§3.8

#5.A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position u of the mass at any time t. Plot u versus t. Find the frequency, period, and amplitude of the motion.

Solution. Notice that the weight $mg = 2$ lb, $L = 6$ in.$= 1/2$ ft. $\gamma = 0$. Thus $m = \frac{mg}{g} = \frac{1}{16}\frac{lb}{ft/sec^2}$, $k = \frac{mg}{L} = 4\frac{lb}{ft}$. The IVP describing the scenario is,

$$\frac{1}{16}u'' + 4u = 0\,, \qquad u(0) = \frac{1}{4}, \qquad u'(0) = 0\,.$$

The characteristic equation of the corresponding homogeneous equation $\frac{1}{16}r^2 + 2 = 0$ has roots

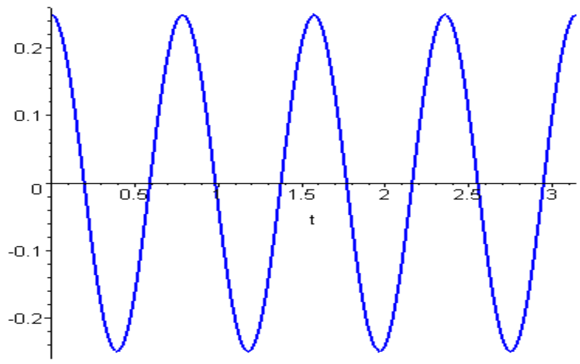
$$8i,\ -8i$$

The general solution of the DE is

$$u(t) = c_1\cos(8t) + c_2\sin(8t)\,.$$

$u(0) = \frac{1}{4}$ implies that $c_1 = \frac{1}{4}$. $u'(0) = 0$ implies that $8c_2 = 0$, or $c_2 = 0$. Thus,

$$u(t) = \frac{1}{4}\cos(8t)\,.$$



The period $T = \frac{2\pi}{8} = \frac{\pi}{4}$, frequency $\omega = 8$ rad/sec., and the amplitude is $1/4$.

#10 A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3in./sec, find its position u at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|u(t)| < 0.01$ in. for all $t > \tau$.

Solution. Notice that the weight $mg = 16$ lb, $L = 3$ in.$= 1/4$ ft. $\gamma = 2$ lb-sec/ft. Thus $m = \frac{mg}{g} = \frac{1}{2}\frac{lb}{ft/sec^2}$, $k = \frac{mg}{L} = 64\frac{lb}{ft}$. The IVP describing the scenario is,

$$\frac{1}{2}u'' + 2u' + 64u = 0\,, \qquad u(0) = 0, \qquad u'(0) = 1/4\,.$$

The characteristic equation of the corresponding homogeneous equation $\frac{1}{2}r^2 + 2r + 64 = 0$ has roots

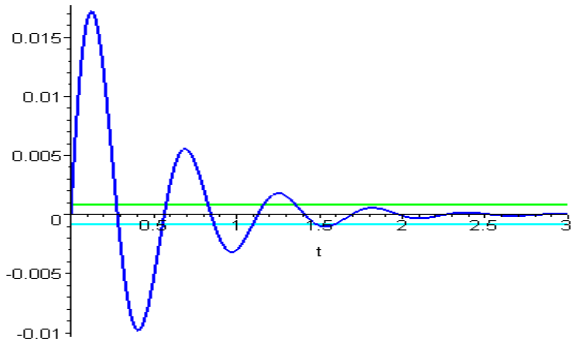
$$-2 + 2\sqrt{31}i,\ -2 - 2\sqrt{31}i$$

The general solution of the DE is

$$u(t) = e^{2t}\left[\,c_1\cos(2\sqrt{31}t) + c_2\sin(2\sqrt{31}t)\,\right].$$

$u(0) = 0$ implies that $c_1 = 0$. $u'(0) = \frac{1}{4}$ implies that $2\sqrt{31}c_2 = \frac{1}{4}$, or $c_2 = \frac{1}{8\sqrt{31}}$. Thus,

$$u(t) = \frac{1}{8\sqrt{31}}e^{-2t}\sin(2\sqrt{31}t)\,.$$



$u(t)$ first returns to its equilibrium is when $2\sqrt{31}t = \pi$, or $t = \frac{\pi}{2\sqrt{31}}$ sec.

$$|u(t)| < 0.01\, in. = \frac{0.01}{12}\, ft\,.$$

Using calculator, we have $\tau = 1.5927088249$. That is, when $t > 1.5927088249$, $|u(t)| < \frac{0.01}{12}\, ft\,.$