9 Complex Dynamics, the Julia Set.

9.1 Linear Functions

Let's consider in this section functions of the form

$$f(z) = az,$$

where a is a non-zero complex constant and z is a complex variable $z = x + iy$. It is clear that $z = 0$ is the only fixed point, for $az = z$ only if $z = 0$.

Let's write a and z in polar form: $a = \rho e^{i\phi}$, and $z_0 = re^{i\theta}$. Then we get

$$f(z_0) = az_0 = \rho re^{i(\theta+\phi)}.$$

We can see that applying the function f stretches (or shrinks) the radius by a factor of ρ, and rotates the argument by and angle of ϕ. Iterating, we get

$$f^2(z_0) = a(az_0) = a^2 z = \rho^2 re^{i(\theta+2\phi)}.$$

So the orbit of z_0 looks like:

$$z_0 = re^{i\theta},\ z_1 = \rho re^{i(\theta+\phi)},\ z_2 = \rho^2 re^{i(\theta+2\phi)},\ \ldots,\ z_n = \rho^n re^{i(\theta+n\phi)},\ \ldots$$

Since $|e^{i(\theta+n\phi)}| = 1$, the modulus of z_n is

$$|z_n| = |\rho^n re^{i(\theta+n\phi)}| = \rho^n r|e^{i(\theta+n\phi)}| = \rho^n r.$$

So the fate of the orbit depends on the radius of a. If $\rho < 1$, then $\rho^n r \to 0$ as $n \to \infty$, and all orbits of $f(z)$ are attracted to 0 as $n \to \infty$. If instead, $\rho > 1$, then $\rho^n r \to \infty$ as $n \to \infty$, and all orbits (except that of 0) are attracted to ∞ as $n \to \infty$. As n increases, the argument $\theta + n\phi$ continues to rotate, so if $\rho \neq 1$, all orbits will either spiral in towards 0 or spiral away from 0 towards ∞.

Example. $f(z) = \frac{1}{2}iz$ rotates angles by $\frac{\pi}{2}$ and shrinks radii by a factor of $\frac{1}{2}$. Therefore, all orbits (except the fixed point at the origin) will spiral counter-clockwise towards the origin.

Example. $f(z) = (1-i)z$ rotates angles by $\frac{-\pi}{4}$ and expands radii by a factor of $\sqrt{2}$. Therefore, all orbits (except the fixed point at the origin) will spiral clockwise away from the origin towards infinity.

Lastly, let's consider the case when $\rho = 1$, and we have the function of the form

$$f(z) = e^{i\phi}z = re^{i(\theta+\phi)}$$

So if z_0 has a radius of r, the radius of z_n will remain r. Thus the orbit of z_0 will remain on the circle of radius r. What the orbit does on this circle, however, depends on ϕ.

If ϕ is a rational multiple of 2π, that is, if

$$\phi = 2\pi\frac{k}{m},$$

for integers k and m, then

$$f^m(z_0) = re^{i(\theta+m\phi)} = re^{i(\theta+2\pi k)} = re^{i\theta} = z_0,$$

so all orbits are periodic of period m.

However, if ϕ is an irrational multiple of 2π, then it can be shown that there are no periodic points (except 0). Moreover, each orbit is a dense subset of the circle radius r. That means the orbit of z_0 will go almost everywhere on the circle, and will get arbitrarily close to any point you'd like on the circle.

Example. Consider the following function:

$$f(z) = z^2 + i.$$

Let $z_0 = \text{-}1 + i$. Since $z_1 = (z_0)^2 + i = (\text{-}1 + i)^2 + i = i$, and $z_2 = (z_1)^2 + i = i^2 + i = \text{-}1 + i = z_0$, z_0 lies on a period 2 cycle:

$$\text{-}1 + i,\ i,\ \text{-}1 + i,\ i,\ \text{-}1 + i,\ \ldots$$

To see if this cycle is attracting, repelling, or neutral, we consider that z_0 is a fixed point for

$$f^2(z) = f(z^2 + i) = (z^2 + i)^2 + i = z^4 + 2iz^2 - 1 + i.$$

We compute $(f^2)'(z) = 4z^3 + 4iz$, so $(f^2)'(\text{-}1 + i) = 4(\text{-}1 + i)^3 + 4i(\text{-}1 + i) = 4(2 + 2i) - 4i - 4 = 4 + 4i$. So

$$|(f^2)'(\text{-}1 + i)| = |4 + 4i| = 4\sqrt{2} > 1,$$

which means the 2-cycle is repelling.

Exercise. Find all fixed points for $f(z) = z^2 + z + 1$ and determine whether they are attracting, repelling, or neutral.

Exercise. Show that $z_0 = e^{2\pi i/3}$ lies on a 2-cycle for $f(z) = z^2$. Is this cycle attracting, repelling, or neutral?

9.2 The Complex Quadratic Family

We will now turn our attention to quadratic functions of the form

$$Q_c(z) = z^2 + c,$$

where c is a complex constant. Let's first consider

$$Q_0(z) = z^2.$$

Let $z_0 = re^{i\theta}$. Then the orbit of z_0 under Q_0 is given by

$$\begin{aligned} z_0 &= re^{i\theta} \\ z_1 &= r^2 e^{i(2\theta)} \\ z_2 &= r^4 e^{i(4\theta)} \\ &\vdots \\ z_n &= r^{2^n} e^{i(2^n\theta)} \end{aligned}$$

The behavior of the orbit of z_0 thus depends on the radius of z_0. If $r < 1$, then $r^{2^n} \to 0$ as $n \to \infty$, and so the orbit of z_0 will tend to the origin. Note that we saw in the previous section that the origin is an attracting fixed point for the map $Q_0(z) = z^2$.

If $r > 1$, then $r^{2^n} \to \infty$ as $n \to \infty$, and so the orbit of z_0 will tend to ∞, and is said to be *unbounded*.

If $r = 1$, that is, if z_0 is on the unit circle, then $r^{2^n} = 1$, so the orbit of z_0 will remain on the unit circle. In fact, $Q_0(z)$ simply doubles angles on the unit circle. But we have already seen that the doubling map on the unit circle is chaotic, hence Q_0 is chaotic on the unit circle.

In fact, the sensitive dependence on ititial conditions is very extreme. Consider a small wedge $W = \{re^{i\theta} | r_1 < r < r_2, \quad \theta_1 < \theta < \theta_2\}$ centered at a point z_0 on the unit circle. Then $Q_0^n(W)$ is a region determined by $r_1^{2^n} < r < r_2^{2^n}$, and $n\theta_1 < \theta < n\theta_2$. For n large enough, the angles exeeds 2π, so the region will wrap around itself. Also the inner radius of $Q_0^n(W)$ tends to 0 while the outer radius tends to infinity. Hence the orbits in W eventually reach every point in the complex plane, except for 0. This is called super-sensitivity.

9.3 The Julia Set

Definitions. The *filled Julia set* for $Q_c = z^2 + c$ is the set of all points whose orbits are bounded. The *Julia set* is the boundary of the filled Julia set.

For $Q_0 = z^2$, these sets are quite simple. As we saw in the last section, if $|z_0| < 1$, the orbit tends to 0, and is therefore bounded; if $|z_0| = 1$, then the orbit remains on the unit circle, and it therefore bounded; and if $|z_0| > 1$, the orbit tends to ∞ and is therefore unbounded. So the filled Julia set for Q_0 is the unit disk, and the Julia set for Q_0 is the unit circle. Notice that the Julia set for Q_0 is precisely the set where Q_0 is chaotic and supersensitive.

Theorem The quadratic function $Q_{-2}(z) = z^2 - 2$ on $\mathrm{C} \setminus [-2, 2]$ is conjugate to $Q_0(z) = z^2$ on $A = \{z| \, |z| > 1\}$. Hence the Julia set for $Q_{-2}(z)$ is the interval [-2,2].

Proof. Consider $h(z) = z + \frac{1}{z}$on A. Show that H is the required conjugacy.

Theorem If $|c| > 2$, then Julia set for the quadratic function $Q_c(z) = z^2 + c$ is a Cantor set, and the map Q_c on its Julia set is conjugate to the shift map on two symbols.

Theorem. The Escape Criterion. Suppose $|z_0| > \max\{|c|, 2\}$. Then $|Q_c^n(z_0)| \to \infty$ as $n \to \infty$. That is, the orbit of z_0 escapes to infinity.

Proof. $|z_1| = |Q_c(z_0)| = |z_0^2 + c|$, so by the triangle inequality,

$$|z_1| = |z_0^2 + c| \geq |z_0|^2 - |c|$$

But since $|z_0| \geq |c|$, we have

$$|z_1| \geq |z_0|^2 - |z_0| = (|z_0| - 1)|z_0|.$$

Now since $|z_0| > 2$, there must be a $\lambda > 0$ such that $|z_0| > 2 + \lambda$, and so $|z_0| - 1 > 1 + \lambda$. Therefore

$$|z_1| \geq (1 + \lambda)|z_0|.$$

That means $|z_1| > |z_0|$, and so Q_c moves points farther away. If we apply the same argument to z_1, and then to $|z_2|$, etc., we will find

$$|z_n| \geq (1 + \lambda)^n |z_0|.$$

Thus, as $n \to \infty$, the orbit of z_0 tends to infinity.

Note that in particular, if $|c| > 2$, then the filled Julia set for Q_c is contained entirely within the disk $|z| < |c|$.

The escape criterion allows us to develop an algorithm for plotting the Julia set for Q_c.

Algorithm for the Filled Julia Set for Q_c. Choose a maximum number of iterations, N. For each point z in the grid, compute the first N points on the orbit of z. If $|Q_c^k(z)| > \max\{|c|, 2\}$, for some $k < N$, then stop iterating and color z white. If $|Q_c^k(z)| \leq \max\{|c|, 2\}$, for all $k \leq N$, then color z black.

Since the white points satisfied the escape criterion, their orbits will be unbounded, and thus they are not in the filled Julia set. The black points have orbits that have remained bounded, (at least for the first N iterations), and so plotting the black points gives an approximation of the filled Julia set.

The escape criterion implies that the filled Julia set for Q_c exists only when z has radius less than 2 and less than $|c|$, so unless you are using a c value with $|c| > 2$, you can restrict your grid to $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

While increasing the number of iterations will improve the accuracy of the picture, in practice it is usually sufficient to set $N = 100$ to get the overall picture, then increase N if more detail is desired.

Exercise. Write a computer program to plot the filled Julia set for Q_c. Use you program to produce filled Julia sets for the following functions.

1. $f(z) = z^2 - 1$
2. $f(z) = z^2 + 0.3 - 0.4i$
3. $f(z) = z^2 + 0.360284 + 0.100376i$
4. $f(z) = z^2 - 0.1 + 0.8i$

Exercise.Use your computer program to plot the filled Julia sets for and $f(z) = z^2 + 0.25$ and $f(z) = z^2 + 0.255$, using $N = 50$, $N = 200$, and $N = 500$ iterations. Describe the effect of N on the filled Julia sets.

Exercise.Use your computer program to plot the filled Julia set for $f(z) = z^2 - 0.75 + 0.1i$ using $N = 100$, $N = 200$, and $N = 500$ iterations.