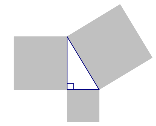
Proving the Pythagorean Theorem

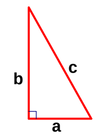
Proposition 47 of Book I of Euclid's *Elements* is the most famous of all Euclid's propositions. Discovered long before Euclid, the Pythagorean Theorem is known by every high school geometry student:

> *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*



Our more modern restatements of the theorem are more algebraic in nature:

> *In a right triangle whose sides have length a and b, and whose hypotenuse has length c, the relationship* $a^2+b^2=c^2$ *holds.*



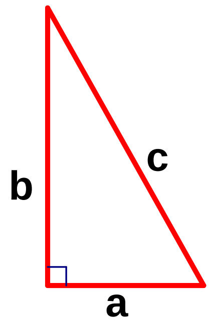
There are literally hundreds of proofs of the Pythagorean Theorem. Your task, with your partners, is to present one proof of the Pythagorean Theorem to the class.

The proofs are very visual, and they all combine algebra and geometry in some way. Unlike the Euclidean propositions you recently presented, however, these proofs are not quite complete, and you will need to supply a few details. Your job will be to explain your proof to the class so that we all understand your diagram and how the Pythagorean Theorem arises from it.

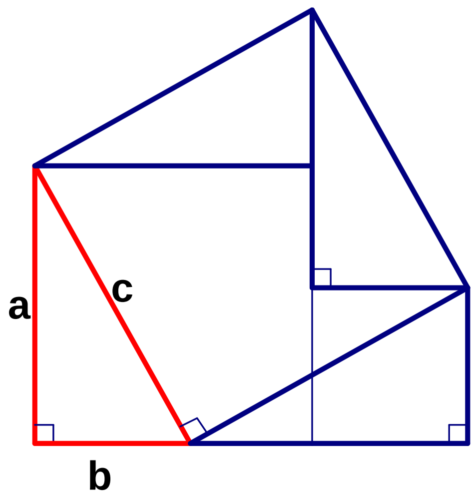
Some Suggestions for your Presentations

1. Introduce yourselves at the start of your presentation. Start with a single, labeled right triangle, and build your diagram from it. That will help us to see where your picture comes from.
2. Make sure to justify your statements. It doesn't need to be quite as formal as the Euclidean propositions, but we want to know, for example, how you know a certain square is a square, or why an area is what it is. *Especially make sure to tell us where in your proof you use the fact that your triangle is a right triangle.*
3. Feel free to draw pictures on the board as you go. You are welcome to use transparencies or PowerPoint if you want. But do make sure that you clearly explain your steps slowly and in detail.
4. Meet with your partners outside of class once or twice to go over the proposition. Practice your delivery!
5. Email me (*szydliks@uwosh.edu*) or see me if you're stuck on your proposition or if you have other concerns. Don't wait!

Pythagoras Proof #1



Starting with our given triangle, make three more copies of the triangle, and assemble them to construct the following figure:



Let A_1 be the area of the entire figure as shown. Let A_2 be the sum of the areas of the two squares with sides of length a and b, respectively. Let A_3 be the area of the square with side of length c. Then

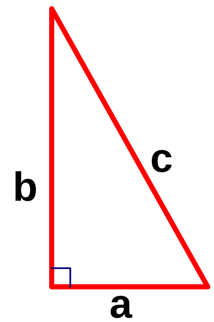
$$A_1 - A_2 = 2(\tfrac{1}{2})ab = ab$$
$$A_1 - A_3 = 2(\tfrac{1}{2})ab = ab$$
$$\text{Thus } A_1 - A_2 = A_1 - A_3$$
$$\text{and so } A_2 = A_3.$$
$$\text{Therefore } a^2 + b^2 = c^2.$$

In your presentation, make sure to explain:

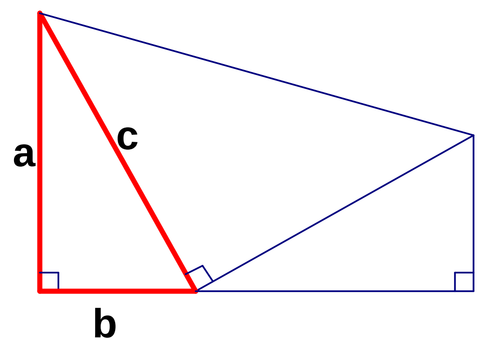
1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you know your squares are really squares.*
3. *how you figure out the areas involved.*

Pythagoras Proof #2

(This proof was first published by James Garfield, our 20th U.S. President.)



Starting with our given triangle, make a second copy of it lying on its side and construct the following figure:



Let A_1 be the sum of the areas of the three triangles. Let A_2 be the area of the trapezoid. Then

$$A_1=½\,(ab)+½\,(c^2)+½\,(ab)=½\,(c^2+2ab)$$
$$A_2=(a+b)(b+a)/2=[(a+b)^2]/2=(a^2+2ab+b^2)/2$$
Now $A_1=A_2$

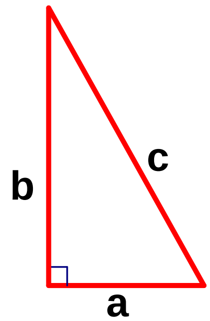
So $(c^2+2ab)/2=(a^2+2ab+b^2)/2$

and therefore $c^2=a^2+b^2$.

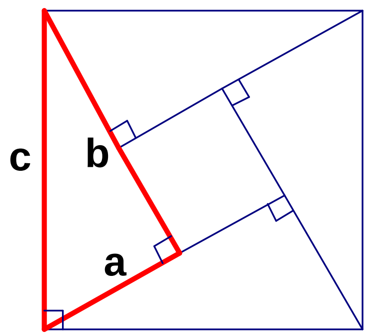
In your presentation, make sure to explain:

1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you know that the entire figure is really a trapezoid and the middle triangle is really a right triangle.*
3. *how you figure out the areas involved, especially the trapezoid.*

Pythagoras Proof #3



Starting with our given triangle, make 3 more copies of it and assemble the four triangles into the following figure:



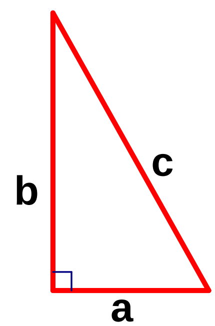
The inside figure is a square with sides of length *b-a*. The outside figure is a square with sides of length *c*. We can write the area of the outside square two different ways which are equal. So

$$c^2=4(\tfrac{1}{2})ab+(b-a)^2$$
$$=2ab+b^2-2ab+a^2$$
$$=a^2+b^2$$

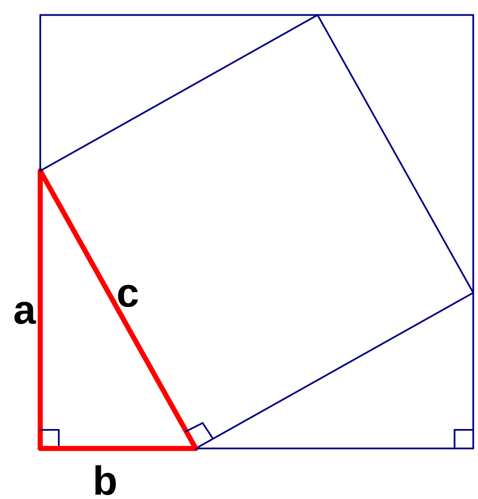
In your presentation, make sure to explain:

1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you know that the squares in your figure are truly squares.*
3. *how you figure out the areas involved.*

Pythagoras Proof #4



Starting with our given triangle, make 3 more copies of it and assemble the four triangles into the following figure:



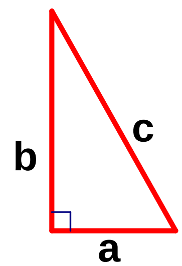
The inside figure is a square with sides of length *c*. The outside figure is a square with sides of length $a+b$. We can write the area of the outside square two different ways which are equal. So

$$(a+b)^2=4(½)ab+c^2$$
So $a^2+2ab+b^2= 2ab+c^2$,
and therefore $a^2+b^2= c^2$.

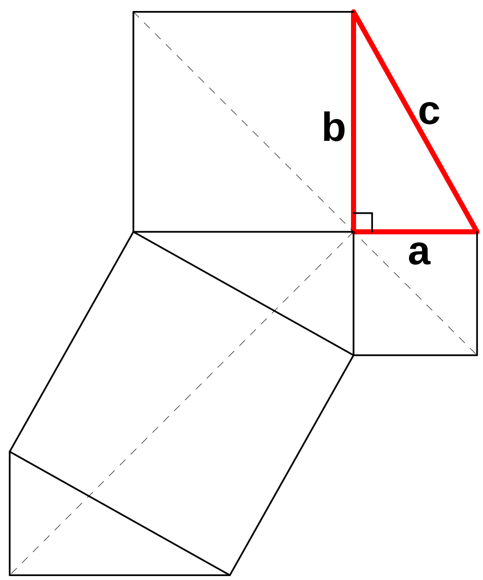
In your presentation, make sure to explain:

1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you know that the squares in your figure are truly squares.*
3. *how you figure out the areas involved.*

Pythagoras Proof #5
(This proof was first given by Leonardo da Vinci)



Starting with our given triangle, build squares on the sides of length a and b. Then construct a square with side length c as shown and then complete the figure by adding one more copy of the given triangle as shown:



Let A_1 be the sum of the areas of the two squares with sides of lengths a and b, respectively, together with the two triangles that share sides with these squares. Let A_2 be the sum of the areas of the square with side lengths c, together with the two triangles that share sides with that square. In both A_1 and A_2, the dashed segment divides the figure into two congruent parts that have area $½(a+b)^2$-$½(ab)$= $½(a^2+ab+b^2)$.

Then $A_1=A_2$.

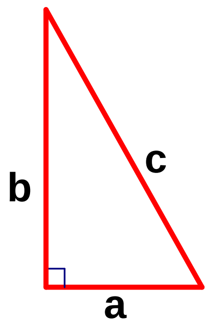
Also, $A_1=a^2+ab+b^2$ and $A_2=c^2+ab$.

Therefore $a^2+b^2=c^2$.

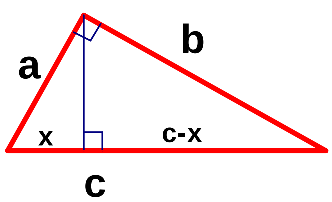
In your presentation, make sure to explain:

1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you figure out the areas involved.*

Pythagoras Proof #6



Starting with our given triangle, turn it on end and drop a perpendicular from the right angle to the hypotenuse:



The perpendicular divides the right triangle into two triangles that are similar to each other and to the original triangle. Label the segments as in the picture. Then, since corresponding sides are proportional, we have

$$\frac{c}{a} = \frac{a}{x} \text{ and } \frac{c}{b} = \frac{b}{c-x}$$

Cross multiplying gives us $a^2=cx$ and $b^2=c(c\text{-}x)$.

Adding these two equations gives us

$$a^2+b^2=cx+c(c\text{-}x)=c^2$$

In your presentation, make sure to explain:

1. *where you use the fact that the given triangle has a right angle (there may be more than one place where you use this fact.)*
2. *how you know all the triangles are similar (as well as what "similar" means).*
3. *which corresponding sides give us the fractions in the equations above.*