8 Fractals: Cantor set, Sierpinski Triangle, Koch Snowflake, fractal dimension.

8.1 Definitions

Definition

- If every point in a set S has arbitrarily small neighborhoods whose boundaries do not intersect S, then S has topological dimension 0.
- The topological dimension of a subset S of R^n is the least non-negative integer k such that each point of S has arbitrarily small neighborhoods whose boundaries meet S in a set of dimension $k-1$.

Examples Find the topological dimension of the following sets:

1. A finite collection of points.
2. $S = \{\frac{1}{n} | n = 1, 2, 3...\}$
3. A line segment.
4. The unit circle S^1.
5. The unit disk.
6. The unit sphere S^2.
7. The unit ball.

Definition

- A set S is self-similar if it can be divided into N congruent subsets, each of which when magnified by a constant factor M yields the entire set S.
- The fractal dimension of a self-similar set S is

$$D = \frac{\log(N)}{\log(M)}.$$

- A fractal is a set whose fractal dimension exceeds its topological dimension.

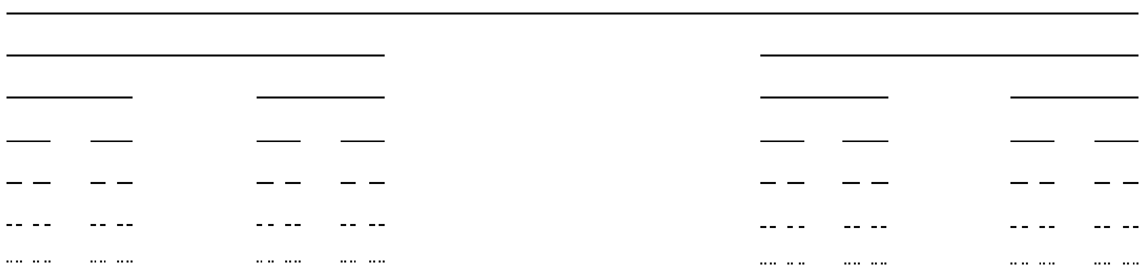
There are many sets which are self-similar that are not fractals. Find the fractal dimension of the following sets:

1. A line segment.
2. A (filled) square
3. A (filled) cube
4. The unit sphere S^2.

8.2 Standard examples of Fractals

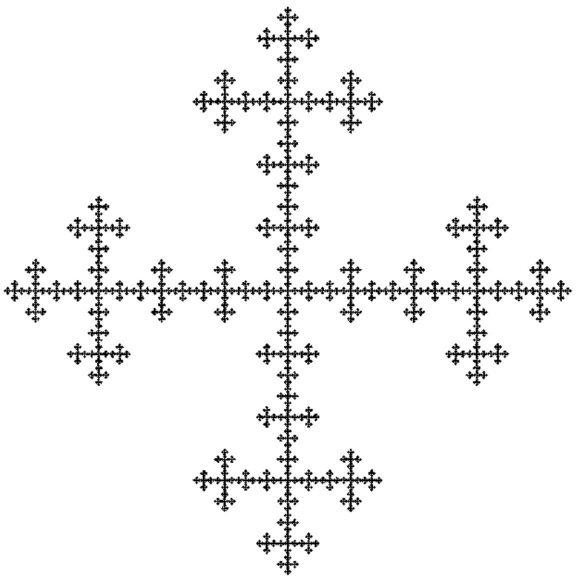
- **The middle thirds Cantor set. Since the Cantor set is totally disconnected, it has topological dimension 0. The Cantor set is self-similar, consisting of $N = 2$ congruent subsets, each when magnified by a factor of M = 3 yields the original set. Hence the fractal dimension of the Cantor set is $D = \frac{\log(2)}{\log(3)} \approx 0.631$. In general, the Cantor set consists of 2^n subsets, each with magnification factor 3^n. So the fractal dimenstion is**

$$D = \frac{\log(2^n)}{\log(3^n)} = \frac{n\log(2)}{n\log(3)} \approx 0.631.$$



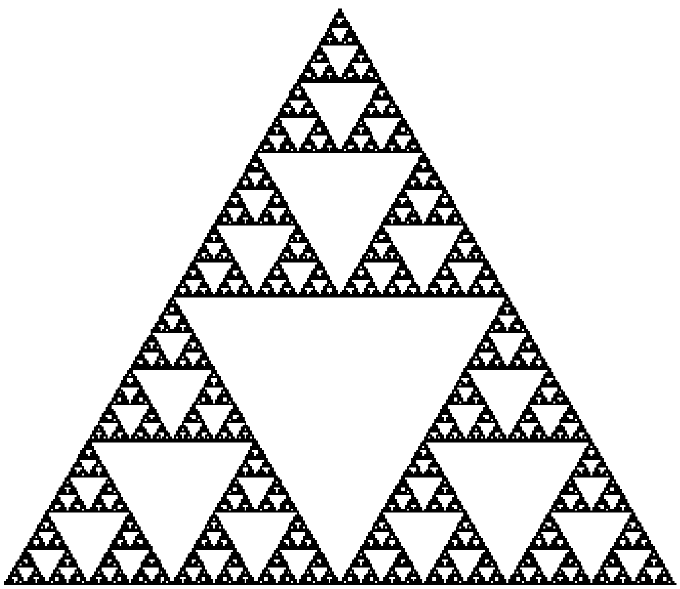
- **The Box Fractal is a higher-dimensional analog to the middle thirds cantor set. Starting with the closed (filled) unit square, at the first stage remove 4 open squares of size $\frac{1}{3}$. At the second stage, remove $4 \cdot 5$ open squares of size $\frac{1}{9}$. At the nth stage, remove $4 \cdot 5^{n-1}$ open squares of size $\frac{1}{3^n}$. Arbitrarily small neighborhoods intersect the box fractal at a finite set of points, so it has topological dimension 1. The box fractal is self-similar. At each stage, there are 5^n subsets, each with magnification factor 3^n, so the fractal dimension is**

$$D = \frac{\log(5^n)}{\log(3^n)} = \frac{n\log(5)}{n\log(3)} \approx 1.465.$$



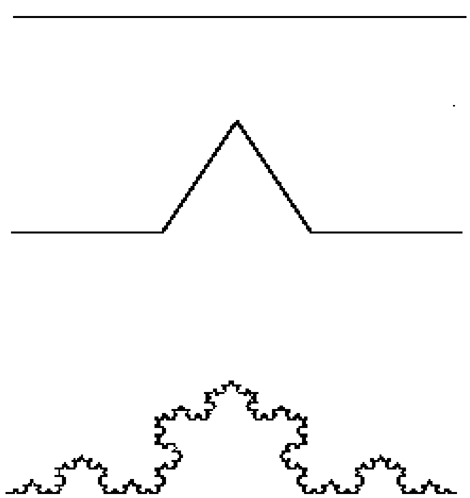
- **The Sierpinski Triangle is constructed like the box fractal, but using a triangles instead. Start with a closed (filled) unit equilateral triangle, at the first stage remove 1 open triangle of size $\frac{1}{2}$. At the second stage, remove 3 open triangles of size $\frac{1}{4}$. At the nth stage, remove 3^{n-1} open triangles of size $\frac{1}{2^n}$. After $n \to \infty$, we are left with a self-similar set with topological dimension 1. The Sierpinski Triangle consists of 3^n subsets with magnification factor 2^n. So the fractal dimension is**

$$D = \frac{\log(3^n)}{\log(2^n)} = \frac{n\log(3)}{n\log(2)} \approx 1.585.$$



- The Koch Curve is constructed very differently. Start with a closed unit interval. At the first stage remove the middle third of the interval and replace it with two line segments of length 1/3 to make a tent. The resulting set consists of 4 line segments of length 1/3. At the next stage, repeat this procedure on all of the existing line segments, resulting in a set that contains 16 line segments of length 1/9. At each stage there are 4^n line segments of length $\frac{1}{3^n}$. When $n \to \infty$, the resulting set is called the Koch curve. The set is self-similar, with 4^n subsets with magnification factor 3^n, so the fractal dimension is

$$D = \frac{\log(4^n)}{\log(3^n)} = \frac{n\log(3)}{n\log(4)} \approx 1.2619.$$



One amazing feature of the Koch curve is that it has infinite length. At each stage of the contruction, there are 4^n line segments of length $\frac{1}{3^n}$, for a total length of $\left(\frac{4}{3}\right)^n \to \infty$ as $n \to \infty$.

8.3 Create your own fractal: Iterated Function Systems.

We will now explore a new way of creating fractals, as the attracting set of an iterated function system.

Let x_0 be any point in the interval [0,1]. Define two functions:

$$F_0(x) = \frac{1}{3}x,$$

and

$$F_1(x) = \frac{1}{3}(x-1) + 1 = \frac{1}{3}x + \frac{2}{3}.$$

The function F_0 will move the point x_0 two-thirds of the way towards 0, while the function F_1 will move the point x_0 two-thirds of the way towards 1. 0 is the only fixed point for F_0, while 1 is the only fixed point for F_1. Let's explore the orbit of the intitial condition x_0 under the system of functions F_0 and F_1, where at each step, we choose to apply either F_0 or F_1 randomly with equal probability.

Where can the orbit of x_0 end up? First note that if x_0 is in the interval (1/3, 2/3), both F_0 or F_1 will map x_0 outside of this interval. Also, F_0 maps the interval [0,1/3] to [0,1/9], and the interval [2/3,1] to [2/9,1/3]. Similarly, F_1 maps the interval [0,1/3] to [2/3,7/9], and the interval [2/3,1] to [8/9,1].

Thus x_1 cannot be in (1/3, 2/3), and in fact, no iterates may be in this interval. Also x_2 cannot be in either (1/9,2/9) or (7/9,8/9), and in fact, no iterates may be in these intervals. Continuing this analyis, we see that the orbit of any initial condition x_0 can only be atttracted to the middle thirds cantor set.

To be more precise, represent sequence of iterations applied to x_0 by a sequence $(s_1 s_2 s_3 ...)$, where $s_i = 0$ if we apply F_0, and $s_i = 1$ if we apply F_1. Then the orbit of an initial condition x_0 is given by:

$$\begin{aligned}
x_1 &= \frac{x_0}{3} + \frac{2s_1}{3} \\
x_2 &= \frac{1}{3}\left(\frac{x_0}{3} + \frac{2s_1}{3}\right) + \frac{2s_2}{3} = \frac{x_0}{3^2} + \frac{2s_1}{3^2} + \frac{2s_2}{3} \\
x_3 &= \frac{1}{3}\left(\frac{x_0}{3^2} + \frac{2s_1}{3^2} + \frac{2s_2}{3}\right) + \frac{2s_3}{3} = \frac{x_0}{3^3} + \frac{2s_1}{3^3} + \frac{2s_2}{3^2} + \frac{2s_3}{3} \\
\vdots &= \vdots \\
x_n &= \frac{x_0}{3^n} + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \frac{2s_3}{3^{n-2}} ...
\end{aligned}$$

Now as $n \to \infty$, the first term goes to zero, so we have:

$$\lim_{n\to\infty} x_n = \sum_{i=1}^{\infty} \frac{t_i}{3^i},$$

where each t_i is either 0 or 2. As we saw in a previous section, these are precisely the points in the middle thirds Cantor set. Also notice that the result is independent of the intitial condition x_0, and only depends on the sequence of functions applied.

This gives us a new way of contructing the middle thirds Cantor set, as the attractor of the iterated function system $\{F_0, F_1\}$. We may implement this experimentally by fixing some initial condition in [0,1], and iterate, choosing F_0 and F_1 at each iteration with equal probability, and we know the orbit will be attracted to a point in the middle thirds Cantor set. We can iterate for say 1000 iterations, then plot x_1000 as being very very close to a point in the Cantor set. Then we can start over (with the same initial condition) and iterate another 1000 times. This sequence of iterations will be attracted to another point in the cantor set.

Let's try this same approach to create some other fractals:

To create the box fractal, we start with any initial condition $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ in the unit square $[0,1] \times [0,1]$, and iterate with the following set of functions:

$$
\begin{aligned}
F_0 &= \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} \\
F_1 &= \frac{1}{3}\begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \\
F_2 &= \frac{1}{3}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \\
F_3 &= \frac{1}{3}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \\
F_4 &= \frac{1}{3}\begin{bmatrix} x-1/2 \\ y-1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}
\end{aligned}
$$

What are the fixed points for each of the F_i?

The Sierpinski Triangle is the attractor of the following iterated function system:

$$
\begin{aligned}
F_0 &= \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} \\
F_1 &= \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
F_2 &= \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}
\end{aligned}
$$

In our above examples, each function is a contraction by a factor of $\beta < 1$ towards some fixed point (x_0, y_0). We can also introduce a rotation by an angle of θ. So we can write a function in an iterated function system (IFS) in following form:

$$
F_i\begin{bmatrix} x \\ y \end{bmatrix} = \beta \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
$$

or in general:

$$
F_i\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}
$$