7 Examples and the Period 3 Theorem.

Even though we have until now focused on the quadratic family $Q_c(x) = x^2 + c$, there are many other simple functions that exhibit chaotic behavior. The techniques we developed with the quadratic family can be applied to analyzing other functions. All of the functions discussed in the examples and homework are also chaotic, including the tent map, doubling map, and the logistic family $F_\lambda(x) = \lambda x(1 - x)$.

Example. The Doubling Function is defined by

$$D(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1/2 \\ 2x - 1 & \text{for } 1/2 \leq x < 1 \end{cases}$$

We will show that $D(x)$ is chaotic on $[0, 1)$. Let’s represent each $x \in [0, 1)$ by its binary expansion:

$$x = 0.b_1 b_2 b_3 \ldots = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \ldots \text{ where each } b_i \in \{0, 1\}.$$  

For $x = \frac{1}{2^n}$, represent $x$ with a binary expansion ending in 0’s, rather than 1’s. Then if $b_1 = 0$, we know $x \in [0, 1/2)$. Similarly, $b_1 = 1$, we know $x \in [1/2, 1)$.

Suppose $x \in [0, 1/2)$ is given. Then $x = 0.2b_2 b_3 b_4 \ldots = \frac{b_2}{2} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \ldots$. Then $D(x) = 2x = 0.2b_2 b_3 b_4 \ldots = \frac{b_2}{2} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \ldots$.

For example, consider $x = 0.201010110 \ldots$. Then $x = 1/4 + 1/16 + 1/64 + 1/128 + \ldots$ and $D(x) = 2x = 2/4 + 2/16 + 2/64 + 2/128 = 1/2 + 1/8 + 1/32 + 1/64 \ldots = 0.21010110 \ldots$.

Now suppose $x \in [1/2, 1)$ is given. Then $x = 0.21b_2 b_3 b_4 \ldots = \frac{1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \ldots$. Then $D(x) = 2x - 1 = 1.2b_2 b_3 b_4 \ldots - 1 = 0.2b_2 b_3 b_4 \ldots$. Hence, on $[0, 1)$, $D(x)$ is equivalent to $\sigma(x)$, the shift map on two symbols.

We have already seen that the shift map $\sigma$ is chaotic on the entire space of sequences of two symbols, hence $D(x)$ is chaotic on the entire interval $[0, 1)$.

Exercise. Let $S^1$ represent the unit circle in the plane. We can describe any point on $S^1$ by its polar angle $\theta$ in radians. Let $D(\theta) = 2\theta$, the doubling map on $S^1$. Prove that the doubling map is chaotic on $S^1$ by finding a semiconjugacy between $D(\theta)$ and $Q_{-2}(x)$. 

Theorem (Period Three implies Chaos). Suppose \( F : \mathbb{R} \to \mathbb{R} \) is continuous. If \( F \) has a periodic point of prime period 3, then \( F \) has periodic points of all other periods.

Proof. First we need to establish two lemmas that follow from the continuity of \( F \).

- **Fixed Point Lemma.** Suppose \( I \) and \( J \) are closed intervals such that \( I \subset J \). If \( J \subset F(I) \), then \( F \) has a fixed point in \( I \).

- **Preimage Lemma.** Suppose \( I \) and \( J \) are closed intervals and \( J \subset F(I) \). Then there exists a closed subinterval \( I' \subset I \) such that \( J = F(I') \).

Let \( F \) have a 3-cycle given by \( a \to b \to c \to a \). We will assume \( a < b < c \), the other cases are handled similarly.

Let \( I_0 = [a, b] \) and \( I_1 = [b, c] \). Then we have \( I_1 \subset F(I_0) \), and \( I_0 \cup I_1 \subset F(I_1) \).

- First we note that since \( I_1 \subset F(I_1) \), \( F \) has a fixed point in \( I_1 \), by the fixed point lemma.

- Now we will find a period 2 cycle. First we have that \( I_1 \subset F(I_0) \), so by the Preimage Lemma, there is a subset \( A_0 \subset I_0 \) such that \( I_1 = F(A_0) \). On the other hand, \( I_0 \subset F(I_1) \), so in fact \( I_0 \subset F^2(A_0) \). Then by the fixed point lemma, there is a fixed point for \( F^2 \) in \( I_0 \). So we have a point of period 2 for \( F \). In fact, since the first iterate of this point leaves \( I_0 \) it cannot be a fixed point for \( F \), so it has prime period 2.

- We will find a periodic cycle of period \( n \) for all \( n > 3 \) by invoking the Preimage Lemma \( n \) times. Since \( I_1 \subset F(I_1) \), there is a closed subinterval \( A_1 \subset I_1 \) such that \( I_1 = F(A_1) \). Now again we have \( A_1 \subset F(A_1) \), so there is a closed subinterval \( A_2 \subset A_1 \) such that \( A_1 = F(A_2) \). Thus \( I_1 = F^2(A_2) \).

Continue this process for \( n - 2 \) steps to produce the following nested collection of closed subintervals:

\[
A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1,
\]

such that \( A_i = F(A_i + 1) \) and \( I_1 = F^{n-2}(A_{n-2}) \).

Now let’s bring in \( I_0 \). We have \( A_{n-2} \subset I_1 \subset F(I_0) \), so there is a closed subinterval \( A_{n-1} \subset I_0 \) such that \( A_{n-2} = F(A_{n-1}) \).

Finally, we also have \( A_{n-1} \subset I_0 \subset F(I_1) \), so there a closed subinterval \( A_n \subset I_1 \) such that \( A_{n-1} = F(A_n) \).
What we have accomplished is the following:

\[ A_n \xrightarrow{F} A_{n-1} \xrightarrow{F} \ldots \xrightarrow{F} A_1 \xrightarrow{F} I_1 \]

where \( A_n \subset I_1 \), and \( I_1 = F^n(A_n) \). But by the fixed point lemma, this means that there is a fixed point for \( F_n \) in \( A_n \). Let us call this fixed point \( x_0 \). Then \( x_0 \) is a periodic point of period \( n \) for \( F \). In fact, since the first iterate of \( x_0 \) lies in \( I_0 \), while the next \( n - 1 \) iterates lie in \( I_1 \), we know that \( x_0 \) has prime period \( n \).

Sarkovskii’s Theorem. Suppose \( F : \mathbb{R} \to \mathbb{R} \) is continuous. If \( F \) has a periodic point of prime period \( n \), then \( F \) has periodic points of prime period \( k \), for all \( k \) after \( n \) in the Sarkovskii ordering as follows:

- \( 3, 5, 7, 9, \ldots \)
- \( 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, \ldots \)
- \( 2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, \ldots \)
- \( 2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, \ldots \)
- etc.
- \( \ldots, 2^n, \ldots, 2^3, 2^2, 2, 1 \).