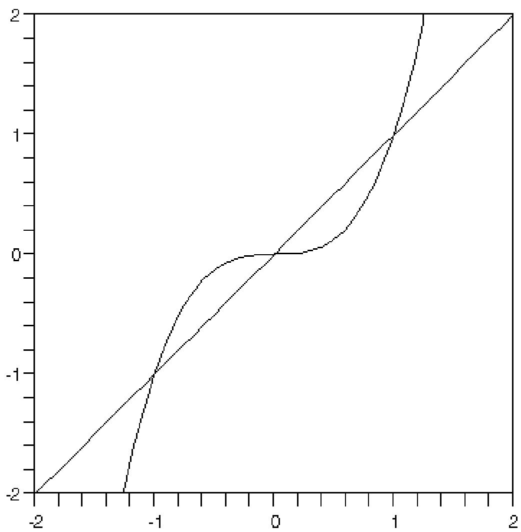
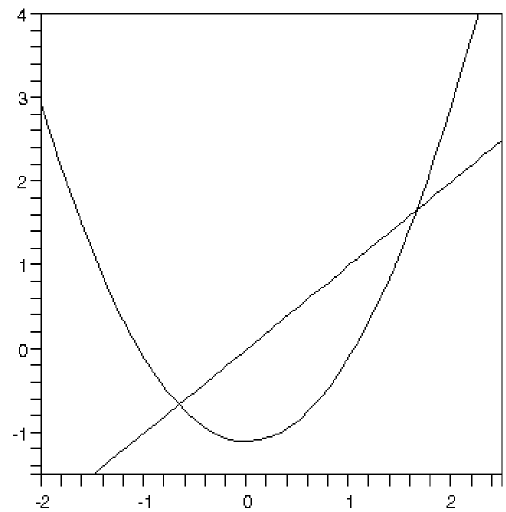
2 Graphical Analysis, and Attracting and Repelling Fixed Points

Graphical analysis is a tool to help visualize orbits for functions of a single real variable $f(x)$. First we note that the graphs of $y = f(x)$ and $y = x$ will intersect at the real fixed points for $f(x)$. So we begin our graphical analysis by plotting $y = f(x)$ and the diagonal $y = x$ on the same axes. To sketch an orbit, we pick an initial condition x_0, then find $y_0 = f(x_0)$ by moving vertically to the graph of $y = f(x)$. Then, to iterate we wish to let $x_1 = y_0$, that is, let the new x-value be the previous y-value. This is switching can be accomplished on the graph by moving horizontally to the diagonal. Then we begin again by moving vertically to the graph of $y = f(x)$, then horizontally to the diagonal $y = x$, etc.

Examples. On the plots below, use graphical analysis to analyze the orbits of $f(x) = x^3$ and $f(x) = x^2 - 1.1$.





Looking at $f(x) = x^3$, from graphical analysis, we can see that f has three fixed points, at $x = 0, 1,$ and -1. Orbits that begin inside the interval (-1,1) tend to the fixed point at $x = 0$, while orbits outside of the interval [-1,1] tend to positive or negative infinity. We say that the fixed point at o is ***attracting***, while the fixed points at 1 and -1 are ***repelling***.

Meanwhile, we can see that $f(x) = x^2 = 1.1$ has two fixed points, at $x \approx -.66$ and $x \approx 1.66$. In both cases, orbits that begin the fixed point do not tend towards the fixed point, and so both are said to be repelling. However, in the case of the fixed point near $x = -.66$, nearby orbits to not seem to be attracted to infinity.

An exellent web-based applet for making these "cobweb" graphs is found at http://www.emporia.edu/math-cs/yanikjoe/Chaos/CobwebPlot.htm

A fixed point z_0 is said to be an *attracting* fixed point for f if there is a neighborhood D of z_0 such that if $z \in D$, then $f^n(z) \in D$ for all $n > 0$, and in fact

$$f^n(z) \rightarrow z_0 \text{ as } n \rightarrow \infty.$$

A fixed point z_0 is said to be an *repelling* fixed point for f if there is a deleted neighborhood D of z_0 such that if $z \in D$, then $f^n(z) \notin D$ for some $n > 0$. This means that an orbit with an initial condition starting even very close to z_0 will eventually need to move away from z_0. Note that the orbit doesn't have to go to infinity or anywhere in particular, it just has to move away from z_0.

Theorem 1. Suppose $f(z)$ has a fixed point at z_0. Then z_0 is:

1. *attracting* if $|f'(z_0)| < 1$,
2. *repelling* if $|f'(z_0)| > 1$, and

If $|f'(z_0)| = 1$, the test is inconclusive, and the fixed point at z_0 is said to be *neutral*. It maybe that z_0 is attracting, repelling, or neither.

Example. Let $f(z) = z^2$. To find the fixed points of f, we solve $f(z) = z$, or

$$z^2 = z$$

$$z(z-1) = 0$$

So the only fixed points are $z = 0$ and $z = 1$. Next we compute $f'(z) = 2z$, and $|f'(z_0)| = |2z_0|$. So at the fixed point $z_0 = 0$,

$$|f'(0)| = |2 \cdot 0| = |0| = 0 < 1,$$

and by the theorem, the origin is an attracting fixed point for $f(z) = z^2$.
At the fixed point $z_0 = 1$,

$$|f'(0)| = |2 \cdot 1| = |2| = 2 > 1,$$

and by the theorem, 1 is a repelling fixed point for $f(z) = z^2$.

Example. Let $f(x) = \frac{1}{x}$. Then there are two neutral fixed points at 1 and -1. Since every other initial condition lies on a cycle of period 2, these fixed points are neither attracting nor repelling.

Example. Let $f(x) = x + x^3$. The fixed point at $x_0 = 0$ is neutral, since $|f'(0)| = 1$, yet all other orbits are repelled away from 0, although the divergence away from the fixed point at 0 is very slow.

If instead z_0 is periodic of period n, then since $f^n(z_0) = z_0$, we can view z_0 is a fixed point for $f^n(z)$. So we can apply the theorem above to f^n to determine the attraction or repulsion of periodic orbits.

Corollary 1. Suppose $f(z)$ has a periodic point z_0 of period n. Then the orbit of z_0 is:

1. *attracting* if $|(f^n)'(z_0)| < 1$,
2. *repelling* if $|(f^n)'(z_0)| > 1$, and
3. *neutral* if $|(f^n)'(z_0)| = 1$.

Example. Consider the following function:

$$f(z) = z^2 + i.$$

Let $z_0 = \text{-}1 + i$. Since $z_1 = (z_0)^2 + i = (\text{-}1 + i)^2 + i = i$, and $z_2 = (z_1)^2 + i = i^2 + i = \text{-}1 + i = z_0$, z_0 lies on a period 2 cycle:

$$\text{-}1 + i,\ i,\ \text{-}1 + i,\ i,\ \text{-}1 + i,\ \ldots$$

To see if this cycle is attracting, repelling, or neutral, we consider that z_0 is a fixed point for

$$f^2(z) = f(z^2 + i) = (z^2 + i)^2 + i = z^4 + 2iz^2 - 1 + i.$$

We compute $(f^2)'(z) = 4z^3 + 4iz$, so $(f^2)'(\text{-}1 + i) = 4(\text{-}1 + i)^3 + 4i(\text{-}1 + i) = 4(2 + 2i) - 4i - 4 = 4 + 4i$. So

$$|(f^2)'(\text{-}1 + i)| = |4 + 4i| = 4\sqrt{2} > 1,$$

which means the 2-cycle is repelling.

Instead of first computing $f^n(z)$, then taking the derivative, we can use the chain rule to compute $(f^n)'(z)$. Assume z_0 lies on a period n-cycle. Then

$$(f^2)'(z_0) = f'(f(z_0))f'(z_0) = f'(z_1)f'(z_0).$$

Repeated application of the Chain Rule gives us

$$(f^n)'(z_0) = f'(z_{n-1}) \cdot \ldots \cdot f'(z_1) \cdot f'(z_0).$$

Exercise. Find all fixed points for the following functions and determine whether they are attracting, repelling, or neutral.

1. $f(z) = z^2 + 1$
2. $f(z) = 3x - 3x^2$
3. $f(z) = z^3 + (i + 1)z$

Exercise. Let $f(x) = -\frac{1}{2}x^3 - \frac{3}{2}x + 1$. Show that 0 lies on a periodic cycle. Is this cycle attracting or repelling or neutral?