Properties and Super Properties[*]

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Abstract

The paper [2] discussed the properties HS and HL, related properties HC and HG, and the corresponding *strong* properties stHS, stHL, stHC, stHG. Here we explore the *super* properties suHS, suHL, suHC, suHG.

1 Introduction: the Super Idea

All topological spaces considered in this paper are T_3 (Hausdorff and regular). The notions of a space being HS (hereditarily separable) and HL (hereditarily Lindelöf) are standard in the literature. The paper [2] introduced the names HC and HG for two related properties; these two concepts also occur in the literature, but under different names. The four properties HS, HL, HC, HG, whose definitions are recalled below, are related by the implications:

$$\begin{tikzcd}[column sep=small] & \mathrm{HS} \arrow[dr] & \\ \mathrm{HG} \arrow[ur] \arrow[dr] & & \mathrm{HC} \\ & \mathrm{HL} \arrow[ur] & \end{tikzcd}$$

The corresponding *strong* properties stHC, stHS, stHL, stHG were also discussed in [2]. As usual, if $\mathcal{P}$ is a property of spaces, then X is *strongly* $\mathcal{P}$ (st$\mathcal{P}$) iff all finite powers of X have $\mathcal{P}$. Now we shall introduce the four *super* properties suHC, suHS, suHL, suHG. The definitions of the strong and super properties yield the implications:

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$$\begin{tikzcd} & \mathrm{stHS} \arrow[dr] & \\ \mathrm{stHG} \arrow[ur] \arrow[dr] & & \mathrm{stHC} \\ & \mathrm{stHL} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{suHS} \arrow[dr] & \\ \mathrm{suHG} \arrow[ur] \arrow[dr] & & \mathrm{suHC} \\ & \mathrm{suHL} \arrow[ur] & \end{tikzcd}$$

We have no formal *general* definition of su$\mathcal{P}$, but we shall give an informal discussion of su$\mathcal{P}$, and then, based on the definitions of HC, HS, HL, HG, give definitions of suHC, suHS, suHL, suHG.

We shall then prove some further implications among these 12 properties. There will remain a few open questions about what is provable under various axioms such as $\mathsf{MA}(\aleph_1)$ or PFA, but each of the possible $12 \cdot 11 = 132$ implications will either be proved in ZFC or refuted by CH.

First, we recall the discussion of HC, HS, HL, HG from [2]. All four of these properties can be described in terms of sequences of ω_1 points and neighborhoods. Given a space X, let $\mathcal{U}$ range over arbitrary sequences $\langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$, where each U_α is open and $x_\alpha \in U_\alpha$. Then X is:

HS iff $\forall \mathcal{U}\, \exists \alpha < \beta\, [x_\alpha \in U_\beta]$	iff X has no left separated sequence;
HL iff $\forall \mathcal{U}\, \exists \alpha < \beta\, [x_\beta \in U_\alpha]$	iff X has no right separated sequence;
HC iff $\forall \mathcal{U}\, \exists \alpha \neq \beta\, [x_\beta \in U_\alpha]$	iff X has no discrete sequence;
HG iff $\forall \mathcal{U}\, \exists \alpha \neq \beta\, [x_\beta \in U_\alpha \ \&\ x_\alpha \in U_\beta]$	iff X has no weakly separated sequence.

The first two lines express a standard characterization of the properties HS and HL; see the article by Roitman [7]. It is clear from the third line that X is HC iff X is hereditarily ccc iff X has countable spread. The concept HG is due, with different language, to Tkačenko [8]; see the discussion in [2]. It is also called the pointed ccc in Gruenhage [1]. Note that all the arrows in the above diagrams for the properties and the strong properties are clear from the definitions.

The *general super idea* is: If $\mathcal{P}$ is a property of the form "Given $\aleph_1$ things, there is a pair of them that is 'nice' in some way", then su$\mathcal{P}$ asserts that given $\aleph_1$ things, there is a subset of $\aleph_1$ of them, all pairs from which are 'nice'.

For example, if $\mathbb{P}$ is a forcing poset, then $\mathbb{P}$ is ccc iff given $p_\alpha \in \mathbb{P}$ for $\alpha < \omega_1$, we have $\exists \alpha \neq \beta\, [p_\alpha \not\perp p_\beta]$. Then super ccc concludes that $\exists I \in [\omega_1]^{\aleph_1}\, \forall \alpha, \beta \in I\, [p_\alpha \not\perp p_\beta]$. So, the succc is the Knaster property K (see [5], Section V.4).

For an arbitrary property $\mathcal{P}$, we have not given a formal *general* definition of su$\mathcal{P}$ because, unlike the st$\mathcal{P}$ notion, the su$\mathcal{P}$ depends on the specific form of the definition of $\mathcal{P}$ in terms of sequences, and equivalent definitions of $\mathcal{P}$ could yield superfications that are *not* equivalent. But using the above definitions of HS, HL, HC, HG , we simply *define* their super versions:

Definition 1.1 *Given a space X, let $\mathcal{U}$ range over sequences $\langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$, where each U_α is open and $x_\alpha \in U_\alpha$. Then X is:*

suHS *iff* $\forall \mathcal{U}\ \exists I \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in I\ [\alpha < \beta \to\ [x_\alpha \in U_\beta]]$;
suHL *iff* $\forall \mathcal{U}\ \exists I \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in I\ [\alpha < \beta \to\ [x_\beta \in U_\alpha]]$;
suHC *iff* $\forall \mathcal{U}\ \exists I \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in I\ [\alpha < \beta\ \to [x_\beta \in U_\alpha \text{ or } x_\alpha \in U_\beta]]$;
suHG *iff* $\forall \mathcal{U}\ \exists I \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in I\ [\alpha < \beta\ \to [x_\beta \in U_\alpha\ \&\ x_\alpha \in U_\beta]]$.

Remarks. We would have obtained an equivalent definition if we required the x_α all to be different points. In either case, these notions are trivial (or true vacuously) when X is countable. Also, each of HS, HL, HC, or HG alone yields a *countably* infinite index set I that satisfies the corresponding super condition; the underlying separation property determines a partition of $[\omega_1]^2$ that, by the Erdös Theorem $\aleph_1 \to (\aleph_1, \aleph_0)^2$ (see [3] p. 115), produces the desired set. For example, for HC, letting U_α be as in the definition, the set $J_0 = \{\{\alpha, \beta\} \in [\omega_1]^2 : x_\beta \notin U_\alpha\ \&\ x_\alpha \notin U_\beta\}$ determines a partition $[\omega_1]^2 = J_0 \cup J_1$. For any $A \subseteq \omega_1$ with $[A]^2 \subseteq J_0$, HC implies $|A| \leq \aleph_0$, and hence $\forall \mathcal{U}\ \exists I \in [\omega_1]^{\aleph_0}\ \forall \{\alpha, \beta\} \in [I]^2\ [x_\beta \in U_\alpha \text{ or } x_\alpha \in U_\beta]$.

Sometimes it is more convenient to apply equivalent expressions obtained by replacing a *list* of ω_1 different elements by a well-ordered *set* of $\aleph_1$ elements.

Proposition 1.2 *Given a space X and $E \in [X]^{\aleph_1}$, along with a well-order $\lhd$ of E in type ω_1, let $\mathcal{U}$ range over sequences $\langle (x, U_x) : x \in E\rangle$, where each U_x is open and $x \in U_x$. Then X is:*
suHS *iff* $\forall \mathcal{U}\ \exists I \in [E]^{\aleph_1}\ \forall \{x, y\} \in [I]^2\ [x \lhd y \to x \in U_y]$;
suHL *iff* $\forall \mathcal{U}\ \exists I \in [E]^{\aleph_1}\ \forall \{x, y\} \in [I]^2\ [x \lhd y \to y \in U_x]$;
suHC *iff* $\forall \mathcal{U}\ \exists I \in [E]^{\aleph_1}\ \forall \{x, y\} \in [I]^2\ [x \in U_y \text{ or } y \in U_x]$;
suHG *iff* $\forall \mathcal{U}\ \exists I \in [E]^{\aleph_1}\ \forall \{x, y\} \in [I]^2\ [x \in U_y\ \&\ y \in U_x]$.

With either view, it is easy to verify the four implications in the preceding super properties diagram.

In Figure 1 we stack the previous three diagrams, and include arrows illustrating further implications. The picture omits superfluous arrows, with the exception of some arrows at the *super* level: the arrow from suHS to suHC follows from suHS $\to$ suHL $\to$ suHC, and the equivalence of suHG, suHS, and suHL is emphasized by some superfluous double-headed arrows. Otherwise, implications in the transitive closure of the directed graph in Figure 1 are not shown. For example, suHS $\to$ stHS follows from suHS $\to$ suHG $\to$ stHG $\to$ stHS. Similarly, suHL $\to$ stHL. Even though suHC sits in the top level, the strongest properties implied by suHC are the HS and HL properties of the bottom level. So suHC $\to$ HC, but suHC does not imply stHC.

With the suHC exception, super $\to$ strong $\to$ the base "level 0"; that is, omitting suHC, the remaining seven downward (su$\mathcal{P} \to$ st$\mathcal{P}$ or st$\mathcal{P} \to \mathcal{P}$) implications of the diagram hold. Moreover, no implications go upwards: level 0 does not imply the weakest strong property stHC, and stHG does not imply the weakest super property suHC.

In Section 2, we shall prove (in ZFC) the implications of the diagram.

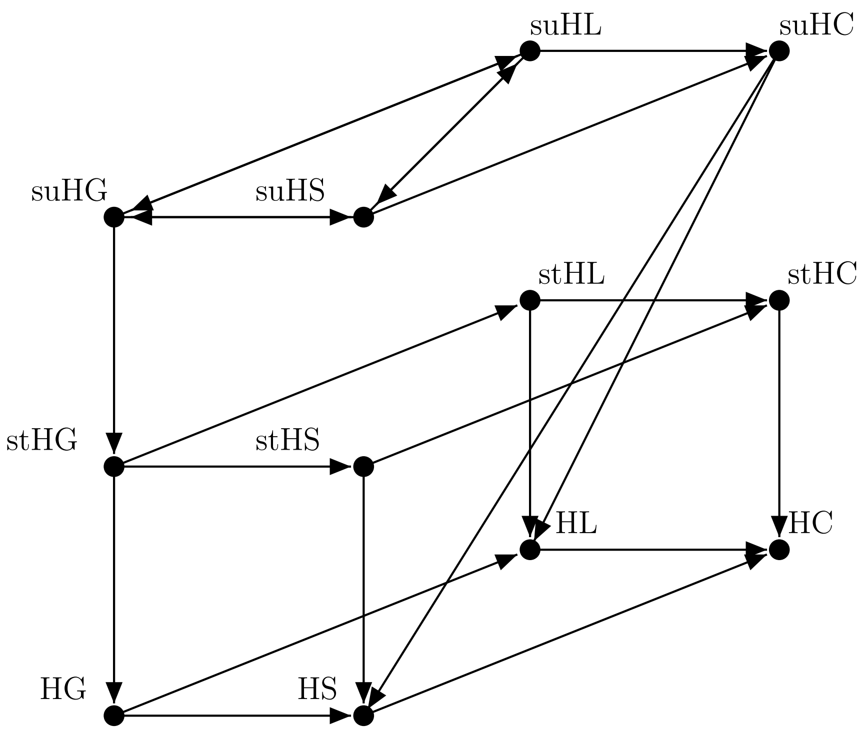

Figure 1: Summary of implications

In Section 3, we shall show that every implication not given by the transitive closure of this diagram has a counterexample under CH; a few of these counterexamples exist in ZFC. All can be obtained using the COMA, an axiom described in [2]; this axiom is a consequence of CH, but is also true in all models of the form $V[1\,\text{Cohen real}\,]$ or $V[1\,\text{random real}\,]$.

In Section 4, we shall make a few remarks on additional implications provable under axioms such as $\mathsf{MA}(\aleph_1)$ or PFA.

2 Proofs of the Implications

We first note that, as is clear from the definitions, all 12 of our properties are hereditary.

The only new implications involve the super properties. As noted in the Introduction, among just the super properties, all but suHS $\to$ suHG and suHL $\to$ suHG and suHL $\leftrightarrow$ suHS are clear from the definitions. Likewise, if $\mathcal{P}$ is any of HS,HL,HC,HG, then su$\mathcal{P} \to \mathcal{P}$ follows from the definitions.

Recall that a property $\mathcal{Q}$ is *productive* iff whenever X and Y both have $\mathcal{Q}$, then $X \times Y$ has $\mathcal{Q}$.

Proposition 2.1 *For $\mathcal{P}$ any of the properties HC,HS,HL, or HG, if* su$\mathcal{P}$ *is productive, then* su$\mathcal{P}$ *implies* st$\mathcal{P}$*.*

The super definitions then give us the following:

Corollary 2.2 *If $\mathcal{P}$ is property HS,HL, or HG, then* su$\mathcal{P}$ *implies* st$\mathcal{P}$*.*

We shall see in Section 3 that suHC is not productive, and suHC $\to$ stHC is refutable in ZFC. So, we shall complete our entire picture by proving the following two facts:

Proposition 2.3 *The super property suHC implies both HS and HL.*

Proposition 2.4 *The three properties suHG, suHS, and suHL are equivalent.*

The new implications yielding suHG from the others follow from the facts about the non-existence of increasing/decreasing ω_1-chains of closed sets in HS/HL spaces.

Proof of Proposition 2.4. Fix $E \in [X]^{\aleph_1}$, along with any well order $\lhd$ of E in type ω_1, and open $U_x \ni x$ for $x \in E$. Choose open $V_x \ni x$ for $x \in E$ such that $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. We must show that $\exists K \in [E]^{\aleph_1}\ \forall \{x, y\} \in [K]^2\ [x \in U_y\ \&\ y \in U_x]$.

Proof of suHS $\to$ suHG: Applying suHS, fix $J \in [E]^{\aleph_1}$ such that $\forall \{x, y\} \in [J]^2\ [x \lhd y \to x \in V_y]$. For $x \in J$, let $F_x = \mathrm{cl}\{t \in J : t \trianglelefteq x\}$. Then $x \in F_x \subseteq \overline{V_x} \subseteq U_x$. These are "almost" a chain in the sense that for $x, y \in J$, $x \lhd y \to F_x \subseteq F_y$. Since X is HS, there are no increasing ω_1-chains of closed sets, so fix $K \in [J]^{\aleph_1}$ such that $F_x = F_y$ for all $x, y \in K$. For any $x, y \in K$, $x \in F_x = F_y \subseteq \overline{V_y} \subseteq U_y$, which establishes suHG.

The proof of suHL $\to$ suHG is similar, replacing $\lhd$ by $\rhd$, and using the fact that if if X is HL, there are no *decreasing* ω_1-chains of closed sets. ☕

Proof of Proposition 2.3. Assume that X is suHC.
Proof of suHC $\to$ HS: Assume HS fails, and we shall produce an uncountable discrete subspace of X, contradicting HC.

Since X is not HS, we have $E \in [X]^{\aleph_1}$, well-ordered by $\lhd$ in type ω_1, along with open $U_x \ni x$ for each $x \in E$ such that for all $x, y \in E$, $x \lhd y$ implies $x \notin U_y$.

Since suHC is hereditary, we may, WLOG, replace X by this well-ordered subspace E. Now each U_x is an open subset of E. Also, each $y \in E$ satisfies $y \in U_y \subseteq [y, +\infty)$, and hence each $[y, +\infty)$ is open. Apply regularity to choose for each x an open V_x with $x \in V_x \subseteq \overline{V_x} \subseteq [x, +\infty)$.

Applying suHC, fix $I \in [E]^{\aleph_1}$ such that $\forall \{x, y\} \in [I]^2\ [x \in V_y \text{ or } y \in V_x]$. When $x \lhd y$, $x \notin V_y$, so $y \in V_x$. Then for all $x \in I$, $V_x \cap I = \overline{V_x} \cap I = [x, +\infty) \cap I$, and hence $[x, +\infty) \cap I$ is relatively clopen in I. Thus I is discrete in its relative topology: Letting $x^+ = \min\{i \in I : x \lhd i\}$, each $(x, +\infty) = [x^+, +\infty) \cap I$ is relatively clopen in I, and hence each $\{x\}$ is also relatively clopen in I.

Proof of suHC $\to$ HL: Assume that HL fails, and we shall derive a contradiction.

Since X is not HL, we can pass to a right separated ω_1-sequence in X. To construct an uncountable discrete subspace as in the preceding HS argument, let $E \in [X]^{\aleph_1}$ be ordered by $\lhd$ in type ω_1^* (the *reverse* of the order type ω_1), along with open $U_x \ni x$ for $x \in E$ such that for all $x, y \in E$, $x \lhd y$ implies $x \notin U_y$.

Then repeat the HS argument to get I. At the end, however, I is ordered in type ω_1^*, so not every $x \in I$ has a successor x^+, because x may correspond to a limit in this reverse ordering. To finish, let J be the set of all $x \in I$ that do have successors x^+ in I, so that J is discrete in its relative topology. Now J corresponds to the countable successor ordinals in ω_1, and hence is uncountable, contradicting suHC. ☕

3 The Non-Implications

Table 1 summarizes counterexamples to verify that *all* possible (ZFC) implications among our twelve properties are depicted by (the transitive closure of) Figure 1. Only the three equivalent properties suHG, suHS, and suHL, which *do* imply the other nine properties, do not start a row in the table. The middle column "properties *not* implied" lists a minimal subset: if property $\mathcal{P}$ does not imply property $\mathcal{R}$, then the row for property $\mathcal{P}$ lists as one of its "properties *not* implied" either $\mathcal{R}$ or a property $\mathcal{Q}$ such that $\mathcal{R}$ implies $\mathcal{Q}$. For example, HG does not imply stHS, but the table lists instead the weaker property stHC as one of its "properties *not* implied".

In the brief counterexample descriptions in the table, the $\dot\cup$ represents a disjoint union, and the "sep" of row one abbreviates separated. Only the HL and suHC rows

property	properties *not* implied	counterexample(s)
HC	HS, HL, stHC	right sep S-space $\dot\cup$ left sep L-space
HS	HL, stHC	S-space $\dot\cup$ Sorgenfrey line
HL	HS, stHC	L-space $\dot\cup$ Sorgenfrey line
HG	stHC, suHC	Corollary to result from [2]
stHC	HS, HL	strong L-space; strong S-space
stHS	HL	strong S-space
stHL	HS	strong L-space
stHG	suHC	Theorem 3.5
suHC	stHC, HG	Sorgenfrey line

Table 1: Counterexamples

of the table use ZFC counterexamples; otherwise, CH or the COMA yields the counterexamples. The stHC starts the only row that requires two counterexamples. We know that stHC $\leftrightarrow$ (stHS or stHL) (see [2]), so we have listed one counterexample to the implication stHC $\to$ HS and another to stHC $\to$ HL.

We begin with suHC, where the counterexample is obtained in ZFC.

Lemma 3.1 *The Sorgenfrey line is suHC.*

Proof. Let X be $\mathbb{R}$ with the Sorgenfrey topology. Fix $E \in [X]^{\aleph_1}$ along with open $U_x \ni x$ for $x \in E$.

Shrinking the U_x, we may assume each $U_x = (q_x, x]$, for some $q_x \in \mathbb{Q}$ with $q_x < x$. Choose $q \in \mathbb{Q}$ and $I \in [E]^{\aleph_1}$ such that $q_x = q$ for all $x \in I$. Then for all $\{x, y\} \in [I]^2$, $x \in U_y$ iff $x < y$. ☕

Proposition 3.2 *The property suHC is not productive.*

Proof. If X is $\mathbb{R}$ with the Sorgenfrey topology, then X^2 is not HC, because its diagonal is discrete. ☕

Corollary 3.3 *The property suHC implies neither stHC nor HG.*

Of the other eight rows, all but the HG and stHG examples involve standard CH constructions of S-spaces, L-spaces, strong S-spaces, or strong L-spaces. From [2], it is clear that the COMA is sufficient to construct these.

This leaves the HG and stHG properties, and assuming the COMA again yields the counterexamples listed in our table. We need to show that HG $\not\to$ stHC and HG $\not\to$ suHC and stHG $\not\to$ suHC. Of course, the second non-implication follows from the third, so we need only consider the first and third.

Corollary 3.4 (COMA) *The property HG does not imply stHC.*

Proof. By [2] (Theorem 1.3), the COMA implies that there are stHG spaces X and Y such that $X \times Y$ contains an S-space and an L-space. Say the S-space is $\{a_\alpha : \alpha < \omega_1\}$ and is right separated, while the L-space is $\{b_\alpha : \alpha < \omega_1\}$ and is left separated. Then $(X \times Y)^2$ contains $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$, which is discrete; so, $(X \times Y)^2$ is not HC. Now the disjoint union $X \dot\cup Y$ is HG, because X and Y are, but not stHC: $X \times Y$ embeds into $(X \dot\cup Y)^2$, and hence the non-HC $(X \times Y)^2$ embeds into $(X \dot\cup Y)^4$. ☕

We do not see how to prove the next theorem just by quoting a result from [2]. Instead, we refine the example generated by Theorem 6.11(C) of [2], and add the (ZFC) Lemma 3.8. The COMA is included as an assumption for Theorem 3.5 and Corollary 3.7 because these results use Theorem 6.11(C) of [2], which explicitly uses the COMA.

Theorem 3.5 (COMA) *The property stHG does not imply suHC.*

We use the terminology of [2], especially of the proof of Theorem 6.11(C). In [2], to build a stHG space, we adapt the construction in Roitman [7]; we employ a map $\Psi : \omega_1 \times \omega_1 \to 2$ to define a subspace $X = \mathcal{F}^\Psi = \{f^\Psi_\beta : \beta \in \omega_1\}$ of the product 2^{ω_1}, with $f_\beta = f^\Psi_\beta$ and each $f_\beta : \omega_1 \to 2$ defined by $f_\beta(\alpha) = \Psi(\alpha, \beta)$.

Here, to build a *non*suHC version, we specify a particular restriction that generates values of Ψ that are SSD or Strongly Symmetric about the Diagonal of $\omega_1 \times \omega_1$ and also force the suHC property to fail. The following gives the features of the SSD of [2] that we require:

Definition 3.6 *For $n \in \omega\backslash\{0\}$, a* (normalized) block pattern *of block size n* (BP_n) *is a sequence $\mathcal{A} = \langle A_\xi : \xi < \omega_1\rangle$, where each $A_\xi \in [\omega_1]^n$ and $\max(A_\xi) < \min(A_\eta)$ for all $\xi < \eta < \omega_1$. List each A_ξ in increasing order as $A_\xi = \{\alpha^i_\xi : i < n\}$. A map $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the* SSD_n *iff for each* BP_n *and all choices of $\langle c_{i,j} : i, j < n\rangle$ with each $c_{i,j} \in \{0, 1\}$:*

$$\exists\{\xi, \eta\} \in [\omega_1]^2 \left[\forall i, j\ [\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j}]\ \ \&\ \ \forall i, j\ [\Psi(\alpha^i_\eta, \alpha^j_\xi) = c_{i,j}]\right].$$

Then Ψ satisfies the SSD *iff $\forall n \in \omega\backslash\{0\}$ Ψ satisfies the* SSD_n*.*

We begin with a corollary to Theorem 6.11(C) of [2]:

Corollary 3.7 (COMA) *There is a (canonical) map $\Psi : \omega_1 \times \omega_1 \to 2$ that satisfies the SSD.*

Proof. Apply Theorem 6.11(C) to get a (canonical) map $\Psi : \omega_1 \times \omega_1 \to 2$ that satisfies the SSD. (In Theorem 6.11, simply use the "restriction" $\mathfrak{T} = (\{0, 1\}, \{0, 1\}, \{0, 1\})$.) ☕

Proof of Theorem 3.5. Let Ψ be the map from Corollary 3.7. By Lemma 5.5 of [2], the associated space $X = \mathcal{F}^\Psi$ is stHG. Lemma 3.8 completes the proof.

The next lemma, which does not use the COMA, shows that $X = \mathcal{F}^\Psi$ is not suHC. Any Ψ whose image of the diagonal of $\omega_1 \times \omega_1$ is a single value gives us neighborhoods that make it easy to see suHC fails. We could apply Theorem 6.11(C) to produce a (canonical) map whose diagonal image is a single value, or we can simply restrict a Ψ to a subset of $\omega_1 \times \omega_1$, as in the proof of the following:

Lemma 3.8 *If the map $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the* SSD_1*, then $X = \mathcal{F}^\Psi$ is not suHC.*

Proof. Suppose $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the SSD_1. Assume, without loss of generality, $\{\alpha < \omega_1 : \Psi(\alpha, \alpha) = 0\}$ is uncountable. Then, passing to a subsequence $\langle \alpha_\xi \rangle$ of ω_1, $\Psi(\alpha_\xi, \alpha_\xi) = 0$ for each $\xi \in \omega_1$.

To see that $X = \mathcal{F}^\Psi$ is not suHC, consider the open sets $U_\beta = \{f_\alpha \in X : f_\alpha(\beta) = 0\}$. Since $f_\beta(\beta) = \Psi(\beta, \beta)$, for each $\beta = \alpha_\xi$ we have $f_\beta \in U_\beta$.

Fix $I \in [\omega_1]^{\aleph_1}$. We use SSD_1 to produce $\{\alpha, \beta\} \in [I]^2$ so that $f_\alpha \notin U_\beta$ and $f_\beta \notin U_\alpha$: Let $\mathcal{A}$ be the normalized block pattern with $A_\xi = \{\alpha_\xi\}$ for each $\xi < \omega_1$. Apply the SSD_1 with $c_{0,0} = 1$ to get $\xi < \eta < \omega_1$ such that $\Psi(\alpha_\xi, \alpha_\eta) = c_{0,0} = 1$ and $\Psi(\alpha_\eta, \alpha_\xi) = c_{0,0} = 1$. Letting $\alpha = \alpha_\eta$ and $\beta = \alpha_\xi$, we have $f_\alpha(\beta) = 1 = f_\beta(\alpha)$, as desired.

4 Some Consistent Implications

Section 2 exhausted the ZFC implications of Figure 1. Here, we mention additional implications that follow from $\mathsf{MA}(\aleph_1)$ or PFA.

Assuming $\mathsf{MA}(\aleph_1)$, the properties stHS and stHL are equivalent [4], and hence stHL implies HS. In contrast, in ZFC, L-spaces exist [6]. Assuming PFA, there are no S-spaces [9], and so HS implies HL.

Finally, we point out an $\mathsf{MA}(\aleph_1)$ result for suHG:

Theorem 4.1 **($\mathsf{MA}(\aleph_1)$)** *Properties stHG and suHG are equivalent.*

The proof, as in [4], applies $\mathsf{MA}(\aleph_1)$ to a partial order $\mathbb{P} \subseteq [\omega_1]^{<\omega}$ to get an uncountable $I \subseteq \omega_1$ whose family of singletons is centered in $\mathbb{P}$; see also §6.3 of [7].

Proof. We already know that in ZFC the property suHG implies stHG. To prove the other direction, suppose X is stHG, and $E \in [X]^{\aleph_1}$, with open $U_x \ni x$ for $x \in E$. Then, assuming $\mathsf{MA}(\aleph_1)$, we must prove that $\exists I \in [E]^{\aleph_1}\ \forall \{x, y\} \in [I]^2\ [x \in U_y \ \&\ y \in U_x]$.

Let $\mathbb{P} = \{p \in [E]^{<\omega} : \forall \{x, y\} \in [p]^2\ [x \in U_y \ \&\ y \in U_x]\}$. Define $p \leq q$ iff $p \supseteq q$; so $\mathbb{1} = \emptyset$. Note that $\mathbb{P}$ contains all singletons. Applying $\mathsf{MA}(\aleph_1)$ gives us $I \in [E]^{\aleph_1}$ such that $\{\{x\} : x \in I\}$ is centered. Then for all $\{x, y\} \in [I]^2$ we have $\{x, y\} \in \mathbb{P}$, and hence $x \in U_y$ and $y \in U_x$.

To finish, we show that $\mathbb{P}$ is ccc (to verify that $\mathsf{MA}(\aleph_1)$ *does* apply). Assume not. We shall produce a weakly separated ω_1-sequence in X^n for some $n \in \omega$, contradicting stHG. Let $\{p_\alpha : \alpha < \omega_1\}$ be an antichain. When $\alpha \neq \beta$, $p_\alpha \perp p_\beta$ implies that there are $x \in p_\alpha \backslash p_\beta$ and $y \in p_\beta \backslash p_\alpha$ such that $\neg[x \in U_y \ \& \ y \in U_x]$.

Passing to a subsequence, we may assume the p_α form a delta system with some root R. Since $\{p_\alpha \backslash R : \alpha < \omega_1\}$ is also an antichain, we may also assume $R = \emptyset$, so that the p_α are pairwise disjoint. Finally, passing to a subsequence, we have some $n \in \omega$ so that $|p_\alpha| = n$ for all $\alpha < \omega_1$.

Now, let $p_\alpha = \{x^i_\alpha : i < n\}$. Let $V_\alpha = \bigcap_{i<n} U_{x^i_\alpha}$. Then V_α is open and $p_\alpha \subseteq V_\alpha$ because $p_\alpha \in \mathbb{P}$. But when $\alpha \neq \beta$, $p_\alpha \perp p_\beta$ implies that $p_\alpha \not\subseteq V_\beta$ or $p_\beta \not\subseteq V_\alpha$. Now, in X^n, letting $\vec{x}_\alpha = (x^i_\alpha : i < n)$, the sequence $\langle \vec{x}_\alpha : \alpha < \omega_1 \rangle$ is weakly separated by $W_\alpha = (V_\alpha)^n$: Each $\vec{x}_\alpha \in W_\alpha$, but when $\alpha \neq \beta$, either $\vec{x}_\alpha \notin W_\beta$ or $\vec{x}_\beta \notin W_\alpha$.

Thus, the fact that the property suHG is productive (see Section 2) gives us the following:

Corollary 4.2 ($\mathsf{MA}(\aleph_1)$) *The property stHG is productive.*

The property stHG is *not* productive under the COMA, again by Theorem 1.3 of [2].

We do not know what other implications may be added, in various models of set theory, to Figure 1.

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