§3.8

#5. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position \( u \) of the mass at any time \( t \). Plot \( u \) versus \( t \). Find the frequency, period, and amplitude of the motion.

**Solution.** Notice that the weight \( mg = 2 \) lb, \( L = 6 \) in. = \( 1/2 \) ft. \( \gamma = 0 \). Thus \( m = \frac{mg}{g} = \frac{2}{16} \frac{lb}{ft/sec} \), \( k = \frac{mg}{L} = 4 \frac{lb}{ft} \).

The IVP describing the scenario is,

\[
\frac{1}{16} u'' + 4u = 0, \quad u(0) = \frac{1}{4}, \quad u'(0) = 0.
\]

The characteristic equation of the corresponding homogeneous equation \( \frac{1}{16}r^2 + 2 = 0 \) has roots

\( 8i, -8i \)

The general solution of the DE is

\[ u(t) = c_1 \cos(8t) + c_2 \sin(8t). \]

\( u(0) = \frac{1}{4} \) implies that \( c_1 = \frac{1}{4} \). \( u'(0) = 0 \) implies that \( 8c_2 = 0 \), or \( c_2 = 0 \). Thus,

\[ u(t) = \frac{1}{4} \cos(8t). \]

The period \( T = \frac{2\pi}{8} = \frac{\pi}{4} \), frequency \( \omega = 8 \) rad/sec., and the amplitude is 1/4.

#10 A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in./sec, find its position \( u \) at any time \( t \). Plot \( u \) versus \( t \). Determine when the mass first returns to its equilibrium position. Also find the time \( \tau \) such that \( |u(t)| < 0.01 \) in. for all \( t > \tau \).

**Solution.** Notice that the weight \( mg = 16 \) lb, \( L = 3 \) in. = \( 1/4 \) ft. \( \gamma = 2 \) lb-sec/ft. Thus \( m = \frac{mg}{g} = \frac{16}{2} \frac{lb}{ft/sec} \), \( k = \frac{mg}{L} = 64 \frac{lb}{ft} \). The IVP describing the scenario is,

\[
\frac{1}{2} u'' + 2u' + 64u = 0, \quad u(0) = 0, \quad u'(0) = 1/4.
\]

The characteristic equation of the corresponding homogeneous equation \( \frac{1}{2}r^2 + 2r + 64 = 0 \) has roots

\( -2 + 2\sqrt{31}i, -2 - 2\sqrt{31}i \)

The general solution of the DE is

\[ u(t) = e^{2t} \left[ c_1 \cos(2\sqrt{31}t) + c_2 \sin(2\sqrt{31}t) \right]. \]

\( u(0) = 0 \) implies that \( c_1 = 0 \). \( u'(0) = \frac{1}{4} \) implies that \( 2\sqrt{31}c_2 = \frac{1}{4} \), or \( c_2 = \frac{1}{8\sqrt{31}} \). Thus,

\[ u(t) = \frac{1}{8\sqrt{31}} e^{-2t} \sin(2\sqrt{31}t). \]
$u(t)$ first returns to its equilibrium is when $2\sqrt{31}t = \pi$, or $t = \frac{\pi}{2\sqrt{31}}$ sec.

$$|u(t)| < 0.01 \text{ in.} = \frac{0.01}{12} \text{ ft}.$$  

Using calculator, we have $\tau = 1.5927088249$. That is, when $t > 1.5927088249$, $|u(t)| < \frac{0.01}{12} \text{ ft}$.  
