4 The Orbit Diagram and Transition to Chaos

Theorem

There is at most only one attracting periodic orbit for $Q_c(x) = x^2 + c$. Moreover, if there is an attracting periodic orbit, then the orbit of $x_0 = 0$, the only critical point of $Q_c(x)$, will be attracted to it. For a proof of this theorem, see Devaney, Chapter 12.

Since the orbit of $x_0 = 0$, called the critical orbit, will find the attracting periodic orbit for $Q_c(x)$, if there is one, will create a diagram experimentally that shows the asymptotic behavior of the orbit of $x_0 = 0$ for various $c$ values.

We plot the parameter $c$ on the horizontal axis, and plot the asymptotic orbit of $x_0 = 0$ on the vertical axis. In practice, skipping the first 100 interactions, then plotting the next 100 works well. Here is the orbit diagram, or bifurcation diagram, for the quadratic family $Q_c(x) = x^2 + c$.

Try writing your own code to create this orbit diagram. There is a nice interactive applet available at http://math.bu.edu/DYSYS/applets/OrbitDiagram.html
5 The Definition of Chaos.

First some preliminary definitions:

**Definition** Suppose $X$ is a set and $Y$ is a subset of $X$. Then $Y$ is **dense** in $X$ if for any point $x \in X$, and any $\epsilon > 0$, there is a point $y \in Y$ such that $|x - y| < \epsilon$.

For example, the rational numbers are dense in the real numbers. Also, the irrational numbers are dense in the real numbers. However, the integers are not dense in the real numbers.

A **dense orbit** of a dynamical system on a set $X$ is an orbit whose points form a dense subset of $X$.

**Definition** A dynamical system is **transitive** if for any pair of points $x$ and $y$, and any $\epsilon > 0$, there is a third point $z$ within $\epsilon$ of $x$ whose orbit comes within $\epsilon$ of $y$.

**Theorem** A dynamical system is transitive if and only if it has a dense orbit.

It is easy to see that if a dynamical system has a dense orbit, then it is transitive, since this orbit comes arbitrarily close to all points. The converse is more difficult to prove.

**Definition** A dynamical system $F$ **depends sensitively on initial conditions** if there is a $\beta > 0$ such that for any $x$ and any $\epsilon > 0$, there is a $y$ within $\epsilon$ of $x$ and a $k$ such that $|F^k(x) - F^k(y)| > \beta$.

This means that for each $x$, there are points arbitrarily close to $x$ whose orbits eventually move far away from the orbit of $x$. Hence, if a dynamical system depends sensitively on initial conditions, then numerical computation will be inherently inaccurate, no matter how many digits of accuracy we use.

For example, $f(x) = x^2 - 1$ depends sensitively on initial conditions only at the two repelling fixed points. All other orbits are either attracted to a 2 cycle or diverge to infinity, and nearby orbits stay near and share the same fate.

**Definition (Devaney)** A dynamical system $F$ is **chaotic** if:

1. Periodic points for $F$ are dense.
2. $F$ is transitive.
3. $F$ depends sensitively on initial conditions.
Theorem The function $Q_{-2}(x) = x^2 - 2$ is chaotic on the interval [-2,2].

To prove this we will first look at a simpler function that we will show is dynamically equivalent to $Q_{-2}(x)$.

Let the V map be defined as $V(x) = 2|x| - 2$. The graphs of $V(x)$ and $Q_{-2}(x)$ on the interval [-2,2] are shown below.

Both functions map the interval [-2,2] onto itself, and both functions map [0,2] one-to-one and onto [-2,2], and [-2,0] one-to-one and onto [-2,2].

Hence both functions can be viewed as a stretching then a folding of the interval [-2,2] back on to itself in a 2 to 1 fashion.
5 THE DEFINITION OF CHAOS.

The graphs of $V^2$ and $V^3$ are shown below:

In general, the graph of $V^n(x)$ consists of $2^n$ linear segments, each with slope $\pm 2^n$, mapping an interval of length $\frac{1}{2^n-2}$ onto the entire interval $[-2,2]$.

Let $I$ represent any one of these intervals.

- $V^n$ has a fixed point in $I$, which means that periodic points are dense in $[-2,2]$.

- Let $x$ be any point in $I$, and $y$ be any other point in $[-2,2]$. Let $n$ be high enough so that $I$ has width less than $2\epsilon$. Since $I$ is mapped onto the entire interval $[-2,2]$, this means that there is some $z \in I$ that is mapped to $y$. Thus $V(x)$ is transitive on $[-2,2]$.

- For any $x \in I$, there is a $y \in I$ such that $|V^n(x) - V^n(y)| \geq 2$. So $V(x)$ depends sensibly on initial conditions on $[-2,2]$.

Hence we have proved that $V(x)$ is chaotic on $[-2,2]$. 
To prove that $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2,2]$, we will come up with a semi-conjugacy between $V(x)$ and $Q_{-2}(x)$.

**Definition** Suppose $F : X \to X$ and $G : Y \to Y$ are two dynamical systems. A mapping $h : X \to Y$ is called a **conjugacy** if $h$ is a homeomorphism (continuous, onto, and one-to-one), and satisfies

$$h \circ F = G \circ h.$$ 

A mapping $h : X \to Y$ is called a **semi-conjugacy** if $h$ is continuous, onto, and at most $n$-to-one), and satisfies

$$h \circ F = G \circ h.$$ 

Does such a semi-conjugacy $h$ exists between $V$ and $Q_{-2}$ on $[-2,2]$?

Yes. One example is the function $h(x) = -2 \cos(\frac{\pi}{2}x)$. It is straightforward to see that $h(x)$ is continuous, onto, and at most 2-to-1 on the interval $[-2,2]$. As an exercise, show that $h \circ V(x) = Q_{-2} \circ h(x)$ on $[-2,2]$.

So we have that $Q_{-2}$ and $V$ are semi-conjugates, and

$$h \circ V = Q_{-2} \circ h.$$ 

Then

- $h$ carries orbits of $V$ to orbits of $Q_{-2}$.
- Since $h$ is at most $n$-to-one, $h$ takes periodic orbits to periodic orbits (not neccesarily preserving the period, however).
- Since $h$ is continuous and onto, $Q_{-2}$ must have periodic points that are dense.
- Similarly, $Q_{-2}$ must have a dense orbit, so it is transitive.
- Since $n$ can be chosen to that $V^n$ maps arbitrarily small intervals onto $[-2,2]$, the same must be true for $Q_{-2}$, so $Q_{-2}$ depends sensitively on initial conditions.

Hence we have proven:

**Theorem** The function $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2,2]$. 