level 3: Decade cards for 10-100 _____

Level 4: 64-73 _____

Levels 1-4: Max. of 4 points (1 pt. for each level that all are correct) ***Score***_____

7. Subitizing Say: *"I'm going to show you some dots really quickly. I want you to tell me how many there are."* Show each dot pattern 1-2 seconds.

Level 1: (regular dot patterns)	2	4	3	5	6
Level 2: (irregular dot patterns)	3	4	5	6	
Level 3: (regular dot patterns)	7	8			
(irregular dot patterns)	7	8			

Not scored

8. Additive Tasks ***Note:*** *Use one color for the covered set and a different color for the added set of counters.*

Level 1, part 1: Count out a set of 18 counters (same color)_____

Level 1, part 2: Say: "*I have ____ counters under here (show the counters and then cover). I am going to slide one more under (the child should watch the counter being slid under). How many are under here now?"*

(a) 3 and then slide one more _____

(b) 7 and then slide one more _____

(c) 11 and then slide one more _____

Level 1: Max. 4 points. (1 pt. each) ***Score***_____

Level 2: Say*: "I have _____ counters under here and _____ more counters over here.* (Leave counters exposed adjacent to cover and wave your hand over both sets.) *How many altogether?"*

(a) 3 covered, 1 exposed _____

(b) 4 covered, 2 exposed _____

(c) 5 covered, 4 exposed _____

(d) 12 covered, 3 exposed _____

(e) numeral 22 covered, 2 exposed _____

Level 2: Max. 5 points. (1 pt. each) ***Score***_____

Level 3: Say: *"I have _____ counters under here* (show and then cover set) *and I have _____ counters under here.* (Show and then cover set.) *How many altogether? "*Wave your hand over both covered sets.

(a) 5 + 2 _____

(b) 7 + 5 _____

(c) 15 + 3 _____

(d) numeral 25 + 3 (counters) _____

Level 2: Max. 4 points. (1 pt. each) ***Score***_____

9. Missing Addend Tasks Teacher covers the counters and asks the student to look away. Say: *"Here are _____. I am putting some more under here. Now there are _____. How many more did I put under here?"* **Note:** *Use two colored counters with the original set in one color and the added items in another.*

(a) 4 to 5 (secretly put 1 under) _____

(b) 5 to 7 (secretly put 2 under) _____

(c) 6 to 9 (secretly put 3 under) _____

(d) 15 to 17 (secretly put 2 under) _____

Parts a-d: Max. 4 points. (1 pt. each if all correct) ***Score*_____**

10. Tens and Ones Tasks Depending on the child's familiarity with the tool, you may use cubes organized in tens sticks, ten frames, base-ten blocks, or a picture of these to assess this understanding.

Part One: Place 4 unit cubes in front of student. Ask "How many?" Then place a group of ten, base ten block, or a ten frame next to the unit cubes, ask "Now, how many?" Continue placing tens until you reach 74. Circle the correct answers given.

4 14 24 34 44 54 64 74

Part Two: Place the following sequence of cubes, base ten blocks, ten frames, or a representation of them, in a line in front of the child: 10, 3, 10, 10, 4, 3, 10, 2, 10, 10. Cover everything. Slowly uncover each set and ask the child to add that next quantity to the total. Circle the correct answers given:

Totals: 10 13 33 37 40 50 52 72

Part Three: Place the following sequence of cubes, base ten blocks, ten frames, or a representation of them, in a line in front of the child: 4, 10, 20, 12, 25. Cover everything. Slowly uncover each set and ask the child to add that next quantity onto the total. Circle the correct answers given:

4 14 34 46 71

Parts 1-3: Max. 3 points. (1 pt. for each part that is correct) ***Score*_____**

Number Development Probe - Levels A1, A2 & B

Student: _____ Date _____Recorder_____

Task	*Points*
Count forward from any number Within 1000 (1 pt.) E.g. "Start counting at 396" (Stop student at 413) Within 10,000 (2 pts.) E.g. "Start counting at 4,989" (Stop student at 5,012)	
Name the number (up to 3) after a given number Within 1000 (1 pt.) E.g. "What number is three after 698?" Within 10,000 (2 pts.) E.g. "What number is three after 2,489?"	
Rote count by 10 From any 2 digit number (1 pt.) E.g. Start at 13 and count by 10s to 63 From any 3 digit number (2 pts.) E.g. Starts at 598 and count by 10s to 648	
Count backward from any number and across decades and hundreds Within 100 (1 pt.) E.g. "Count beginning at 63" (Stop at 49) Within 1000 (2 pts.) E.g. "Count beginning at 214" (Stop at 196)	
Name the number (up to 3) before a given number Within 100 (1 pt.) E.g. "What number comes 3 before 51?" Within 1000 (2 pts.) E.g. "What number comes 3 before 511?"	
Read numbers Within 100-1000 (1 pt.) E.g. 315 and 756 Within 1,000-10,000 (2 pts.) E.g. 5,028 and 9,206	
Sequence numbers from smallest to largest Within 100-1000 (1 pt.) E.g. Sequence 218, 303, 330, 456, 465 Within 1,000-10,000 (2 pts.) E.g. Sequence 5,012, 5,210 and 5,102	
Compose and decompose numbers Within 20 (1 pt.) E.g. Find four expressions that equal (or ways to make) 13 Within 100 (2 pts.) E.g. Find four expressions that equal (or ways to make) 25	
Total	

*Name:*_____ *Date:*_____ *Recorder:*_____

Problem Solving Probe A

Materials:

- ☑ Pencil and paper
- ☑ Unifix cubes

Directions:

- ☑ Read each problem to the child using the child's name in the problem. If the child is unable to solve the problem mentally offer Unifix cubes, paper and pencil.
- ☑ Record child's strategy
- ☑ Assign: 0 pts. if child cannot solve the problem
  1 pt. if child can model the solution to the problem with a drawing or blocks
  2 pts. if child uses any strategy beyond direct modeling or knows a number fact

Story Problems	***Points***
JRU (Join Result Unknown) _____ had 10 markers. The teacher gave him/her 6 more. How many markers did _____ have then? *Strategy:*	
SRU (Separate Result Unknown) _____ had 12 coins. He/she gave 3 to a friend. How many coins did _____ have left? *Strategy:*	
M (Multiplication) _____ has 4 cups. There are 10 grapes in each cup. How many grapes does _____ have in all? *Strategy:*	
MD (Measurement Division) _____ had 30 books to put in a bookcase. He/she put 10 books on each shelf. How many shelves did he/she use? *Strategy:*	

Problem Solving Probe A (cont.)

Story Problems	***Points***
JCU (Join Change Unknown) _____ has 20 dollars. How many more dollars does he/she need to have 26 dollars in all? *Strategy:*	
CDU (Compare Difference Unknown) _____ has 46 stickers. Sara has 40 stickers. How many more stickers does _____ have than Sara? *Strategy:*	
PPW, WU (Part Part Whole, Whole Unknown) _____ had 10 red balloons and 8 blue balloons. How many balloons did _____ have altogether? *Strategy:*	
SCU (Separate Change Unknown) _____ had 15 flowers. He/She gave some to a friend. Now he/she has 12 left. How many flowers did _____ give to her/his friend? *Strategy:*	
PD (Partitive Division) _____ has 12 toys. He/she has to put them away on 3 shelves. He/she wants to put the same number on each shelf. How many toys will he/she put on each shelf? *Strategy:*	
PD with a fraction as an answer There is 1 brownie on a plate. 4 friends are going to share the brownie. Each friend will get the same amount. How much brownie will each friend get? Prompt if needed: What number name would you give to the part that one friend gets? (2 pts. If correct) *Strategy:*	

*Name:*_____ *Date:*_____ *Recorder:*_____

Problem Solving Probe B

Materials:

- ☑ Pencil and paper
- ☑ Base-ten blocks

Directions:

- ☑ Read each problem to the child using the child's name in the problem. If the child is unable to solve the problem mentally offer base-ten blocks, paper and pencil.
- ☑ Record child's strategy
- ☑ Assign: 0 pts. if child cannot solve the problem
  1 pt. - if child can model the solution to the problem with a drawing or blocks or uses a counting strategy.
  2 pts. - if child uses a number sense strategy that does not involve counting by 1s.

Story Problems	***Points***
JRU (Join Result Unknown) _____ had 25 dollars. Then s/he earned 75 more dollars. How many dollars did _____ have then? *Strategy:*	
SRU (Separate Result Unknown) _____ had 54 coins in a collection. He/she gave 19 to a friend. How many coins did _____ have left? *Strategy:*	
M (Multiplication) Two spiders climbed up the wall. Each spider has 8 legs. How many legs do the two spiders have in all? *Strategy:*	
MD (Measurement Division) _____ had 30 books to put in a bookcase. He/she put 5 books on each shelf. How many shelves did he/she use? *Strategy:*	

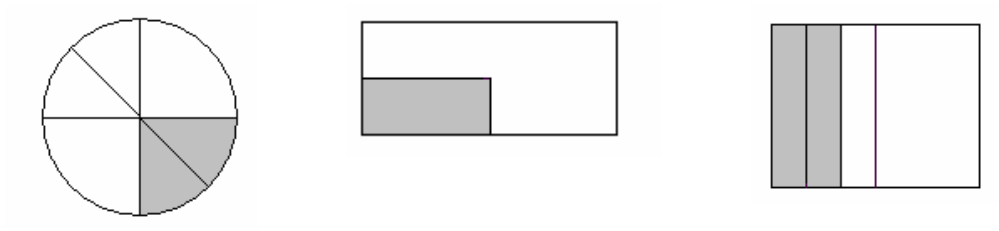
Problem Solving Probe B (cont.)

Story Problems	***Points***
JCU (Join Change Unknown) _____ has 65 dollars. How many more dollars does he/she need to have 100 dollars altogether? *Strategy:*	
CDU (Compare Difference Unknown) _____ has 100 stamps in a collection. Sara has 48 stamps in her collection. How many more stamps does _____ have than Sara? *Strategy:*	
PPW, WU (Part Part Whole, Part Unknown) _____ grew 71 flower plants in her garden. 36 had pink flowers and the rest had yellow flowers. How many plants had yellow flowers? *Strategy:*	
SCU (Separate Change Unknown) _____ had 61 paperclips. He/She gave some to the teacher. Now he/she has 29 left. How many paperclips did _____ give to the teacher? *Strategy:*	
PD (Partitive Division) _____ has 30 games. He/she has to put them away on 6 shelves. He/she wants to put the same number of games on each shelf. How many games will he/she put on each shelf? *Strategy:*	
PD with a fraction as an answer There 10 brownies on a plate. 8 friends are going to share the brownies. Each friend will get the same amount. How much of the brownies will each friend get? Prompt if needed: What number name would you give to the part that one friend gets? (2 pts. If correct) *Strategy:*	

Beginning Fraction Knowledge (Interview) – Grades 4 & 5

Name_____Date_____

1. The drawings below represent three kinds of pizza. Each pizza has a different shape. About what fraction amount is each shaded part? How do you write that fraction?



_____ _____ _____

2. You are at a party. You could either sit at a table where 4 friends equally share a small cake or you could sit at a table where 5 friends equally share a small cake. At which table would you get more cake? Why?

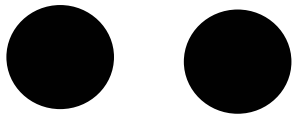
3. Twelve (12) people equally share three (3) sub sandwiches. How much does each person get?

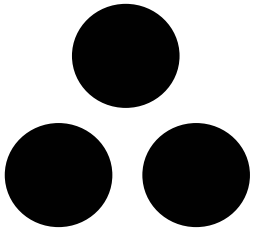
4. Daviette has 8 coins in his collection. ¼ of the coins are pennies. How many coins are pennies?

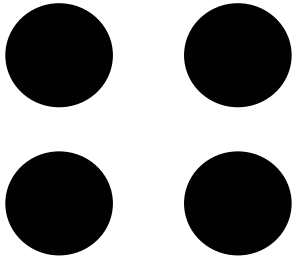
5. Grant has 3 fish. One-half of his pets are fish. How many total pets does Grant have?

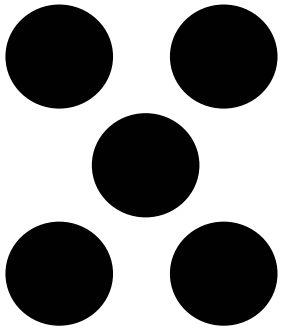
6. Breezy used ½ of a bottle of paint for an art project. Bao used ¼ of a bottle of paint for his project. How much did Breezy and Bao use in all?

Dot Cards (p. 1 of 4)

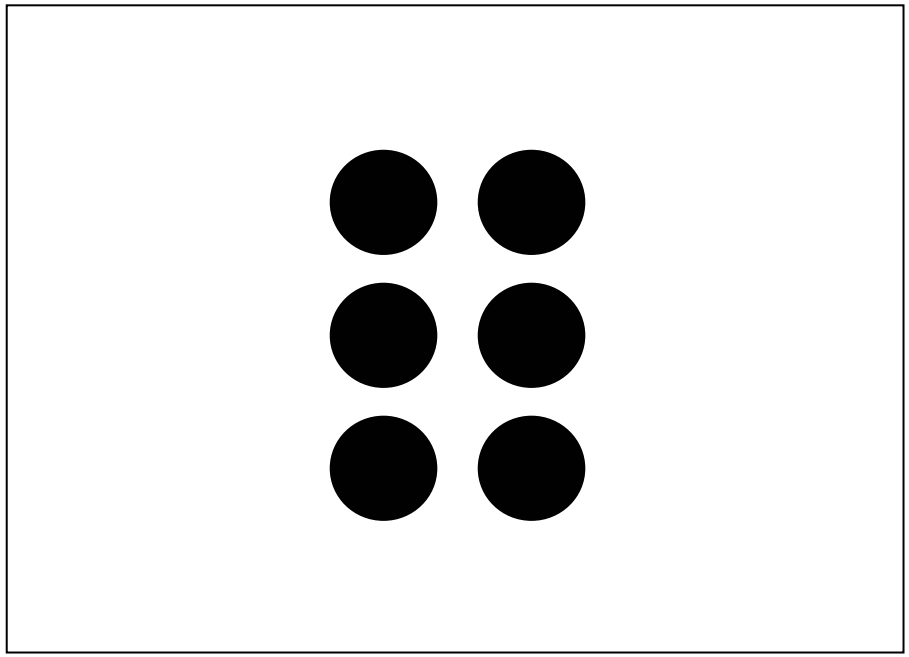


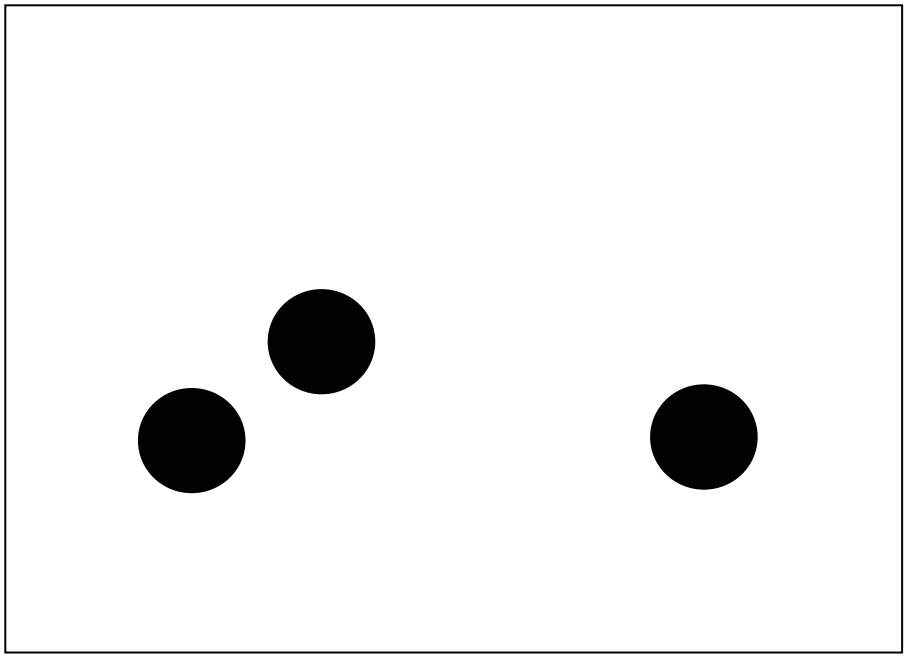


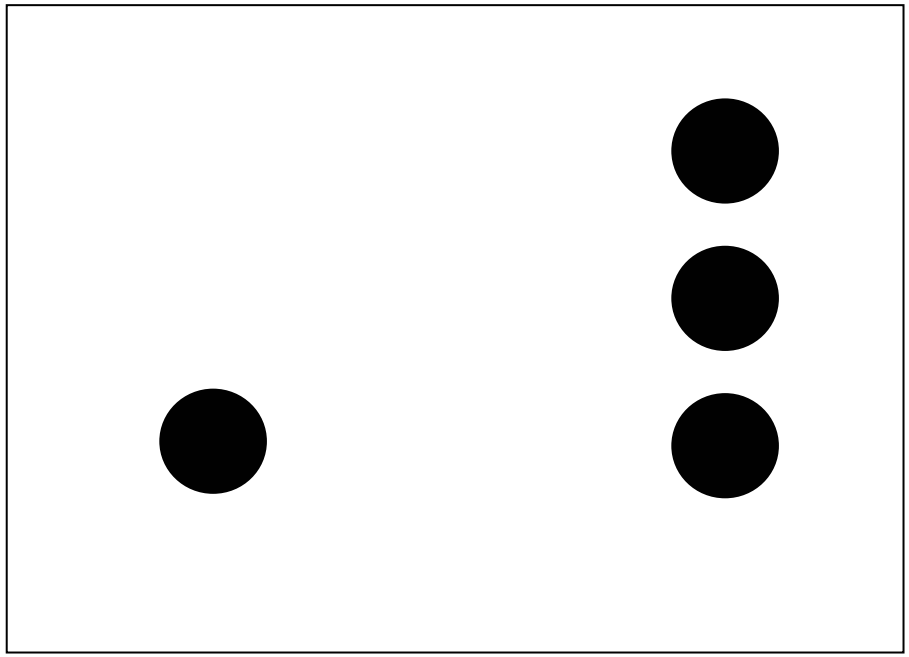


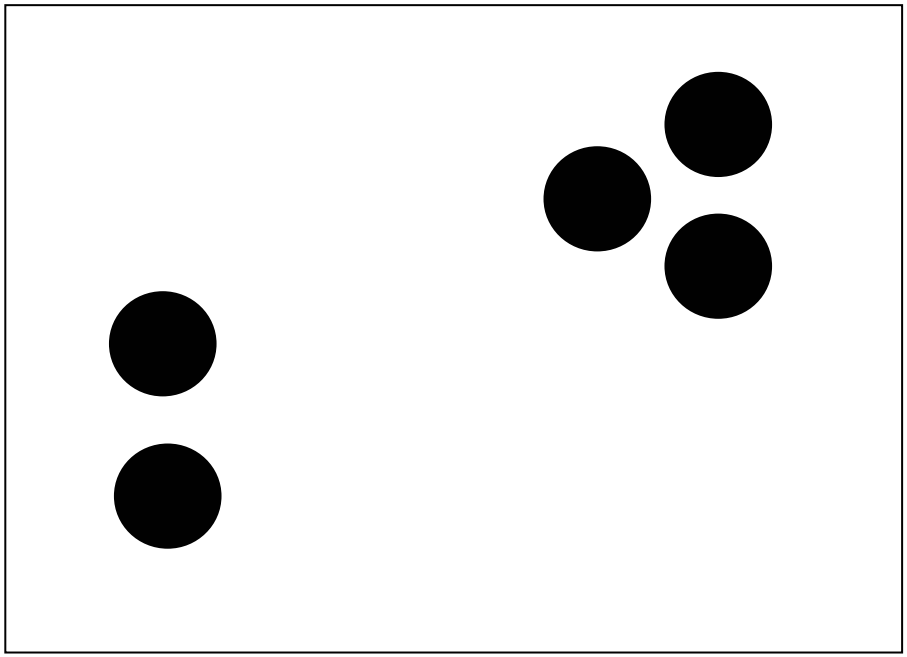


Dot Cards (p. 2 of 4)

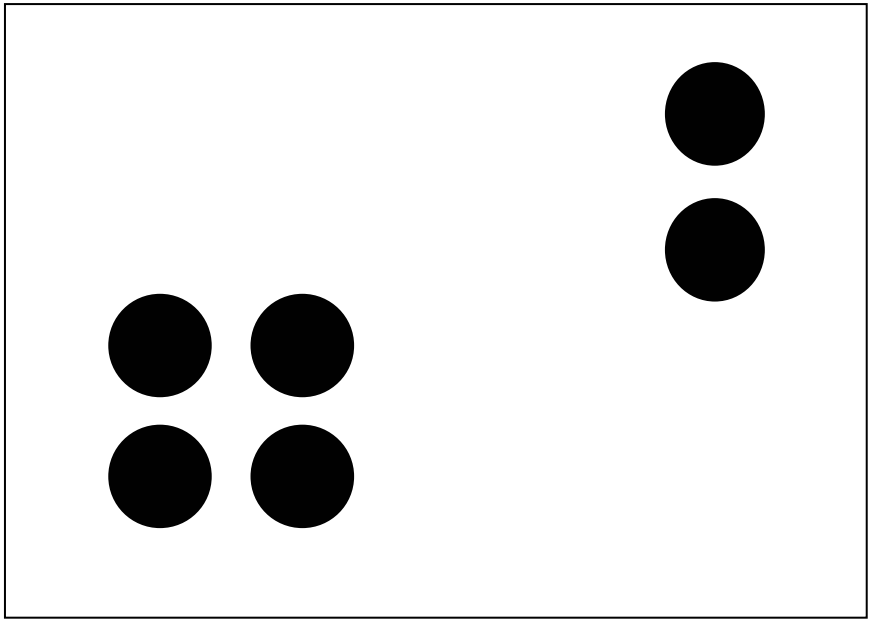


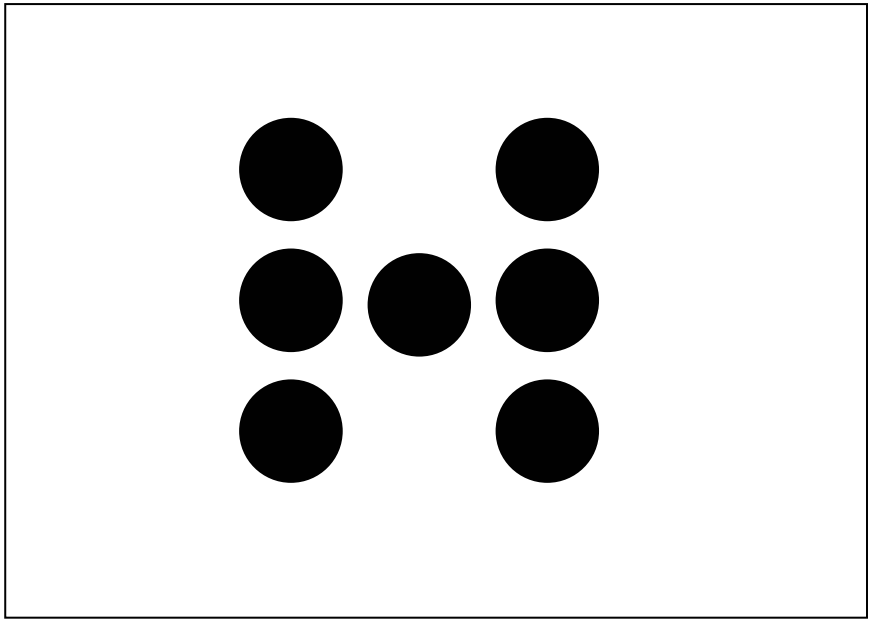


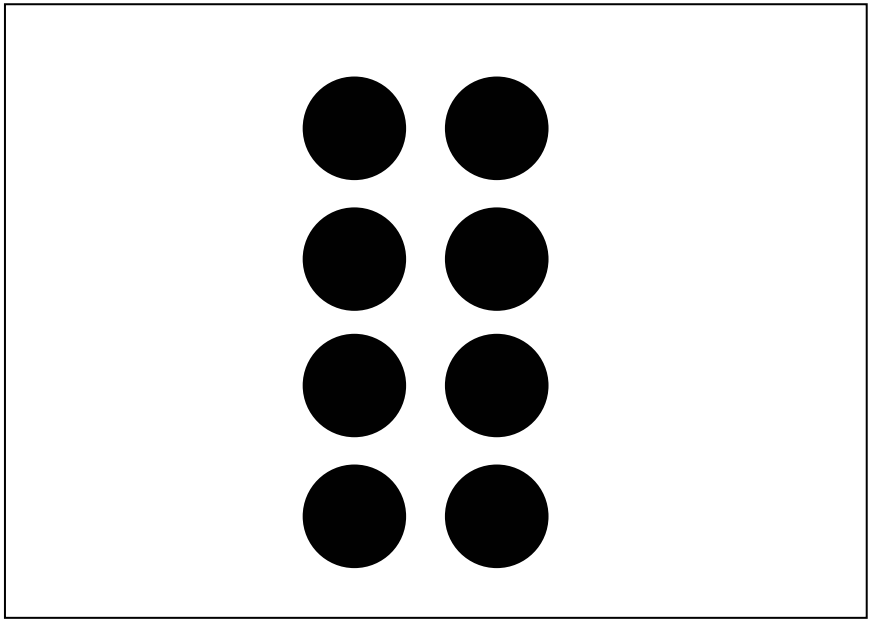


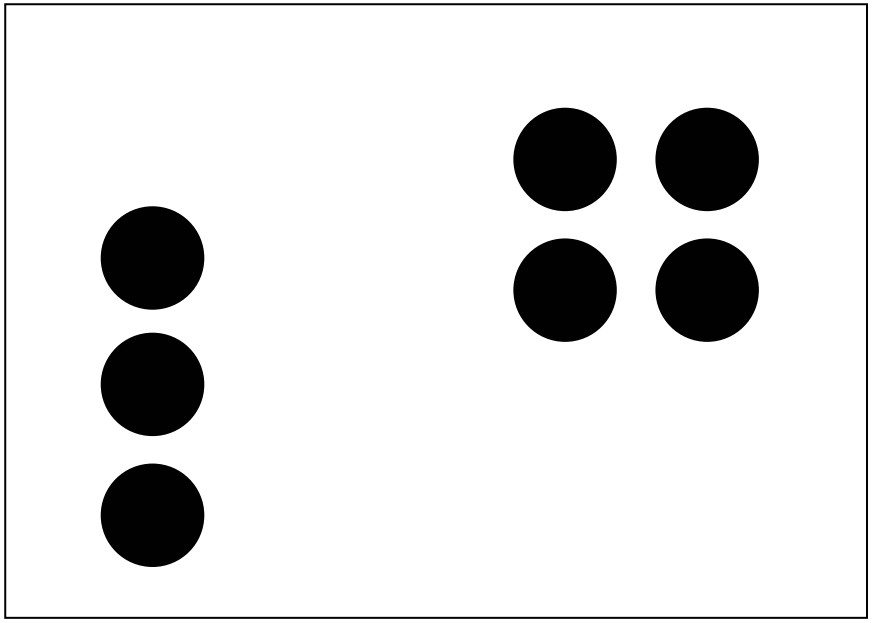


Dot Cards (p. 3 of 4)

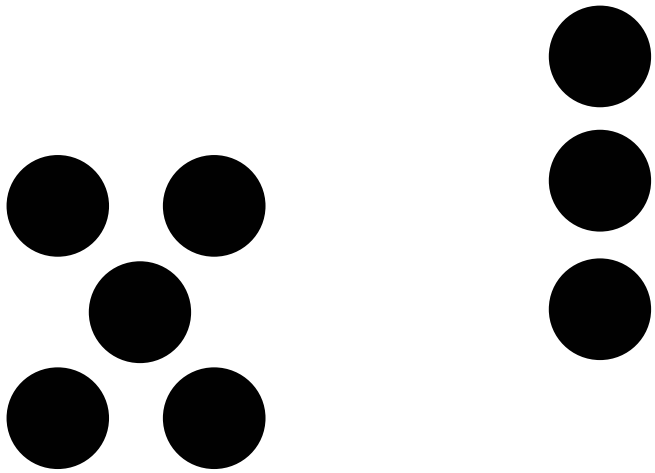








Dot Cards (p. 4 of 4)



Number Development Assessment

Numeral Identification Strips Levels 1 – 4

8 3 5 7 9 2 4 6 1 10

24 29

12 20 83 14 81 13 21 15

340 213 850 620 380

Number Development Assessment

Sequencing Level 1

1	2	3	4	5
6	7	8	9	10

Number Development Assessment

Sequencing Level 2

8	9	10	11	12
13	14	15	16	17

Number Development Assessment

Sequencing Level 3

10	20	30	40	50
60	70	80	90	100

Number Development Assessment

Sequencing Level 4

64	65	66	67	68
69	70	71	72	73

Pre-K Developmental Guidelines

A child's number knowledge may align with components in more than one level.

	Pre K	Pre K-K	Level K
Story Problem Solving	Solves problems embedded in daily routines using direct modeling including JRU, SRU, M and MD--totals to 5	Solves problems embedded in daily routines using direct modeling including JRU, SRU, M and MD--totals to 20	Solves problems using direct modeling including JRU, SRU, M and MD--totals to 30
Activities and Progress Monitor	Informal learning through play Progress monitor (not applicable)	Informal learning through play Progress monitor (not applicable)	Teacher guided story problem solving Problem Solving Probe A
Representation	Draws pictures or uses other informal symbols or objects to represent quantity in collection--not necessarily with 1:1 correspondence	Draws pictures or uses other informal symbols or objects to represent quantity in collection-- with 1:1 correspondence but may be inaccurate with larger collections	Draws pictures or uses other informal symbols or objects to represent quantity in a collection up to 30 with 1:1 correspondence
Activities and Progress Monitor	Informal learning through play Progress Monitor not applicable	Informal learning through play Progress Monitor not applicable	Representation Assessment A
Fractions	Uses informal strategies to solve fair sharing problems with collections of up to 10 items between 2 people	Uses informal strategies to solve fair sharing problems with collections of up to 20 items between 2 people; knows fair shares have the same number	Solving equal sharing problems involving "half" but may not label the parts using standard terms
Activities and Progress Monitor	Informal learning through play or sharing situations, Progress Monitor not applicable	Informal learning through play or sharing situations, Progress Monitor not applicable	Informal learning through play or sharing situations, Progress Monitor not applicable
Place Value	Trade several small items for a larger one	Trade several small items for a larger one	Trade several small items for a larger one

	Pre K	Pre K-K	Level K
Activities and Progress Monitor	Informal learning through play Progress monitor (not applicable)	Informal learning through play Progress monitor (not applicable)	Informal learning through play Progress monitor (not applicable)
Number Knowledge	Forward counting: can rote count to 10 but does not have 1:1 correspondence consistently	Forward counting: can count 1-20, can name number directly after for numbers 1-20 but may have to count from 1 to do so. Backward counting—can count from 10, can name number before with numbers 1-5 but may have to count from 1 Number Identification: 1-10 consistently, 11-20 inconsistently Number Sequencing/Ordering: 1-20 Decomposing/Composing: can subitize regular dot patterns to 6 Object counting: can count out sets up to 10 accurately.	Forward counting: can count 1-30 starting from any number, can name number directly after for numbers 1-30 Backward counting—Can consistently backward count 10-1 starting from any number; Can inconsistently count backward 30-0; Can inconsistently name number directly before 1-30 but may have to count from smaller number. Number Identification: 1-30 consistently Number Sequencing/Ordering: 1-30 Decomposing/Composing: Can subitize finger patterns to 10, irregular dot patterns, use pairs of dots to determine quantity Object counting: can count out sets up to 30 accurately.
Activities and Progress Monitor	*Chapter 10 in LMPG* *Number Development Assessment*	*Chapter 10 in LMPG* *Number Development Assessment*	*Chapter 10 in LMPG* *Number Development Assessment*

Problem Solving and Strategy Use

A child's number knowledge may align with components in more than one level.

Solution Strategies	Level A 1	Level A 2	Level B
	Solves JRU, SRU, Multiplication and Measurement Division story problems (with numbers to 5)	Solves story problems with numbers to 100 including: JRU, SRU, M, JCU, SCU, CDU, PPW-WU, MD, PD	Solves story problems with numbers to 1000 including: JRU, SRU, M, JCU, CDU, SCU, PPW-PU, MD, PD
Direct modeling	Directly models problems	Directly models problems	Directly models problems
Counting	Counts on 1, 2 or 3 from the first addend within 30. Counts back no more than 3 when minuend is ten or fewer.	Counts on 1, 2 or 3 to addends up to 100. Counts back 1, 2 or 3 from numbers to 100. Counts by 2, 5, or 10.	
Number relationships and number facts.		Uses number relationships for sums to ten (addition math facts) Adds 10 to a single digit number using mental computation.	Knows or easily derives addition and subtraction facts. Mentally adds 10 to or subtracts 10 from any number 10-100. Mentally adds any decade to or subtracts any decade from numbers 10-100. e.g. 56+30, 72-40. Decomposes a single-digit number when adding or subtracting within 100 e.g. 48 + 6 = (48 + 2) + 4 or 56 – 9 = 56 - 6 -3 Uses repeated addition or skip counting to solve X2, X5, X10
Activities	**See Activities for Level A 1**	**See Activities for Level A 2**	**See Activities for Level B**
Progress Monitor	Problem Solving Probe A and/or Addition Fact Interview A	Problem Solving Probe A and/ or Addition Fact Interview A	Problem Solving Probe B and/or Addition Fact Interview B

Fraction Developmental Guidelines (4th/5th grade intervention only)

A child's number knowledge may align with components in more than one level.

	Level A 1	Level A 2	Level B
Components	Solves equal sharing problems resulting in "half" Names one part as "one half" correctly	Solves partitioning or equal sharing problems where the solution has a unit fraction Identifies unit fractions using words (e.g. "one fourth")	Solves fractions as operator problems (e.g. What is ¼ of 12?) Identifies and writes unit fractions
Solution Strategies	**NA**	Draws fractional parts of a set or a single unit	Draws fractional parts of a set or a single unit
Activities	**See Intervention Activities for Level A 1**	**See Intervention Activities for Level A 2**	**See Intervention Activities for Level B**
Progress Monitoring Tool	PD story problem with fraction answer from Problem Solving Probe A	PD story problem with fraction answer from Problem Solving Probe A	PD story problem with fraction answer from Problem Solving Probe B Beginning Fraction Knowledge Assessment

Place Value Developmental Guidelines

A child's number knowledge may align with components in more than one level.

	Level A 1	Level A 2	Level B
Components	Recognizes that 23 and 32 are different quantities & can count out sets by 1s to match the numeral Not able to recognize that the digit in the tens place represents a group of ten in a model Given a set of base ten blocks, child can count the set using 10s and 1s Relates numbers to 10 Given a double digit number, child can assemble a set of base ten blocks to match number	Adds ten to a single digit number using mental computation Relates numbers to the nearest decade Given a set of base ten blocks, the child can count the set using 100s, 10s and 1s Given a triple digit number, the child can assemble a set of base ten blocks to match	Adds ten to a double digit number using mental computation Adds 20 or 30 to a double digit number using mental computation Can combine units into a groups of ten and leftover 'ones.' (e.g. 23 ones is the same as two groups of ten and 3 ones)—by modeling first and then moving to mental computation Can decompose a number into tens and ones (e.g. 43 = 10+10+10+3)—by modeling first and then moving to mental computation
Activities	**See Intervention Activities for Level A 1**	**See Intervention Activities for Level A 2**	**See Intervention Activities For Level B**
Progress Monitor	Early Base Ten Assessment Number Development Assessment: Subtest 10 Problem Solving Probe A	Number Development Assessment: Subtest 10 Problem Solving Probe A	Number Development Assessment: Subtest 10 Problem Solving Probe B

Number Knowledge Developmental Guidelines

A child's number knowledge may align with components in more than one level.

	Level A 1	Level A 2	Level B
Components	Forward counting—1-100 starting from any number; naming up to 3 numbers directly after (1-100) Backward counting—30-0 (starting from any number); Naming up to 3 numbers directly before (1-30) Number Identification: 1-100 Sequencing/Ordering numbers: 1-100 Decomposing/composing numbers to 10 Conceptual subitizing quantities to 10	Forward counting—1-1000 starting from any number; naming up to 3 numbers directly after (1-1000); rote counting by 2s, 5s, 10s, rote counts by 10 from any 2 digit number Backward counting—100-0 (starting from any number); Naming up to 3 numbers directly before (1-100); Number Identification: 1-1000 Sequencing/Ordering numbers: 1-1000 Decomposing/composing numbers to 20	Forward counting—1-1000 starting from any number by 10s and 100s; counting by 1s from any number up to 10,000; naming up to 3 numbers directly after (1-10000); Backward counting—1000-0 (starting from any number); Naming up to 3 numbers directly before (1-1000); Number Identification: 1-10,000 Sequencing/Ordering numbers: 1-10,000 Decomposing/composing numbers to 100
Activities	**See Intervention Activities for Level A 1**	**See Intervention Activities for Level A 2**	**See Intervention Activities for Level B**
Progress Monitoring Tool	Number Development Assessment	Teacher Observation checklist	Teacher Observation checklist

Providing Intervention Instruction

Use the "Developmental Guidelines" found on pages 33-38 to identify specific math content and number ranges for each child. Suggestions for planning and specific intervention activities are located on pages 44-46.

Who should provide intervention instruction?

Decisions concerning intervention instruction are the responsibility of the teacher or the instructional team. However, other adults or students can be recruited to carry out the intervention.

When should we offer intervention?

There is no one answer as to when or how often teachers should provide intervention. Instructional teams in communication with students and families will decide on an appropriate approach.

What are some examples of instructional approaches?

Supporting student access to learning

Intervention may take the form of extra time, fewer problems to solve, or strategic practice.

Advantages: Small allowances for individuals can create greater student growth.

Considerations: Teachers must design or choose activities that focus on the underlying concepts needed for students to progress taking into account their prior knowledge and personal learning needs.

Targeted instruction during math class

Intervention may take the form of targeted individual or small group instruction designed to address students' conceptual or skill development.

Advantages: Intervention specifically targets child's learning needs. Students have more success within their zone of proximal development.

Considerations: The teacher must customize instruction based on assessment of each student's knowledge and skills. Emphasis should be on accelerated learning.

Additional instruction within the school day

In this scenario, additional time within the school day is identified for instruction.

Advantages: Child receives targeted instruction necessary for progress and also participates with peers during the regular math period.

Considerations: With this approach, it becomes necessary to prioritize time differently. Instructional teams, parents, and students must recognize that there will be other opportunities missed while receiving this "double dose" of mathematics during the day.

Before the class studies a topic

Just before the class begins a unit of study, intervention could take the form of pre-teaching on foundational concepts related to the unit.

Advantages: *The student gains background knowledge in preparation for instruction with the class.*

Considerations: Teacher must pre-assess the student and plan specific lessons for the student before the rest of the class begins the unit.

Strategies for Effective Math Intervention

1. Determine content and concept level

Use the MMSD K-12 Mathematics Standards and the Developmental Guidelines in this chapter to determine the essential mathematics content the student needs and the appropriate level to begin instruction.

2. Use familiar contexts

Use story contexts that are familiar to students.

3. Provide enough time

Allow students enough time to grapple with new ideas and make sense of new learning.

4. Encourage communication

Provide many opportunities to express thinking verbally. Use the think-pair-share strategy —— students first collect their own thoughts and then talk with a partner before sharing with the whole group.

5. Share "mental math" strategies

Ask students to try solving easier computations mentally and share their mental strategies with their classmates.

6. Teach ways to represent solutions

Teach a variety of representation options such as an empty number line, arrow language, ratio tables, or successive number sentences. Encourage students to use the method that makes the most sense for their thinking processes. Then, interview the student about their written work to confirm that what they have written does represent their thinking. For more information on teaching representation see pp. 45-55 and *Learning Math in the Primary Grades*, Ch. 6, pp. 109-116.

7. Discuss connections

Ask students to talk about the connections they make among math concepts and with real world examples.

8. Discuss new vocabulary

Teach vocabulary in the context of a learning activity. Common language such as "half," "equal," "even," "odd," and "product" have unique mathematical definitions. Talk about various meanings and use new vocabulary consistently. Encourage students to use the new vocabulary in order to develop academic language.

9. Provide targeted practice

Have multiple games and activities for any concepts students may be learning. Choose or design games at the student's independent level so they can develop strategic play and get plenty of successful practice.

10. Monitor progress frequently

Set specific short-term learning goals. Intervention instruction is not a parallel low track for some students. Intervention instruction requires frequent assessment and evaluation. The goal of all the intervention must be to accelerate learning to match grade-level peers.

Considerations for Planning Intervention Activities

Planning for intervention begins with assessing and describing each student's knowledge based on the *Developmental Guidelines* pp 33-38.

The *Components* within the guidelines are benchmarks that describe how children typically develop number concepts. Use the *Developmental Guidelines* in conjunction with the intervention activities listed on the following pages to plan instruction.

A child's intervention plan may need to focus on multiple components of Number Knowledge simultaneously. For example, a child could need instruction on forward counting, backward counting and number identification. Within the component of "forward counting" the child may need to begin work at the rote counting by 2s level.

Number Knowledge Developmental Guidelines A 2	
Components	**Forward counting** 1 – 1000 starting from any number Name a number after or before any number (1-1000) Rote counting by 2's Rote counting by 5's, Rote counting by 10's Rote counting by 10 from any 2-digit number

Begin with what is the "known" for the child and work for student understanding, articulation of patterns and growing confidence. Intervention instruction requires informed regular assessment. Teachers evaluate these assessments to make gradual changes and reflect on the efficacy of the instruction to date.

Intervention Activities for Level A1

*The following is a **sample** of activities chosen to help students progress to Level A1. Refer to Learning Mathematics in the Primary Grades and Learning Mathematics in the Intermediate Grades for additional activities.*

Story Problems Activities

In Learning Math in the Primary Grades:

- Developing Covered Tasks, p. 184-187
- Counting on with Number Cards and Covered Tasks pp. 185-186
- Counting back by ones, pp. 110-111
- Number Work *(Plus Game for counting on 1, 2, or 3)*, p. 153
- Activities to Support Problem Solving, pp. 192-193

Representation Activities

- Solution Strategies, Ch. 6, p. 37
- Representing Solutions, Ch. 6, p. 45

In Learning Math in the Primary Grades:

- Teaching Students Ways to Write Down Their "Thinking Steps", p. 109
- Using the Empty Number Line to Represent Counting Strategies, pp. 110-114
- Compare Empty Number Line Representations, p. 130

Fraction Activities:

- Using Story Problems to Build Fraction Knowledge, Ch. 6, pp. 59-60

Place Value Activities

- Components of Place Value Understanding *(see Verbal Counting Sequence)*, pp. 52-54
- What do you notice? Ch. 7, p. 16
- What do you know about? Ch. 7, p. 16
- Nickname, Real Name, Ch. 7, p. 13
- Place-Value Arrow Cards, Ch. 6, p. 46 *(Cards are in LMPG, pp. 222-24)*

In Learning Math in the Primary Grades:

- Building Five and Ten *(extension to 10 plus a single digit)*, p. 188, *(Also pp. 185, 186, & 225)*
- Number Building *(use arrow cards to help students see hidden zeroes)*, p. 180

Number Knowledge Activities

In Learning Math in the Primary Grades:

- Count Some More, p. 131
- Chapter 10
  - Number Building, p. 180
  - Number Card Match and Number Sort, p. 181
  - Sequencing Activities, pp. 182-183
  - Activities to Support Composing and Decomposing Numbers, pp. 187-191

Intervention Activities for Level A2

*The following is a **sample** of activities chosen to help students progress to Level A2. Refer to Learning Mathematics in the Primary Grades and Learning Mathematics in the Intermediate Grades for additional activities.*

Story Problem Activities

- Choosing Numbers to Build Number Sense *(Begin with the numbers in the Developmental Guidelines. For joining problems, extend to addends in the decades.)*, Ch. 6, pp. 28-29
- Using Story Problems to Build Place Value Knowledge *(adjust numbers as needed)*, pp. 30-32

Representation Activities

- Representation for Base-Ten Blocks, Ch. 6 p. 47
- Empty Number Line, Ch. 6, pp. 48-49
- Equations *(number sentences)*, Ch. 6, p. 55

Fraction Activities

- Using Story Problems to Build Fraction Knowledge, Ch. 6, pp. 59-60
- Sample Fraction as Operator Problems, Ch. 6, p. 61

Place Value Activities

- Place Value Components Chart: *Verbal Counting Sequence, Build/Count Mixed Quantities, Read & Write Numbers, Sequence Numbers, Tens and Tens Within Tens,* Ch. 4, pp. 52-53
- Counting and Compare, Ch. 7, p. 9

In Learning Math in the Primary Grades:

- Nickname, Real name, Ch. 7, p. 13

Number Knowledge Activities

- What Do You Know About, Chapter 7, p, 16
- Number Squeeze, Ch. 7, p. 15
- Place Value Components Chart, *(See Place Value Activities for this level.)* Ch. 4, pp. 52-53

In Learning Math in the Primary Grades:

- After and Before, p. 128
- Count Some More, p.131
- Plus, p. 53 *(This game can be played as plus or minus 1-3, without counting.)*

Intervention Activities for Level B

*The following is a **sample** of activities chosen to help students progress to Level B1. Refer to Learning Mathematics in the Primary Grades and Learning Mathematics in the Intermediate Grades for additional activities.*

Problem Solving Activities *(See intervention activities for Level A2)*

- Estimation, Ch. 6, pp. 34-35
- Mental Calculation, Ch. 6, p. 36

Representation Activities

- Representing Solutions, Ch. 6, p. 45
- Place Value Arrows, Ch. 6, p. 46
- Base 10 Blocks, Ch. 6, p. 47
- Empty Number Line and Teaching Tips, Ch. 6, pp. 49-50
- Arrow Language and Teaching Tips, Ch. 6, pp. 50-51
- Array Model and Area Model for Multiplication, Ch. 6, pp. 52-53
- Equations, Ch. 6, p. 55,
- Number Work Comparing Empty Number line and Arrow Language, Ch. 7, p. 8

Fraction Activities

- Using Story Problems to Build Fraction Knowledge, Ch. 6, pp. 59-60
- Sample Equivalence Fraction Type Problem, Ch. 6, p. 61

Place Value Activities

- Components of Place Value Understanding: *Sequence Numbers from Smallest to Largest, Decompose and Compose Numbers,* Ch. 4, pp. 52-54
- What's the Rule? *(Change the input number by adding 10 or any number ending in 0's.)* Ch. 7, p. 16
- Counting and Compare, Ch. 7, p. 9

In Learning Math in the Primary Grades:

- Looking For Tens, p. 134
- Dash to the Decade, pp. 155-156

Number Knowledge Activities

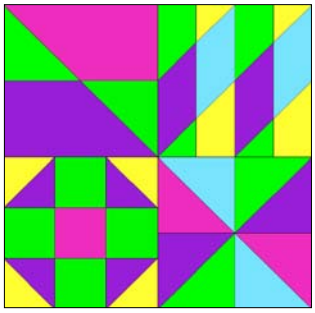
- Components of Place Value Understanding: *Verbal Counting Sequence, Build/Count Mixed Quantities,* pp. 52-54
- Counting and Compare, Ch 7, p. 9

In Learning Math in the Primary Grades:

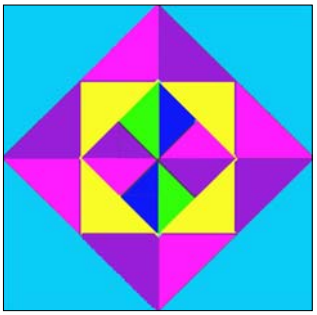
- Target 20, p. 159

Planning and Progress Record

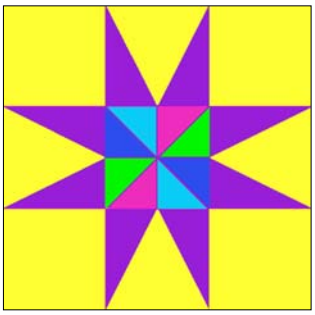
What the Student Knows	Short Term Goal	Activity (include number range)	Results of Instruction
Date:			
Date:			
Date:			
Date:			
Date:			
Date:			
Date:			



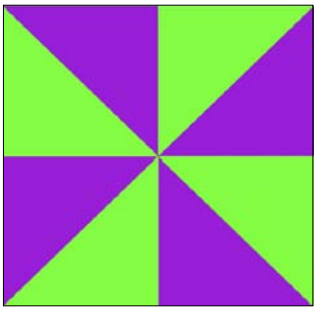
PROBLEM SOLVING



NUMBER WORK



INSPECTING EQUATIONS



FLUENCY & MAINTENANCE

Appendix

Sample Story Problem Pages

The following pages provide examples of ways to set up story problem pages. When setting up story problem pages consider:

- ☑ space to solve
- ☑ how students or teacher will indicate number choices
- ☑ number of solution strategies
- ☑ instructional level or fluency and maintenance
- ☑ place for students to write answer only or a sentence
- ☑ related problem sets on one page
- ☑ problem set title for easy reference

Name_____ Date_____

Boats

There are ____ new first-year students, and first year students always ride in boats across the lake to get to Hogwarts. If each boat can hold 4 students, how many boats does Hagrid need to get so all the first-year students can ride across the lake?

- ❑ 24
- ❑ 48
- ❑ 148

My estimate for the answer to this problem is:_____

Show one way to solve the problem.

Show another way to solve the problem.

_____

Write a sentence that answers the question in your math story problem.

Name_____ Date_____

Math Potatoes

Mr. Tang has _____ potatoes on every page of his book Math Potatoes. There are ____pages in his book. How many potatoes did he have to draw?

25, 14	20, 10
5, 4	20, 4

Name_____ Date_____

Roller Coaster

A roller coaster can carry 48 people at the same time. Each car can carry 6 people. How many cars long is the roller coaster?

I think the answer to this problem is between_____ and _____.

Show one way to get the answer to this problem.

Show another way to get the answer to this problem.

Write a sentence that answers the question in this problem:

_____

Swimming

Carlos swam 75 lengths of the pool. Mark only swam 56 lengths. How many more lengths would Mark have to swim to swim as many as Carlos?

I think the answer to this problem is between_____ and _____.

Show one way to get the answer to this problem.

Show another way to get the answer to this problem.

Write a sentence that answers the question in this problem:

_____

Name_____ Date_____

Number of Pages Read

Carol	492
David	363
Talia	2,005
Xiang	1,899

a) How many more pages did Talia read than Carol?

b) How many pages did Carol and David read altogether?

c) How many fewer pages did Xiang read than Talia?

Number of times a ball was thrown into a basketball hoop.

	Underhand	Overhand
Keenan	28	52
David	13	26
Talia	6	17
Xiang	19	50
Maurice	12	95

a) How many more overhand throws did Maurice make than Talia?

b) Do people throw better overhand or underhand? Explain.

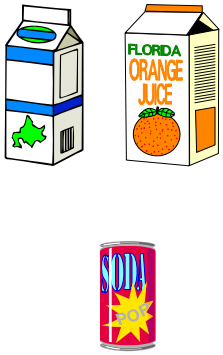
c) Who made the most baskets in all?

Name_____ Date_____

Party Punch

The fifth grade students at Mark Elementary School brought the following list of drinks to make punch for a party.

Pineapple Juice: 1.36L
Minute Maid Orange Juice: 0.95L
Grapefruit Juice: 1.89L
Grape Juice: 1.5L
Tropicana Orange Juice: 2.84L
Sprite: 0.75L
Gatorade: 0.95L
Cranberry Juice 1.14L
Apple Juice 1.36L



What is the smallest amount of punch you can make by mixing 3 of the drinks?

Show your solution strategy below.

What is the largest amount punch you can get mixing 2 of the drinks?

Show your solution strategy below.

Name_____ Date_____

Collections

1. Esteban has 60 toy trucks. 15 use batteries. What fraction of the trucks uses batteries? Show your thinking steps:

2. Fabio has 80 toy plastic animals. 16 are lizards. What fraction are lizards? Show your thinking steps:

3. Gaspar has 24 stamps in his collection. 6 are Canadian. The rest were from Spain. What fraction was from Spain? Show your thinking steps:

4. Herminie bought grapes as a snack. 53 were green the rest were black. If she bought 150 grapes, about what fraction were black? Show your thinking steps:

Name_____ Date_____

Fruit Fun

Problem	Estimation	Answer
Thirty-three pomegranates with 210 seeds in each. How many seeds altogether?		
Sixteen oranges with 12 sections in each. How many sections altogether?		
Twenty-five bunches of bananas with 12 in a bunch. How many bananas in all?		
Three trees with 128 kumquats in each. How many kumquats altogether?		
Fifteen limes for 1 Key Lime pie. How many limes for 5 pies?		
Eighteen pineapple bushes. Six pineapples on each bush. How many altogether?		

Math Story: A Walk for Rocks

Name_____ Date_____

Read this story problem all the way to the end. Then make a plan to solve it.

One day I decided to take a walk. As I headed out of the house I grabbed a paper bag. I began to collect rocks for my rock collection. I had walked 824 steps and for every two steps I had collected 3 rocks.

I was getting tired and stopped to look at my rocks. I sorted my rocks into shiny, dull, smooth, and rough groups. Each group had the same number of rocks when I was finished. I left the dull rocks and continued on my walk.

I walked for 49 minutes. My bag was getting heavy, so for every 7 minutes I walked I threw one rock out of my bag.

I turned around and headed home. When I got home I dumped my rocks out on the deck. I lost one fourth of my rocks through the cracks in the deck.

My friend came over and brought her rocks. She had one-half as many as I had, but we put them all together and shared.

We decided to put the rocks into piles of shiny, dull, rough, smooth, and unusual.

We had the same number of rocks in each pile.

My friend took home the shiny, dull, and rough rocks. I put the smooth and unusual rocks in my bag. When I brought the rocks in the house my Mom asked, "How many rocks do you have in your bag?"

How many do I have? _____

What will you do to solve this problem?

Will you:

act out the solution?_____

draw the solution?_____

represent the problem with number sentences?____

How will you prove that your answer is correct?

Will you:

make a drawing?_____

use math symbols?_____

explain my number sentences in writing?_____

Now solve the problem and keep track of your solution paper as you go!

Writing a Math Story

Now write your own math story about a collection. It will need to have about 7 events. Each event must be connected to the event that came before. The story must use joining or combining, separating, making groups and fractional parts. Make it challenging but not too challenging, about the same as *A Walk for Rocks.* Solve your math story twice so that you are sure that it has a whole number solution (unless some rocks break into parts). When you are finished with your first draft you will need an editor.

I will ask _____ to edit my story. Last, give your story a title.

My story will be about collecting:_____

Editing Checklist:

My story has:	*Me*	*Editor's check*
about 7 events	_____	_____
joining or combining	_____	_____
separating	_____	_____
making groups	_____	_____
fractional parts	_____	_____
a title	_____	_____
correct spelling and punctuation	_____	_____
a math question at the end of the story	_____	_____
a solution	_____	_____

Math story by _____

The title of my story is:_____

The answer to the math question in my story is:_____

Ratio and Proportion Problems

Part 1

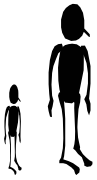
Try to solve each of these problems in several ways. Focus on *reasoning* to solve each problem.

1. You're having a party, and you have ordered pizza for everyone to share. There are 12 people at the party, and you ordered 4 pizzas. Three more people join the party. How much less pizza does each person get than you originally planned, if the extra people want an equal share?
2. Julia used exactly 15 cans of paint for 18 chairs. How many chairs can she paint with 25 cans of paint?
3. The Mayville Park Commission wants to make new parks. They found out that 15 oak trees shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 91 picnic tables. How many oak trees should they plant to shade all the new tables?
4. Jayda and Manuel are making lemonade. Jayda used 10 cups of lemon and 5 teaspoons of sugar. Manuel used 7 cups of lemon and 3 spoons of sugar. Whose lemonade will be sweeter, or will they both taste the same? How can Jayda adjust her recipe to make it taste the same as Manuel's lemonade?
5. Your school is having a lottery as a fund-raiser. Carolina bought 15 tickets. A total of 88 tickets were sold. In another lottery for the basketball team, Sandy bought 12 out of a total of 62 tickets. Who has a higher probability of winning? Why? What does probability tell you? What does it mean?
6. When Marianna was 12 years old, she planted a 3-foot-tall maple tree in her front yard. In the middle of the summer, it cast a shadow 15 inches long. The tree now casts a shadow that is 55 inches long. How tall is the tree now? If the tree has grown at a constant rate since Marianna planted it, how much has it grown per year?
7. A small lake is over-populated with fish (there were 2400 of them), and there isn't enough food for all of them. To lower the population, the Department of Natural Resources caught 400 fish in one day. There is another lake, smaller than the first one, with the same problem. It contains 1800 fish. How many fish need to be caught to reduce the fish population by the same proportion as the larger lake?

Review these seven problems—how are these problems similar or different? Compare your problem-solving strategies for each of them – what strategies did you use? Did you use the same strategies for all of the problems?

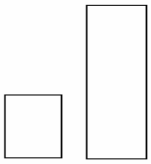
Ratio and Proportion, Part 2

1. Here are Mr. Short and Mr. Tall. Mr. Short's height is measured with paperclips. Mr. Short's height is 8 paperclips.



   When we measure their heights with matchsticks, Mr. Short's height is 4 matchsticks, but Mr. Tall's height is 6 matchsticks. How many paperclips are needed for Mr. Tall's height?

2. These two rectangles are the same shape, but one is larger than the other. If you were given the height of the small one, how would you find the height of the larger rectangle?



3. If 3 people pay $26.25 to go to the movies, how much would 8 people pay?
4. Ms. Brown's and Ms. Hart's classes each have the same number of students. In Mr. Brown's class, for every three boys there are five girls. In Ms. Hart's class, for each boy there are three girls. In which class would you expect to see more girls? Explain your answer.
5. A recipe makes soup for eight people and requires 3 pints of water. You decide to change the amounts to make soup for six people and you added only 2 pints of water. Is your soup thicker or thinner than the original recipe? Why do you think so?
6. Sam the snake is 4 feet long. When he is fully-grown, he will be 6 feet long. Spot is now 5 ft. long but when she is fully grown, she will be 7 feet long. Which snake is closer to being full-grown?
7. Two families are driving to Noah's Ark water park. Both families leave at the same time from the same street. The driver of the red car drives an average of 186 miles in 3 hours and the driver of the blue car drives an average of 320 miles in 5 hours. If each car travels at a constant rate for the entire trip, who will get to the cabin first?

Review these seven problems—how are these problems similar or different? Compare your problem-solving strategies for each of them – what strategies did you use? Did you use the same strategies for all of the problems?

Ratio and Proportion, Part 3

Problem Set One

a) It takes a student 3 hours to solve 9 math problems. If it takes the same time to solve each problem, how long time would it take to solve 18 math problems?

b) A group of students is organizing a field trip, and they estimate that it will take 5 hours to walk 7 km. How long will it take the students to walk 21 km? What assumptions do you need to make to solve this problem? Are those assumptions realistic?

c) Sandy is planning to backpack in northern Wisconsin this summer. She estimates that in 8 hours she could cover 12 km. How many hours does she have to walk if the trek is 42 km?

d) Lisa bought chocolate-covered raisins for a class party. Two pounds of the candies cost $3. How many pounds of candy did Lisa buy for $17?

Problem Set Two

a) In an after-school program there are 5 girls for every 15 boys. How many girls if there are 45 boys?

b) In designing an after school program at Greenhill School, there are 2 boys for every 4 girls. How many boys should be in the program if there are 22 girls?

c) A builder is building a new apartment that will have two-bedroom and three-bedroom apartments. It was decided that for every 6 three-bedroom apartments there should be 14 two-bedroom apartments. How many three bedroom apartments are needed if there are going to be 35 two-bedroom apartments?

Review these seven problems—how are these problems similar or different? Compare your problem-solving strategies for each of them – what strategies did you use? Did you use the same strategies for all of the problems? Were some easier or harder? Why do you think so?

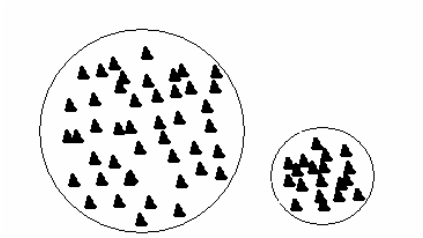
Ratio and Proportion, Part 4

1. One of the drinks served in the school cafeteria is fruit punch. The cook mixes 5 cans of pineapple juice to every 6 cans of orange juice. How many cans of pineapple juice are needed if the cook uses 19 cans of orange juice?

2. There are twenty-five students in your class, including you. On your birthday your mom made 5 pounds of cookies for students to share. How much did each person get?

3. Which has more girls? The family of five with two girls and three boys? Or the family of four with two girls and two boys?

4. Here are two cookies with chocolate chips. How could you decide which cookie is more chocolaty without tasting the cookies?



5. If 3 pizzas serve 9 people, how many pizzas will be needed to serve 108 people?

6. Which is the better deal? Orange juice costs $1.70 for a pack. Apple juice costs $1.10 but the pack is smaller than the orange juice.

7. Which is the better deal, two pounds of apples for $4.26 or three pound of apples for $7.87?

8. Jill wants to buy a CD player that costs $210. Her mother agreed to pay $5 for every $2 that Jill saved. How much will each contribute?

Review these eight problems—how are these problems similar or different? Compare your problem-solving strategies for each of them – what strategies did you use? Did you use the same strategies for all of the problems?

MATH JOURNALS

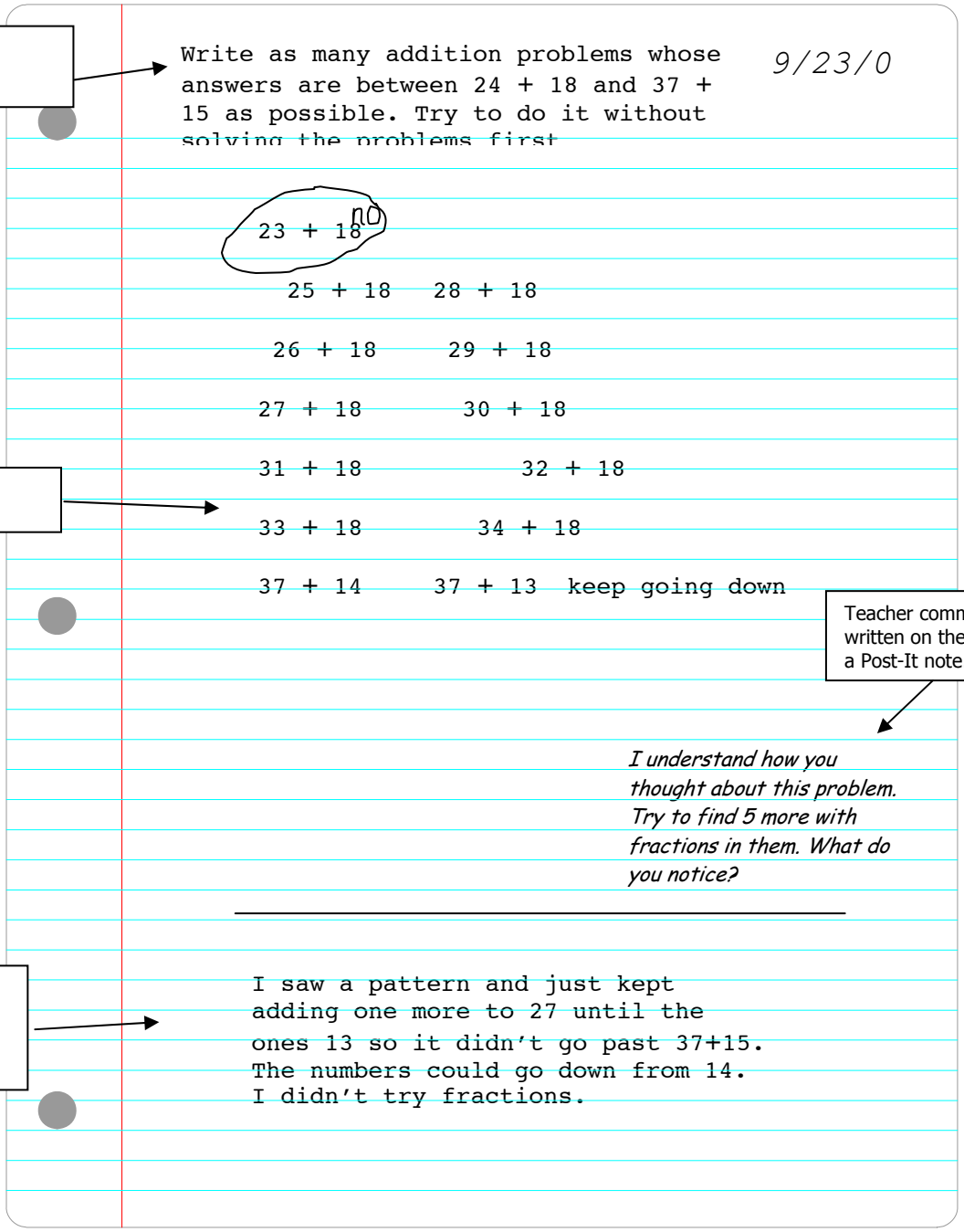
Math journals provide an easily accessible place for students to solve problems. Graph paper journals provide an easy to use grid for modeling problems and keeping work organized on a page. In addition, journals provide a record of work and a great place for student and teacher to communicate. Math journals (notebooks) are most useful when a few norms are established for their use. The following two examples allow for both independent use of space and more formal space for communicating to others. *Provide simple explanations for ways to use the journal.*

Problem statement: neat for communication with others

Experiments: messy for student

Teacher comments written on the page or a Post-It note.

Reasoning: neat use symbols or words to help others understand student's thinking



Estimation Box: a number that shows about what you think the answer is close to, some times I'll give you some choices to think about and add a label, check to see how close you were after you solve

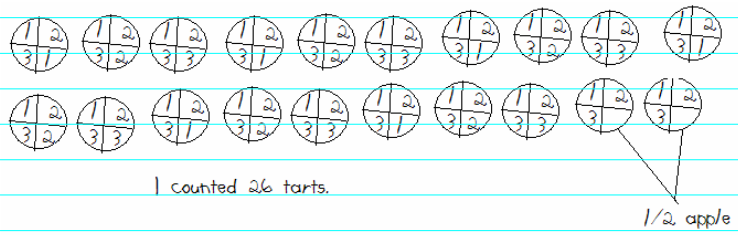
10/15/07

23 tarts

Mom makes small apple tarts using three-fourths of an apple for each small tart. She has 20 apples. How many small apple tarts can she make?

Strategy

Problem: neat (for communication with the teacher and classmates), Students think about the problem as they write it, divide the rest of the page into two parts (vertically or horizontally) and label strategy 1 and strategy 2 in each part



Strategies: solve in ways that make sense to the student, teacher may ask the student to try new approaches, begin with the numbers in the problem for both strategies, if answers disagree ask students to find out why or try a third way, messy is OK if students can write neatly when asked and can use symbols appropriately

Strategy

3/4 + 3/4 = 1 1/2 apples for 2 tarts

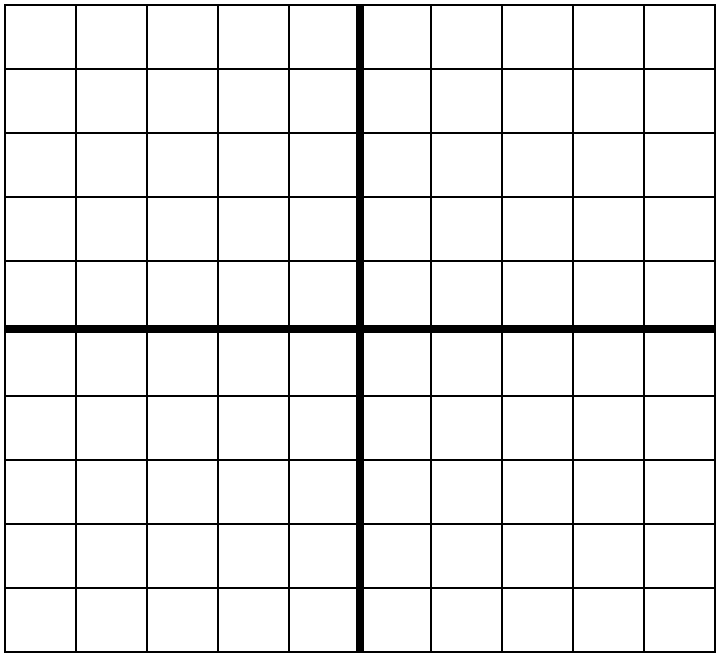
1 1/2 + 1/1/2 = 3 apples for 4 tarts

6 apples for 8 tarts
12 apples for 16 tarts
18 apples for 24 tarts plus 1 1/2 apples to make 26 tarts with 1/2 apple left to eat!!

Mom can make 26 apple tarts with an apple leftover to snack on.

Write a sentence that answers the question in the problem to be sure you have answered the problem

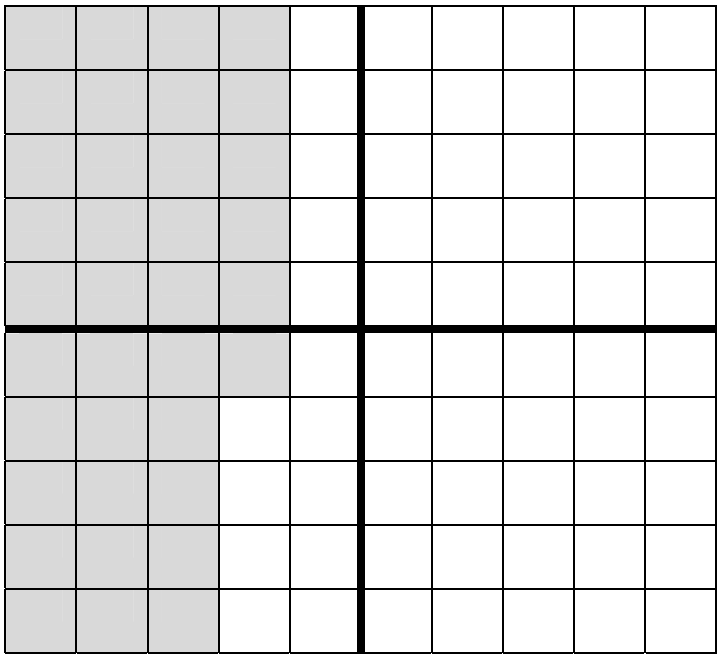
What Do You Notice? *(overhead template)



What Do You Notice? Example Page

Shade in a group of tens and some ones.

Display as an overhead for Number Work



How Many Shaded in All? 36

10 + 10 + 10 + 5 + 1	20 + 15 + 1
(6 × 4) + (4 × 3)	12 + 4 + 20
4 + 4+ 4 + 4 + 4 + 4 + 3 + 3 + 3 + 3	20 + 10 + 6
5 + 5 + 5 + 5 + 5 + 5 + 5 + 1	100 – 50 – 10 – 4
(7 × 5) + 1	100 - 64

Hundreds Charts (Variations)

10	20	30	40	50	60	70	80	90	100
9	19	29	39	49	59	69	79	89	99
8	18	28	38	48	58	68	78	88	98
7	17	27	37	47	57	67	77	87	97
6	16	26	36	46	56	66	76	86	96
5	15	25	35	45	55	65	75	85	95
4	14	24	34	44	54	64	74	84	94
3	13	23	33	43	53	63	73	83	93
2	12	22	32	42	52	62	72	82	92
1	11	21	31	41	51	61	71	81	91

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109
110	111	112	113	114	115	116	117	118	119
120	121	122	123	124	125	126	127	128	129

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

-99	-98	-97	-96	-95	-94	-93	-92	-91	-90
-89	-88	-87	-86	-85	-84	-83	-82	-81	-80
-79	-78	-77	-76	-75	-74	-73	-72	-71	-70
-69	-68	-67	-66	-65	-64	-63	-62	-61	-60
-59	-58	-57	-56	-55	-54	-53	-52	-51	-50
-49	-48	-47	-46	-45	-44	-43	-42	-41	-40
-39	-38	-37	-36	-35	-34	-33	-32	-31	-30
-29	-28	-27	-26	-25	-24	-23	-22	-21	-20
-19	-18	-17	-16	-15	-14	-13	-12	-11	-10
-9	-8	-7	-6	-5	-4	-3	-2	-1	0
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4
4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5
5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6
6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7
7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9	8
8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9	9
9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9	10
10.1	10.2	10.3	10.4	10.5	10.6	10.7	10.8	10.9	11
11.1	11.2	11.3	11.4	11.5	11.6	11.7	11.8	11.9	12
12.1	12.2	12.3	12.4	12.5	12.6	12.7	12.8	12.9	13
13.1	13.2	13.3	13.4	13.5	13.6	13.7	13.8	13.9	14
14.1	14.2	14.3	14.4	14.5	14.6	14.7	14.8	14.9	15
15.1	15.2	15.3	15.4	15.5	15.6	15.7	15.8	15.9	16
16.1	16.2	16.3	16.4	16.5	16.6	16.7	16.8	16.9	17
17.1	17.2	17.3	17.4	17.5	17.6	17.7	17.8	17.9	18
18.1	18.2	18.3	18.4	18.5	18.6	18.7	18.8	18.9	19
19.1	19.2	19.3	19.4	19.5	19.6	19.7	19.8	19.9	20

0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.3
0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.4
0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.5
0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.6
0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.7
0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.8
0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89	0.9
0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1
1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.1
1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.2
1.21	1.22	1.23	1.24	1.25	1.26	1.27	1.28	1.29	1.3
1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.5
1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.6
1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.7
1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.8
1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.9
1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2

Name_____ Date_____

Math and Me (student survey)

Write yes (or y) in the blank if you agree with the statement, and no (or n) if you don not agree with the statement. Explain why you agree or disagree.

1. _____ Some people have a math mind and some don't. Why?

_____

_____

_____

2. _____ You must always know how to answer a problem. Why?

_____

_____

_____

3. _____ Math requires a good memory. Why?

_____

_____

_____

4. _____ I like to do math alone. Why?

_____

_____

_____

5. _____ Math is always done by working intensely until the problem is solved. Why?

_____

_____

_____

6. _____ There is one best way to do a problem. Why?

_____

_____

_____

7. _____ I like to do math with others. Why?

_____

_____

_____

8. _____ Mathematicians do problems quickly, in their heads. Why?

_____

_____

_____

9. _____ There is a magic key to doing math. Why?

_____

_____

_____

10. _____ Your imagination helps you with math. Why?

_____

_____

_____

11. _____ It's OK to count on your fingers. Why?

_____

_____

_____

12. How do you feel about mathematics right now?

_____

_____

_____

13. What is your favorite math topic?

_____

_____

_____

Thank you for writing about math and you!

CAN DO Descriptors (for English Language Proficiency)

The *CAN DO Descriptors* offer teachers and administrators working with English language learners a range of expectations for student performance within a designated English language proficiency level of the WIDA English language proficiency standards.

The *CAN DO Descriptors* are broad in nature, focusing on language functions generally found in the school setting, rather than language skills related to specific academic topics. A distinguishing feature of these descriptors, although not explicitly mentioned, is the presence of visual or graphic support to enable English language learners' access to the language and content requisite for success in school. Given the broad nature of these Descriptors and the fact that they are not distinguished by grade level cluster, educators need to keep in mind the variability of students' cognitive development, age and grade level differences, and their diversity of educational experiences.

The *CAN DO Descriptors* are an extension of the Performance Definitions for the English Language Proficiency Standards. The Descriptors apply to *ACCESS for ELLs™* scores and may assist teachers and administrators in interpreting the meaning of the score reports. In addition, the Descriptors may help explain the speaking and writing rubrics associated with the English language proficiency test.

The *Descriptors* are not instructional or assessment strategies, per se. They are samples of what English language learners may do to demonstrate comprehension in listening and reading as well as production in speaking and writing within a school setting. Unlike the strands of model performance indicators, the descriptors do not scaffold from one English language proficiency level to the next, meaning that they do not form a developmental strand encompassing a shared topic or theme. Rather, each English language proficiency level is to be viewed as a set of independent descriptors.

Presented in matrix format similar to the English language proficiency standards, educators should have ease in examining the *Descriptors* across the language domains for the five levels of English language proficiency. English language proficiency level 6, Reaching, is reserved for those students who have reached parity with their English proficient peers.

For the most part, the *Descriptors* are drawn from the English Language Proficiency Standards' Framework for Large-Scale Assessment that serves as the anchor for the English language proficiency test. Teachers are encouraged to supplement these bulleted points with additional ones from the Framework for Classroom Instruction and Assessment. In that way, educators will have a full complement of what English language learners CAN DO as they move along the second language acquisition continuum.

The WIDA English Language Proficiency Standards for English Language Learners in Grades K-12(2004) can be found on the WIDA Consortium website (www.wida.us).

CAN DO descriptors for the levels of English Language Proficiency

For the given level of English language proficiency level, English language learners can:

Language Domain	Level 1-Entering	Level 2-Beginning	Level 3- Developing	Level 4-Expanding	Level 5-Bridging	
Listening	• Point to stated pictures, words, phrases • Follow one-step oral directions • Match oral statements to objects, figures, or illustrations	• Sort pictures, objects according to oral instructions • Follow two-step oral directions • Match information from oral descriptions to objects, illustrations	• Locate, select, order information from oral descriptions • Follow multi-step oral directions • Categorize or sequence oral information using pictures, objects	• Compare and contrast functions, relationships from oral information • Analyze and apply oral information • Identify cause and effect from oral discourse	• Draw conclusions from oral information • Construct models based on oral discourse • Make connections from oral discourse	Level 6: Reaching
Speaking	• Name objects, people, pictures • Answer wh-questions	• Ask wh- questions • Describe pictures, events, objects, people • Restate facts	• Formulate hypotheses, make predictions • Describe processes, procedures • Re/ tell stories or events	• Discuss stories, issues, concepts • Give speeches, oral reports • Offer creative solutions to issues, problems	• Engage in debates • Explain phenomena, give examples, and justify responses • Express and defend points of view	
Reading	• Match icons and symbols to words, phrases, or environmental print • Identify concepts about print and text features	• Locate and classify information • Identify facts and explicit messages • Select language patterns associated with facts	• Sequence pictures, events, processes • Identify main ideas • Use context clues to determine meaning of words	• Interpret information or data • Find details that support main ideas • Identify word families, figures of speech	• Conduct research to glean information from multiple sources • Draw conclusions from explicit and implicit text	
Writing	• Label objects, pictures, diagrams • Draw in response to oral directions • Produce icons, symbols, words, phrases to convey messages	• Make lists • Produce drawings, phrases, short sentences, notes • Give information requested from oral or written directions	• Produce bare-bones expository or narrative texts • Compare/ contrast information • Describe events, people, processes, procedures	• Summarize information from graphics or notes • Edit and revise writing • Create original ideas or detailed responses	• Apply information to new contexts • React to multiple genres and discourses • Author multiple forms of writing	

Math Glossary

Additive reasoning – strategies based on addition and properties of addition.

Arrow language – a method for recording the steps of a computation. The operation and number is written above an arrow to show each step in the strategy. Sometimes referred to as a "thinking train." Arrow language can be used with any operation or series of operations. (See Chapter 6: Representation)

Associative property – an operation is associative if you can *group* numbers in any way without changing the answer. Addition and multiplication are associative with whole numbers, decimals, and fractions. E.g. (6+5)+3=6+(5+3), (2x6)x4=2x(6x4).

Commutative property -an operation is commutative if you can *change the order* of the numbers involved without changing the result. Addition and multiplication are both commutative with whole numbers, decimals, and fractions. Subtraction and division are not commutative. 8=3=3+8, 4x6=6x4.

Conjecture –a statement which has been proposed as a true statement. Intermediate students make conjectures about mathematical ideas that are new to them. E.g. patterns, properties of number, operations with even and odd numbers. (See Chapter 8: Inspecting Equations)

Computational fluency – the ability to compute flexibly, accurately, and efficiently:

> *Flexibility* requires the knowledge of more than one approach to solving a particular kind of problem (depending on operation and number size) in order to choose the most appropriate strategy for that problem and to check results.
>
> *Accuracy* depends on knowledge of number facts and number relationships, double-checking, and careful recording.
>
> *Efficiency* requires a strategy that is easily carried out and makes use of intermediate results or sub-products to solve a problem.

Coordinate geometry - the study of geometry using a coordinate system like the coordinate plane (x-, y- axis).

Geometric transformations (isometries) – There are only five geometric changes that preserve most properties of a shape including size. They are slides (translations), flips (reflections), turns (rotations), no change (identity), footprints (glide reflection).

Empty number line – a plain line (without marks or numbers) that is used to show a linear (right to left) organization of number. Marks or numbers are added to the line to show the steps of the computation (See Chapter 6: Representation) Useful to show computation strategies or explore number relationships such as difference or numbers" between" consecutive numbers.

Function machine – an imaginary "machine" that changes numbers according to a function. Function machines assign every INPUT value an OUTPUT value for a given function. Intermediate grade functions include all operations, doubling, halving. (See Chapter 7: Number Work)

Justification – the reasoning behind a mathematical idea or solution to a problem.

Line plots – an easy way to organize data along a number line where Xs are written above the number to represent the frequency for that data value.

Magnitude –an estimate of size or magnitude expressed as a power of ten. This is one of many definitions.

Multiplicative reasoning – strategies based on multiplication and the properties of multiplication.

Number sense – some features of number sense include subitizing small quantities, noticing number patterns, counting, comparing numbers, decomposing/composing numbers, understanding operations, linear organization of number, relational thinking, additive reasoning, multiplicative reasoning.

Open number sentence – an equation that has one or more variables designated by letters or a box. (See Chapter 8: Inspecting Equations)

Part-whole situations – a problem setting where a given set is composed of two or more subsets. E.g. the set is balloons; the subsets are green, yellow, red, and blue.

Proportional reasoning – reasoning using multiplication relationships between numbers.

Quantitative relationships – numerical relationships.

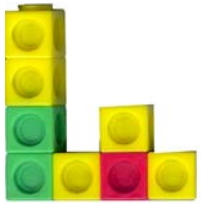
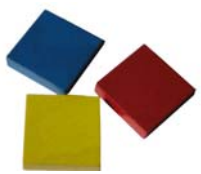
T/F number sentence – an equation (without variables). Used for inspecting equations. Students determine whether the equation is true or false and justify their reasoning. (See Chapter 8: Inspecting Equations)

Unitizing - the construction of a reference unit from a given ratio relationship.

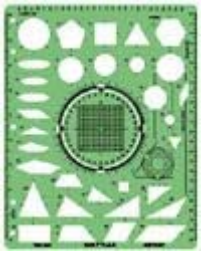
Storage Labels

Number and Operations	
	**number cubes**
	**place value arrows**
	**money**
	**place value blocks**

	**calculators**
Geometry	
	**Power Polygons**
	**geometric solids**
	**Geoblocks**

	Multi-links		square tiles
	Polydron Frames		pattern blocks
	mirrors		pentominoes
	geoboards		wooden cubes
	tangrams		Attribute blocks

Measurement			
	balances		measuring tapes
	rulers		Geometric-solids (fillable)
	angle rulers		circle compass

	protractors		yard/meter sticks
	1” & 1 cm cubes		geometry template

Professional Resources

Bickmore-Brand, J. (Ed.). (1990). *Language in Mathematics.* Portsmouth, NH: Heinemann.

Bushman, L. (2007). *Making Sense of Mathematics: Children Sharing and Comparing Solutions to Challenging Problems*. Reston, VA: National Council of Teachers of Mathematics.

Bushman, L. (2003). *Share & Compare: A Teacher's Story about Helping Children Become Problem Solvers in Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Bachman, V. (2003). *Third-grade math: A month-to-month guide*. Sausalito, CA: Math Solutions Publications.

Barton, M. L. & Heidema, C. (2002). *Teaching Reading in Mathematics* (2nd ed.). A supplement to *Teaching Reading in the Content Areas* teacher's manual. Aurora, CO: McREL Eisenhower Regional Consortium for Mathematics and Science.

Block, C. C. & Mangieri J. N. (2006). *The vocabulary enriched classroom: Practices for improving the reading performance of all students in grades 3 and up*. New York: Scholastic.

Bright, G. W. & Clements, D. H. (2003). *Classroom activities for learning and teaching measurement, 2003 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Bresser, R. & Holtzman, C. (2006). *Minilessons for Math Practice: Grades 3-5*. Sausalito, CA: Math Solutions Publications.

Carpenter, T. P., Fennema, E., Loef Franke, M., Levi, L. & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Chapin, S. H., O'Connor, C. & Anderson, N. C. (2003). *Classroom Discussions: Using Math Talk to Help Students Learn Grades 1-6*. Portsmouth, NH: Heinemann.

Chapin, S. H. & Johnson, A. (2006). *Math Matters: Understanding the Math You Teach Grades K-6*. Sausalito, CA: Math Solutions Publications.

Chappell, M. F., Schielack, J. F. & Zagorski, S. (2004). *Empowering the beginning teacher of mathematics: Elementary school*. Reston, VA: National Council of Teachers of Mathematics.

Coggins, D. (Ed.). (1999). *A mathematics sourcebook for elementary teachers: Key concepts, teaching tips, and learning pitfalls*. Novato, CA: Arena Press.

Confer, C. (2002). *Math by all means: Geometry grades 3-5*. Sausalito, CA: Math Solutions Publications.

Corwin, R. B. & Trafton, P. (1996). *Talking Mathematics: Supporting Children's Voices.* Portsmouth, NH: Heinemann.

Cuoco A. A. & Curcio, F. R. (2001). *Representation in School Mathematics, 2001 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Dacey, L., Cavanagh, M., Findell, C. R., Greenes, C. E., Sheffield, L. J. & Small, M. (2003). *Navigating through measurement in grade 3 through grade 5*. Reston, VA: National Council of Teachers of Mathematics.

Dana, M. *Inside-out math problems: Investigate number relationships & operations*. Grand Rapids, MI: Instructional Fair. (Out of print, see an elementary math resource teacher)

Dana, M. (1998). *Math Practice Games*. Grand Rapids, MI: Instructional Fair. (Blackline resource for Grades 1, 2, 3, 4, & 5 available from a math resource teacher)

Dixon-Krauss, L. (1996). *Vygotsky in the classroom: Mediated literacy instruction and assessment*. Boston: Allyn and Bacon.

Erickson, T. (1989). *Get it Together: Math Problems for Groups (Grades 4-12)*. Berkeley, CA: EQUALS / Lawrence Hall of Science.

Findell, C. R., Small, M., Cavanagh, M., Dacey, L., Greenes, C. R. & Sheffield, L. J. (2001). *Navigating through geometry in grade 3 – grade 5*. Reston, VA: National Council of Teachers of Mathematics.

Fosnot, C. T. & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Fosnot, C. T. & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition, and subtraction*. Portsmouth, NH: Heinemann.

Fosnot, C. T. & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Portsmouth, NH: Heinemann.

Franke, M., Kazemi, E. & Battey, D. (2007). Understanding Teaching and Classroom Practice in Mathematics. In Lester, Jr., F. K. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte, NC: Information Age Publications.

Fulton, B. & Lombard, B. (2003). *The Language of Math: Helping Students Speak, Write, and Think Mathematically*. Millville, CA: Teacher to Teacher Press.

Fuson, K. C., Kalchman, M. & Bransford, J. D. (2005). Mathematical understanding: An introduction. In Donovan, M. S. & Bransford, J. D. (Eds.), *How Students Learn Mathematics In the Classroom*. 215-256. Washington, DC: The National Academies Press.

Graves, M. (2006). *The vocabulary book: learning & instruction.* New York: Teacher's College Press.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K.C., Wearne, D. & Hanlie, M. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Holdaway, D. (1979). *The Foundations of Literacy*. Gosford, NSW, Australia: Scholastic.

Hope, J. A., Reys, B. J. & Reys, R. E. (1987). *Mental Math in the Middle Grades: Blackline Masters*. Parsippany, NJ: Dale Seymour Publications.

Hyde, A. (2006). *Comprehending math: Adapting reading strategies to teach mathematics, K-6*. Portsmouth, NH: Heinemann.

Kamii, C. (2004). *Young Students Continue to reinvent arithmetic* (2nd ed.). New York: Teachers College Press.

Kenney, J. M. (2005). *Literacy strategies for improving mathematics instruction*. Alexandria, VA: Association for Supervision and Curriculum Development.

Kilpatrick, J., Martin, G. W. & Schifter, D. (Eds.). (2003). *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J., Swafford, J., & Findell, C. R. (Eds.). (2001). *Adding it up*. Washington, DC: National Academy Press.

Kilpatrick, J., Swafford, J., & Findell, C. R. (Eds.). (2002). *Helping students learn mathematics*. Washington, DC: National Academy Press.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding*. Mahwah, NJ: Lawrence Elbaum Associates.

Lampert, M. L. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lilburn, P. & Rawson, P. (1995). *Let's Talk Math: Encouraging Children to Explore Ideas*. Portsmouth, NH: Heinemann.

Litwiller, B. & Bright, G. (Eds.). (2002) *Making sense of fractions, ratios, and proportions: 2002 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.

Madison Metropolitan School District. (2004). *MMSD mathematics grade level K-5 standards*. Retrieved May 14, 2009 from http://mathweb.madison.k12.wi.us/standards

Madison Metropolitan School District. (2004). Strategic Plan. Madison, WI.

Malloy, C. E. (2004). Equity in mathematics education is about access. In Rubenstein, R. N. & Bright, G. W. (Eds.). *Perspectives on the Teaching of Mathematics Sixty-sixth Yearbook*. (pp. 1-14). Reston, VA: National Council of Teachers of Mathematics.

Murray, M. (2004). *Teaching mathematics vocabulary in context: Windows, doors, and secret passageways*. Portsmouth, NH: Heinemann.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: An on-line version retrieved March 16, 2006 from http://standards.nctm.org/

Newmann, V. (1994). *Math Journals: Tools for Authentic Assessment*. San Diego, CA: Teaching Resource Center.

O'Connell, S. (2005). *Now I Get It: Strategies for Building Confident and Competent Mathematicians, K-6*. Sausalito, CA: Math Solutions Publications.

Pilar, R. (2004). Spanish-English language issues in the mathematics classroom. In Ortiz-Franco, L., Hernandez, N. G. & De La Crus, Y. *Changing the faces of mathematics perspectives on Latinos* (pp. 23-33). Reston, VA: National Council of Teachers of Mathematics.

Richardson, K. (1999). *Developing number concepts: Place value, multiplication, and division*. Parsippany, NJ: Pearson Education.

Richardson, K. (2002). *Math time: The learning environment*. Peabody, MA: Educational Enrichment, Inc. Didax Educational Resources.

Ronfeldt, S. (2003). *Third-grade math: A month-to-month guide*. Sausalito, CA: Math Solutions Publications.

Sakshaug, L. E., Olson, M. & Olson, J. (2002). *Children are mathematical problem solvers.* Reston, VA: National Council of Teachers of Mathematics.

Schuster, L. & Anderson, N. C. (2005). *Good Questions for Math Teaching: Why Ask Them and What to Ask (Grades 5-8)*. Sausalito, CA: Math Solutions Publications.

Sheffied, S. (2002). *Lessons for introducing place value*. Sausalito, CA: Math Solutions Publications.

Schifter, D. & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.

Small, M., Sheffield, L. S., Cavanagh, M., Dacey, L., Findell, C. R. & Greenes, C. E. (2004). *Navigating through problem solving and reasoning in grades 3-5*. Reston, VA: National Council of Teachers of Mathematics.

Smith, M. (2004). Beyond presenting good problems: How a Japanese teacher implements a mathematics task. In Rubenstein, R. & Bright, G. W. (Eds.). *Perspectives on the Teaching of Mathematics Sixty-sixth Yearbook* (pp. 98-106). Reston, VA: National Council of Teachers of Mathematics.

Sowder, J. & Schapelle, B. (Eds.). (2002). *Lessons learned from research.* Reston, VA: National Council of Teachers of Mathematics.

Spiegel, D. L. (2005). *Classroom Discussions: Strategies for Engaging All Students, Building Higher-Level Thinking Skills, and Strengthening Reading and Writing Across the Curriculum*. New York: Scholastic.

Stenmark, J. K. & Bush, William S. (2001). *Mathematics assessment: A practical handbook K-2*. Reston, VA: National Council of Teachers of Mathematics.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Sullivan, P. & Lilburn, P. (2002). *Good Questions for Math Teaching: Why Ask Them and What to Ask (Grades K-6)*. Sausalito, CA: Math Solutions Publications.

Sutton, J. & Krueger, A. (2002). *EDThoughts: What we know about mathematics teaching and learning*. Aurora, CO: Mid-continent Research for Education and Learning.

Trafton, P. R. & Thiessen, D. (1999). *Learning through problems: Number sense and computational strategies*. Portsmouth, NH: Heinemann.

Van de Walle, J. A. (2001). *Elementary and middle school mathematics*. New York: Longman.

Van de Walle, J. A. & Lovin, L. H. (2006). *Teaching student-centered mathematics: Grades K-3.* Boston: Allyn and Bacon Pearson Education.

von Rotz, L. & Burns, M. (2002). *Lessons for algebraic thinking*. Sausalito, CA: Math Solutions Publications.

Wheatley, G. H. & Reynolds, A. M. (1999). *Coming to know number: A mathematics activity resource for elementary school teachers*. Tallahassee, FL: Mathematics Learning.

White, D. Y. (2000). Reaching all students mathematically through questioning. In Strutchens, M. E., Johnson, M. L. & Tate, W. F. *Changing the face of mathematics: Perspectives on African-Americans* (pp. 21-32). Reston, VA: National Council of Teachers of Mathematics.

Whitin, P. & Whitin, D. J. (2000). *Math is Language Too: Talking and Writing in the Mathematics Classroom*. Reston, VA: National Council of Teachers of English.

Wisconsin Department of Public Instruction. *Wisconsin model academic standards.* Madison, WI: Retrieved March 16, 2006 from http://dpi.wi.gov/standards/matintro.html

Wisconsin Department of Public Instruction. (2005). *Wisconsin knowledge and concepts examination assessment framework.* Madison, WI: Retrieved May 14, 2009 from http://dpi.wi.gov/oea/pdf/math_framework.pdf

Wood, T., Nelson, B. S. & Warfield, J. (2001). *Beyond Classical Pedagogy*. Mahwah, NJ: Lawrence Erlbaum Associates.