Super Properties and Net Weight*

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February 27, 2020

Abstract

We show that it is consistent with $\mathsf{MA}(\aleph_1)$ to have a super HG (suHG) space of uncountable net weight. This suHG property and the weaker Hereditarily Good (HG) (or pointed ccc) property are discussed in papers [3, 4]; the HG is a natural strengthening of both Hereditarily Separable (HS) and Hereditarily Lindelöf (HL).

1 Introduction

All topological spaces considered in this paper are T_3 (Hausdorff and regular).

Paper [3], which introduced the Hereditarily Good acronym HG, related HG to the weaker properties Hereditarily Separable (HS) and Hereditarily Lindelöf (HL). For background discussion on the HG property, also called the pointed ccc in Gruenhage [1], see [3]. As in papers [3, 4], to determine whether a space X is HG, we begin by considering each assignment for X:

Definition 1.1 *Given a space X, a κ–*assignment for X *is a sequence $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \kappa \rangle$, where each U_α is open in X and each $x_\alpha \in U_\alpha$. Then, an* assignment for X *is an ω_1–assignment. A space X has the property HG iff for all assignments $\mathcal{U}$ for X, $\exists \alpha \neq \beta\ [x_\beta \in U_\alpha\ \&\ x_\alpha \in U_\beta]$.*

An assignment for X that makes HG fail need not make the underlying sequence left or right separated; so in [2], Hajnal and Juhász call such a sequence *weakly separated.* On the other hand, if X is HG, then it has no left or right separated ω_1 sequences, and so X is both HS and HL.

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*2010 Mathematics Subject Classification: Primary 54D70; Secondary 54D20, 54A35. Key Words and Phrases: weakly separated, forcing, network, net weight, butterfly.

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§The authors thank the referee for suggestions for improving the exposition.

Tkačenko [14] introduces the notion HG (with different language) and asks how the associated cardinal function R compares to the net weight. In the terminology of Hajnal and Juhász [2], X is HG iff $R(X) \le \aleph_0$.

We say that X is *strongly HG*, or stHG, iff X^n is HG for all $n \in \omega$ (equivalently, X^ω is HG). Also, X is stHG iff $R(X^\omega) = \omega$ in the terminology of [2, 9].

A related concept is HC (hereditarily ccc), obtained by replacing the "&" with an "or": X is HC iff for all assignments $\mathcal{U}$ for X, $\exists \alpha \ne \beta\ [x_\beta \in U_\alpha \text{ or } x_\alpha \in U_\beta]$. In particular, this holds whenever X is HS or HL. As with HG, X is stHC iff X^n is HC for all $n \in \omega$; but this holds *iff* X is stHS or stHL.

Furthermore, X is *super HG*, or suHG, iff for all such assignments,

$$\exists S \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in S\ [x_\beta \in U_\alpha \ \&\ x_\alpha \in U_\beta] \quad . \qquad (*)$$

The corresponding weaker notion suHC is obtained by replacing the "&" with an "or" in $(*)$.

Remarks. We would have obtained an equivalent definition of HG and suHG if we had required all the x_α to be different points. In any case, these notions are trivial (or true vacuously) when X is countable. Also, HG alone yields a *countably* infinite set S that satisfies the condition $(*)$. To see this, apply the Erdös Theorem $\aleph_1 \to (\aleph_1, \aleph_0)^2$ (see [7], p. 115).

Note that $(*)$ is equivalent to the briefer $\exists S \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in S\ [x_\alpha \in U_\beta]$, and to $\exists S \in [\omega_1]^{\aleph_1}\ \forall \alpha, \beta \in S\ [\{x_\alpha, x_\beta\} \subseteq U_\alpha \cap U_\beta]$.

The suHG trivially implies HG; moreover, any finite product of suHG spaces is suHG, and so suHG implies stHG. Paper [4] shows that CH produces an example of a stHG space that is not suHG (or even suHC) (see Section 6 for more on this example), but [4] also proves the following:

Theorem [4]4.1 $\mathsf{MA}(\aleph_1)$ *implies that every stHG space is suHG.*

Recall that a *network* for X is a family $\mathcal{N} \subseteq \mathcal{P}(X)$ such that $U = \bigcup\{N \in \mathcal{N} : N \subseteq U\}$ for all open $U \subseteq X$. The *net weight*, $nw(X)$, is the least cardinality of a network for X. Every base is a network, and $\{\{x\} : x \in X\}$ is a network, so clearly $nw(X) \le \min(|X|, w(X))$.

In some cases, $\mathsf{MA}(\kappa)$ will pin down the net weight:

Theorem 1.2 $\mathsf{MA}(\kappa)$ *implies that every stHG space X with $|X| \le \kappa$ and $w(X) \le \kappa$ has countable net weight.*

Section 3 includes a brief proof. Theorem 1.2 is also essentially a consequence of [9] Theorem 2.1, which shows in ZFC that X is stHG iff there is a ccc poset that forces $nw(X)$ to be countable; the end of the [9] Introduction points out a $\mathsf{MA}(\kappa)$ consequence that implies Theorem 1.2.

Given any countable network and assignment for a space X, a pigeonhole argument makes it easy to see the following:

Proposition 1.3 *If $nw(X) \leq \aleph_0$, then X is suHG.*

So a space having countable net weight is a *trivial* example of a suHG space. Paul Szeptycki asked whether a suHG space can have uncountable net weight. There are still open questions about what happens in various models of set theory, but we give an example with uncountable net weight in *one* model:

Theorem 1.4 *It is consistent with* ZFC *to have* MA *and* $\mathfrak{c} = \aleph_2$ *and a first countable suHG space X with $|X| = w(X) = nw(X) = \aleph_2$.*

Since we are asking for MA to hold, our X here could not satisfy $|X| = w(X) = \aleph_1$ because of Theorem 1.2, applied with $\kappa = \aleph_1$.

The following steps outline the proof of Theorem 1.4:

1. Start with $V \models \mathfrak{c} \leq \aleph_2 = 2^{\aleph_1}$.
2. Form $V[H]$ by adding $\aleph_2$ Cohen reals; so $V[H] \models \mathfrak{c} = \aleph_2 = 2^{\aleph_1}$. This $V[H]$ will contain a first countable space X that is stHG but not suHG (or even suHC), with $|X| = w(X) = nw(X) = \aleph_2$.
3. Iterate ccc forcing $\aleph_2$ times, as in the standard MA construction, but preserving the stHG of X. In the final $V[H][G]$, X will be suHG (by the above noted Theorem [4]4.1).
4. Verify that $nw(X)$ is still $\aleph_2$ in $V[H][G]$.

Section 2 describes the space X of item (2); X is a set of $\aleph_2$ Cohen reals, which we identify with a super Luzin set in the plane, and we give it a generalized butterfly topology (which includes the "standard" butterfly and bow-tie as special cases). Section 3 describes the iteration used in item (3). Section 4 explains how we keep the net weight big in some ccc iterations; in view of the above mentioned result of [9], some care is needed here, since some ccc forcings will make the net weight countable. This will handle item (4) and will complete the proof of Theorem 1.4.

Section 5 mentions a possible generalization of the generalized butterfly topologies. Section 6 shows that the method of [4] for proving the consistency of stHG $\not\to$ suHG cannot produce a first countable space, whereas (2) above does.

We do not know whether one can replace the $\mathsf{MA} + \mathfrak{c} = \aleph_2$ in Theorem 1.4 by the PFA; if so, it cannot be with the same generalized butterfly spaces. These spaces are *cometrizable*; that is, there is a coarser separable metric topology on them (here, the euclidean topology) such that each point has a neighborhood base for the finer topology consisting of sets that are closed in the metric topology. By Gruenhage [1], the PFA implies that every cometrizable space of uncountable net weight contains a subspace that contradicts the HG: an uncountable discrete subspace or an uncountable subspace of the Sorgenfrey line; Theorem 8.5 of Todorčević [15] presents this result just using the OCA (a consequence of the PFA).

A stronger result holds for our generalized butterfly spaces, which are a special kind of cometrizable space. In Section 2, where we define our butterfly notation, we shall show (Theorem 2.9) that under OCA, each of our spaces either has a countable network or an uncountable *closed* discrete set.

We also do not know whether just from CH, one can construct a space, as in Theorem 1.4, that is suHG and has uncountable net weight. In the opposite direction, we show in Theorem 2.5 that under CH, for the same butterfly and bow-tie spaces used in the proof of Theorem 1.4, suHG *does* imply countable net weight. This reverses the usual pattern of results in this area, where CH is used to produce examples (e.g., strong S-spaces and strong L-spaces), and under $\mathsf{MA}(\aleph_1)$ no such example exists (e.g., stHS = stHL).

2 Butterflies and bow-ties and Luzin sets

Our space X of Theorem 1.4 will be a subset of the plane with a butterfly or bow-tie topology; so this section includes definitions and relevant results for such topologies. In particular, Theorem 2.5 points out that for some butterfly spaces, under CH, suHG *implies* countable net weight.

The literature includes many examples but no standard definition of "general butterfly". In [3] we refine a metric space to define the generalized butterfly; now we strengthen our previous definition slightly:

Definition 2.1 *Let (X, d) be a separable metric space with topology $\mathcal{T}$. A* butterfly refinement *$\widehat{\mathcal{T}}$ of $\mathcal{T}$ is obtained as follows. Let $\Delta = \{(x,x) : x \in X\} \subseteq X \times X$ denote the* diagonal*, and for any $E \subseteq X \times X$, let E_x denote the slice $\{y : (x,y) \in E\}$. For each $n \in \omega$, choose sets U^n so that*

1. *$\Delta \subseteq U^n \subseteq X \times X$;*
2. *each $U^n \setminus \Delta$ is $\mathcal{T} \times \mathcal{T}$ open;*
3. *for each x, $\mathrm{diam}(U^n_x) \searrow_n 0$; and*
4. *$\mathrm{cl}(U^{n+1}, \mathcal{T} \times \mathcal{T}) \subseteq U^n$.*

Then $\widehat{\mathcal{T}}$ is the topology on X with open base $\{U^n_x : x \in X \;\&\; n \in \omega\}$. A generalized butterfly space *is a butterfly refinement of some separable metric space.*

The set $\{U^n_x : x \in X \;\&\; n \in \omega\}$ is indeed a base for a T_3 topology on X that refines $\mathcal{T}$; for each x, $\{U^n_x : n \in \omega\}$ is a local base at x, making $\widehat{\mathcal{T}}$ first countable. Fixing $n \in \omega$ gives us what we shall call a *uniform* assignment for X; this is one of the form $\mathcal{U}_n = \langle (x_\xi, U^n_\xi) : \xi < \omega_1 \rangle$, where $U^n_\xi = U^n_{x_\xi}$. Uniform assignments suffice in verifying that a butterfly refinement satisfies any one of the properties HG, suHG, HC, and suHC.

Definition 2.1 yields a generalized butterfly space that satisfies [3] Definition 4.4. For each $x \in X$ and $n \in \omega$: $x \in U^n_x = \{y : (x,y) \in U^n\}$ by (1), and $U^n_x \backslash \{x\}$ is $\mathcal{T}$ open by (2), and $\mathrm{cl}(U^{n+1}_x, \mathcal{T}) \subseteq U^n_x$ by (4).

If each U^n is $\mathcal{T} \times \mathcal{T}$ open, then $\widehat{\mathcal{T}} = \mathcal{T}$. More generally, $\widehat{\mathcal{T}} \supseteq \mathcal{T}$ is always true (by (3)), and $\widehat{\mathcal{T}} = \mathcal{T}$ iff for each $x \in X$ and $n \in \omega$ $x \in \mathrm{int}(U^n_x, \mathcal{T})$.

The sequence $\langle U^n : n \in \omega \rangle$ of Definition 2.1 determines two subtypes of butterfly refinement:

Definition 2.2 *Let* $(X, \widehat{\mathcal{T}})$ *be the butterfly refinement determined by* $\langle U^n : n \in \omega \rangle$. *The butterfly refinement is*

symmetric *iff it satisfies* $\forall x, y \in X\ \forall n \in \omega\ [x \in U^n_y \leftrightarrow y \in U^n_x]$ *(equivalently, each* U^n *is a symmetric subset of* $X \times X$*);*

nice *iff* $\forall x \in X\ \forall n \in \omega\ [x \in \mathrm{cl}(\mathrm{int}(U^n_x, \mathcal{T}), \mathcal{T})\ \&\ x \in \mathrm{cl}(\mathrm{int}(X \backslash U^n_x, \mathcal{T}), \mathcal{T})]$.

An equivalent definition of nice is: For all $x \in X$ and for all $S \subseteq X$: if S is $\mathcal{T}$-dense in some $\mathcal{T}$-neighborhood of x, then S meets each U^n_x and $X \backslash U^n_x$. The equivalence follows from a basic fact about an arbitrary topological space X, setting A in the next lemma to the sets U^n_x and $X \backslash U^n_x$:

Lemma 2.3 *For any space* X*,* $x \in X$*, and* $A \subseteq X$: $x \in \mathrm{cl}(\mathrm{int}(A))$ iff *for all* $S \subseteq X$*, if* S *is dense in some neighborhood of* x *then* $S \cap A \neq \emptyset$*.*

Proof. Fix $x \in X$ and $A \subseteq X$.

$\Rightarrow$: Assume that $x \in \mathrm{cl}(\mathrm{int}(A))$: Suppose that $S \cap W$ is a dense subset of W, where W is an open neighborhood of x. Then $W \cap \mathrm{int}(A)$ is a nonempty open subset of W. Thus $\emptyset \neq S \cap (W \cap \mathrm{int}(A)) \subseteq S \cap A$.

$\Leftarrow$: Assume that $x \notin \mathrm{cl}(\mathrm{int}(A))$: Let W be an open neighborhood of x with $W \cap \mathrm{int}(A) = \emptyset$. Let $S = W \backslash A$. Then S is dense in W and $S \cap A = \emptyset$. ☕

We now give four examples in the plane. We shall use the following notation: In the plane, if $\vec{x} = (x_0, x_1) \neq \vec{y} = (y_0, y_1)$, we denote the slope of the line through $\vec{y}$ and $\vec{x}$ by $\mathrm{sl}(\vec{y}, \vec{x})$. So if $y_0 \neq x_0$, then $\mathrm{sl}(\vec{y}, \vec{x}) = (y_1 - x_1)/(y_0 - x_0)$; when $x_0 = y_0$ and $x_1 \neq y_1$, let $\mathrm{sl}(\vec{y}, \vec{x}) = \infty$. Then "$|\mathrm{sl}(\vec{y}, \vec{x})| < r$" implies that $\vec{y}, \vec{x}$ have different horizontal coordinates. We shall use $\mathrm{sl}(\vec{y})$ to denote $\mathrm{sl}(\vec{y}, \vec{0})$. Let $\|\vec{x}\|$ be the usual euclidean norm on $\mathbb{R}^2$.

In the following four examples, each U^n is *translation invariant* (that is, $U^n = U^n + (\vec{z}, \vec{z}) = \{(\vec{x} + \vec{z}, \vec{y} + \vec{z}) : (\vec{x}, \vec{y}) \in U^n\}$ for each $\vec{z}$), so it suffices to present basic neighborhoods $V^n = U^n_{\vec{0}} \subseteq \mathbb{R}^2$. For each example, we specify sets $V^n \subseteq \mathbb{R}^2$ with $\vec{0} \in V^n$ and $V^n \backslash \{\vec{0}\}$ open and $V^n = -V^n$. We may picture V^n as a butterfly centered at $\vec{0}$. Define $U^n = \{(\vec{x}, \vec{y}) : \vec{x} - \vec{y} \in V^n\}$. Then $(\vec{x}, \vec{y}) \in U^n$ iff $(\vec{x} + \vec{z}, \vec{y} + \vec{z}) \in U^n$, since both are equivalent to $\vec{x} - \vec{y} = (\vec{x} + \vec{z}) - (\vec{y} + \vec{z}) \in V^n$.

Example 2.4 *The following are nice symmetric translation invariant butterfly refinements of* $(\mathbb{R}^2, \mathcal{T})$*, where* $\mathcal{T}$ *is the usual euclidean topology.*

1. The "standard" butterfly *has*
$U^n_{\vec{x}} = \{\vec{x}\} \cup \{\vec{y} \neq \vec{x} : \|\vec{y} - \vec{x}\| < 2^{-n} \ \&\ |\mathrm{sl}(\vec{y}, \vec{x})| < 1 + 2^{-n}/9\}$*, or*
$V^n = \{\vec{0}\} \cup \{\vec{y} \neq \vec{0} : \|\vec{y}\| < 2^{-n} \ \&\ |\mathrm{sl}(\vec{y})| < 1 + 2^{-n}/9\}$.

2. The Type I bow-tie *has*
$V^n = \{\vec{0}\} \cup \{\vec{z} \neq \vec{0} : |z_0| < 2^{-n} \ \&\ |\mathrm{sl}(\vec{z})| < 1 + 2^{-n}/9\}$.

3. The Type II bow-tie *has*
$V^n = \{\vec{0}\} \cup \{\vec{z} \neq \vec{0} : |z_0| < 2^{-n} \ \&\ |\mathrm{sl}(\vec{z})| < 2^{-n}\}$.

4. The bulbous butterfly *has, setting* $\vec{c}_n = (2^{-n}, 0)$*:*
$V^n = \{\vec{0}\} \cup \{\vec{x} : \|\vec{x} - \vec{c}_n\| < 2^{-n} \text{ or } \|\vec{x} + \vec{c}_n\| < 2^{-n}\}$.

For the "standard" butterflies, the large butterflies have slightly larger slope than the smaller butterflies, which ensures that $\mathrm{cl}(U^{n+1}, \mathcal{T} \times \mathcal{T}) \subseteq U^n$. With this topology, each horizontal line has its euclidean topology $\mathcal{T}$ and each vertical line is closed and discrete. For a slanted line L whose slope has absolute value $s \in (0, \infty)$, L has its euclidean topology when $s \leq 1$, and is closed and discrete when $s > 1$.

These "horizontal bow-tie neighborhoods" appear in [12], which traces the example to [5]. The straight edges of the bow-ties here make them somewhat simpler to describe than the "standard" butterflies. Each V^n has width $2 \cdot 2^{-n}$, decreasing to 0 as $n \nearrow \infty$. Type I and Type II give two natural options for the slopes of the sides. With Type I, as with our "standard" butterflies, if L is a line where the magnitude of the slope of L is $s \in [0, \infty]$, then L has its euclidean topology when $s \leq 1$ and L is closed and discrete when $s > 1$. With the skinny Type II, the slopes decrease to 0 as $n \nearrow \infty$. With this topology, each horizontal line has its euclidean topology and each non-horizontal line is closed and discrete.

In the bulbous butterfly, each V^n is the union of two adjacent discs, with radius 2^{-n} and centers $\pm\vec{c}_n$. Note that $V^n = -V^n$ and $\vec{0} \in V^n$ and $\mathrm{int}(V^n) = V^n \backslash \{\vec{0}\}$ and $\overline{V^{n+1}} \subseteq V_n$. In $\widehat{\mathcal{T}}$, vertical lines are closed and discrete, and non-vertical lines have their euclidean topology.

Remarks. The notions of "butterfly refinement" and "symmetric" are hereditary, but "nice" is not. In the plane, the euclidean topology $\mathcal{T}$ itself is a symmetric butterfly that is not nice; that is, we could set each $V^n = \{\vec{z} : \|\vec{z}\| < 2^{-n}\}$, making $\widehat{\mathcal{T}} = \mathcal{T}$ (so our butterflies look like ladybugs), to get a trivial example of a symmetric translation invariant butterfly that is not nice. Also, the topology of the Sorgenfrey line is a nice non-symmetric butterfly refinement of the euclidean topology on $\mathbb{R}$.

We now consider suHG butterflies under CH. We shall prove the following:

Theorem 2.5 *Assume* CH*, and fix* $S \subseteq X$*, where* X *is a symmetric butterfly space. Assume that* S *is suHG. Then* $nw(S) \leq \aleph_0$*.*

In the proof of Theorem 2.5, we shall use $\mathfrak{P}_n$ to denote the family of subsets of X produced by applying suHG to uniform assignments:

Definition 2.6 *Let* $(X, \widehat{\mathcal{T}})$ *be the* symmetric *butterfly refinement determined by* $\langle U^n : n \in \omega \rangle$*. For* $n \in \omega$*, let* $\mathfrak{P}_n$ *denote the family of all subsets* $P \subseteq X$ *that satisfy* $\forall x, y \in P\, [x \in U^n_y]$ *(equivalently,* $P \times P \subseteq U^n$ *).*

Note that $m < n \to \mathfrak{P}_n \subseteq \mathfrak{P}_m$, because $U^n \subseteq U^m$. For the "standard" butterfly, $P \in \mathfrak{P}_n$ iff $\|\vec{y} - \vec{x}\| < 2^{-n}$ and $|\mathrm{sl}(\vec{y}, \vec{x})| < 1 + 2^{-n}/9$ for all $\vec{x}, \vec{y} \in P$ such that $\vec{x} \neq \vec{y}$. For such P, $\mathrm{cl}(P, \mathcal{T})$ need not be in $\mathfrak{P}_n$ because it may contain pairs $\vec{x}, \vec{y}$ with $\|\vec{y} - \vec{x}\| = 2^{-n}$ or $|\mathrm{sl}(\vec{y}, \vec{x})| = 1 + 2^{-n}/9$. But these will still satisfy $\|\vec{y} - \vec{x}\| < 2^{-(n-1)}$ and $|\mathrm{sl}(\vec{y}, \vec{x})| < 1 + 2^{-(n-1)}/9$, so $\mathrm{cl}(P, \mathcal{T}) \in \mathfrak{P}_{n-1}$. This form of backwards closure generalizes:

Lemma 2.7 *Let* $(X, \widehat{\mathcal{T}})$ *be the* symmetric *butterfly refinement determined by* $\langle U^n : n \in \omega \rangle$*. Fix* $n \in \omega$ *and fix* $P \in \mathfrak{P}_{n+1}$*. Then* $\mathrm{cl}(P, \mathcal{T}) \in \mathfrak{P}_n$*.*

Proof. Using $P \times P \subseteq U^{n+1}$, we have $\mathrm{cl}(P, \mathcal{T}) \times \mathrm{cl}(P, \mathcal{T}) = \mathrm{cl}(P \times P, \mathcal{T} \times \mathcal{T}) \subseteq \mathrm{cl}(U^{n+1}, \mathcal{T} \times \mathcal{T}) \subseteq U^n$. ☕

Recall that a separable metric space X has $|X| \leq \mathfrak{c}$; so $|\mathfrak{P}_{n+1}| \leq 2^{\mathfrak{c}}$ and it equals $2^{\mathfrak{c}}$ for each of the four butterflies in Example 2.4. But Lemma 2.7 implies that there are $\mathfrak{c}$ sets in $\mathfrak{P}_n$ (namely, the closed ones) that cover all the sets in $\mathfrak{P}_{n+1}$.

Lemma 2.8 *Assume* CH *and assume that* X *is a symmetric butterfly. Fix* $n \in \omega$ *and* $S \in [X]^{\aleph_1}$ *such that*

$$\forall W \in [S]^{\aleph_1}\ \exists Z \in [W]^{\aleph_1}\ [Z \in \mathfrak{P}_{n+1}] \qquad (\bigstar)$$

Then S *is a countable union of sets in* $\mathfrak{P}_n$*.*

Proof. Note that each $\mathfrak{P}_n$ is hereditary in the sense that if $P \in \mathfrak{P}_n$ and $R \subseteq P$, then $R \in \mathfrak{P}_n$. Since $\mathfrak{P}_n$ is hereditary, it is enough to show that S is contained in a countable union of sets in $\mathfrak{P}_n$.

By CH, there are only $\aleph_1$ $\mathcal{T}$–closed sets. Let $\{F_\alpha : \alpha < \omega_1\}$ list all the $\mathcal{T}$–closed subsets of X in $\mathfrak{P}_n$. Assume that S is *not* contained in a countable union of $\mathfrak{P}_n$ sets. Then, for $\alpha < \omega_1$, choose $w_\alpha \in S \setminus (\bigcup_{\xi<\alpha}(F_\xi \cup \{w_\xi\}))$, and let $W = \{w_\alpha : \alpha < \omega_1\}$. Then $W \cap F_\delta$ is countable for all δ.

Assume that $(\bigstar)$ holds, and fix $Z \in [W]^{\aleph_1}$ such that $Z \in \mathfrak{P}_{n+1}$. By Lemma 2.7, $\mathrm{cl}(Z, \mathcal{T}) \in \mathfrak{P}_n$. Fix δ such that $\mathrm{cl}(Z, \mathcal{T}) = F_\delta$. Then $Z \subseteq W \cap F_\delta$, contradicting $|W \cap F_\delta| < \aleph_1$. ☕

Proof of Theorem 2.5. Under CH, $|X| \leq \mathfrak{c} = \aleph_1$. Assume that $|S| = \aleph_1$ (otherwise $nw(S) \leq \aleph_0$ is trivial). Apply Lemma 2.8 to see that for each $n \in \omega$, S is a countable union of $\mathfrak{P}_n$ sets:

First, we verify $(\bigstar)$: Fix $W = \{w_\alpha : \alpha < \omega_1\} \in [S]^{\aleph_1}$. Consider the assignment $\{(w_\alpha, U^{n+1}_{w_\alpha}) : \alpha < \omega_1\}$ for points in S. By suHG, there is a $J \in [\omega_1]^{\aleph_1}$ such that $w_\alpha \in U^{n+1}_{w_\beta}$ for all $\alpha, \beta \in J$. Set $Z = \{w_\alpha : \alpha \in J\}$ to get $Z \in \mathfrak{P}_{n+1}$ and $Z \in [W]^{\aleph_1}$.

Then, for *each* $n \in \omega$, we have $S = \bigcup_{\ell \in \omega} Z^n_\ell$, for some $Z^n_\ell \subseteq S$ and $Z^n_\ell \in \mathfrak{P}_n$.

Let $\mathcal{N} = \{Z^n_\ell : n, \ell \in \omega\}$, which is a countable family of subsets of S. Then $\mathcal{N}$ is a network for S: Fix a neighborhood of x in S, which we may assume is $U^m_x \cap S$ for some $m \in \omega$. Fix ℓ such that $x \in Z^m_\ell$, and let $N = Z^m_\ell$. Then $x \in N \subseteq S$ is obvious. To see that $N = Z^m_\ell \subseteq U^m_x$, use the fact that $x \in N \in \mathfrak{P}_m$: for any $y \in N$ the definition of $\mathfrak{P}_m$ puts $y \in U^m_x$.

Since we have proved that the S of our Theorem 2.5 has countable net weight, one might ask if it also must have the stronger property of being second countable. But this need not be true. It is well-known that there are spaces with a countable net weight that are not second countable; for example, take any space that is countable but not second countable.

There is also a natural example among our butterflies. Let X be the plane and let $\widehat{\mathcal{T}}$ be one of the topologies of Example 2.4. Let $S = \mathbb{R} \times \mathbb{Q}$. Then $nw(S, \widehat{\mathcal{T}}) = \aleph_0$ because S is the union of countably many (horizontal) copies of $\mathbb{R}$ with its euclidean topology. The reason that $(S, \widehat{\mathcal{T}})$ is not second countable is similar to the one that the Sorgenfrey line has weight $\mathfrak{c}$: The natural base for $(S, \widehat{\mathcal{T}})$, as in Definition 2.1, is $\mathcal{B} := \{U^n_x \cap S : x \in S \;\&\; n \in \omega\}$. If $(S, \widehat{\mathcal{T}})$ were second countable, then some countable $\mathcal{B}_0 \subset \mathcal{B}$ would be a base. Expanding $\mathcal{B}_0$, we may assume that $\mathcal{B}_0 = \{U^n_x \cap S : x \in E \times \mathbb{Q} \;\&\; n \in \omega\}$, where $E \in [\mathbb{R}]^{\aleph_0}$. But this is impossible, because each vertical slice through U^n_x, other than the one through x, is either $\emptyset$ or an open interval in a vertical copy of $\mathbb{Q}$; so whenever $z \in (\mathbb{R} \backslash E) \times \mathbb{Q}$, no $U^n_x \cap S \in \mathcal{B}_0$ satisfies $z \in U^n_x \cap S \subseteq U^0_z \cap S$.

We next prove the OCA result mentioned in the Introduction:

Theorem 2.9 *If* OCA *holds and* X *is a symmetric butterfly space, then* $nw(X) \leq \aleph_0$ *or* X *has an uncountable closed discrete subset.*

Remarks. The Sorgenfrey line shows that we cannot drop the assumption that the butterfly is symmetric. Since "butterfly" and "symmetric" are hereditary, this theorem also applies to every subset of X.

Proof. Recall that OCA states: Whenever $(X, \mathcal{T})$ is a separable metric space and we partition the unordered pairs as $[X]^2 = O \,\dot{\cup}\, C$, where O is open and C is closed, then *either* X is the union of countably many C–homogeneous sets *or* there is an uncountable O–homogeneous set.

Now, for each $n \in \omega$, let $C^n = \{\{x, y\} \in [X]^2 : (x, y) \in \mathrm{cl}(U^n, \mathcal{T} \times \mathcal{T})\}$; so $O^n = \{\{x, y\} \in [X]^2 : (x, y) \notin \mathrm{cl}(U^n, \mathcal{T} \times \mathcal{T})\}$. There are then two cases:

If for *each* n, $X = \bigcup_{\ell \in \omega} H^n_\ell$, where each H^n_ℓ is C^n–homogeneous, then $nw(X, \widehat{\mathcal{T}}) \leq \aleph_0$. To prove this, we show that $\{H^n_\ell : n, \ell \in \omega\}$ is a network. To see this let V be any open set and z any point in V; we shall show that $z \in H^{n+1}_\ell \subseteq V$ for some n, ℓ. First, fix n such that $z \in U^n_z \subseteq V$. Then, fix ℓ such that $z \in H^{n+1}_\ell$. If $x \in H^{n+1}_\ell$ then by homogeneity, $(x, z) \in \mathrm{cl}(U^{n+1}, \mathcal{T} \times \mathcal{T}) \subseteq U^n$, so $x \in U^n_z$ and $z \in U^n_x$. So, $H^{n+1}_\ell \subseteq V$.

If for *some* n, D is uncountable and O^n–homogeneous, then D is closed and discrete. To see this, note that for each $\{x, y\} \in [D]^2$, $x \notin U^n_y$ and $y \notin U^n_x$, so that D is obviously discrete. To prove that it is closed, suppose that $z \in \mathrm{cl}(D, \widehat{\mathcal{T}})\backslash D$. Since $(X, \widehat{\mathcal{T}})$ is first countable, let $\langle x_\ell : \ell \in \omega\rangle$ converge to z in $\widehat{\mathcal{T}}$ with all $x_\ell \in D$. Fix ℓ such that $x_\ell \in U^n_z$. Then $z \in U^n_{x_\ell}$. By convergence, fix $r > \ell$ such that $x_r \in U^n_{x_\ell}$. Then, again by symmetry, $(x_\ell, x_r) \in U^n$, contradicting O^n–homogeneity. ☕

Next, we consider how Luzin sets behave with respect to our various butterflies. Recall that, for any $n \in \omega\backslash\{0\}$, a set $X \subseteq \mathbb{R}^n$ is a *Luzin set* iff X is uncountable and no uncountable subset of X is nowhere dense with respect to the euclidean topology $\mathcal{T}$. Actually, $\mathcal{T}$–Luzin = $\widehat{\mathcal{T}}$–Luzin for our nice butterflies (see Section 5), but we do not need that fact here.

The next lemma is used in Lemmas 2.11, 2.12, and 2.13. Part one uses the euclidean topology $\mathcal{T}$ on $\mathbb{R}^2$, and part two applies to a nice butterfly refinement $\widehat{\mathcal{T}}$.

Lemma 2.10 *Suppose that $\kappa \geq \omega_1$ and $J \in [\kappa]^{\aleph_1}$ and $0 < m < \omega$. Suppose that $X \subseteq (\mathbb{R}^2)^m$ is a Luzin set and $\{x_\xi : \xi \in J\} \subseteq X$ with $x_\xi \neq x_\eta$ for $\xi \neq \eta$. For part (2) below, suppose that $\widehat{\mathcal{T}}$ is a nice butterfly refinement of the euclidean topology $\mathcal{T}$ on $\mathbb{R}^2$ and $\mathcal{U} = \langle (x_\xi, \vec{U}_\xi) : \xi < \kappa\rangle$ is a κ-assignment for X, where each x_ξ is an m-tuple $(x^1_\xi, \ldots, x^m_\xi) \in X$ and each $\vec{U}_\xi$ is a product of m $\widehat{\mathcal{T}}$ basic open neighborhoods. Then there is a set $I \in [J]^{\aleph_1}$ and a nonempty open $W \subseteq (\mathbb{R}^2)^m$ such that:*

(1) *$\{x_\xi : \xi \in I\}$ is a dense subset of W; and*

(2) *for any $\xi \in I$, there are $\eta_1, \eta_0 \in I\backslash\{\xi\}$ such that $x_{\eta_1} \in \vec{U}_\xi$ and $x_{\eta_0} \in X\backslash\vec{U}_\xi$.*

Proof. Let $E_0 = \{x_\xi : \xi \in J\}$ and $W_0 = \mathrm{int}(\mathrm{cl}(E_0)) = \bigcup\{V \subseteq (\mathbb{R}^2)^m : V \in \mathcal{T} \ \&\ V \subseteq \mathrm{cl}(E_0, \mathcal{T})\}$. Then W_0 is a nonempty open subset of $\mathrm{cl}(E_0)$, because X is Luzin. So we can choose $\xi_0 \in J$ with $x_{\xi_0} \in W_0 \cap E_0$. For any $\gamma < \omega_1$, let $E_\gamma = E_0\backslash\{x_{\xi_\alpha} : \alpha < \gamma\}$ and $W_\gamma = \mathrm{int}(\mathrm{cl}(E_\gamma))$. Then apply Luzin again to see that W_γ is a nonempty open subset of $\mathrm{cl}(E_\gamma)$ to get $\xi_\gamma \in J$ with $x_{\xi_\gamma} \in W_\gamma \cap E_\gamma$.

Let $I_0 = \{\xi_\gamma : \gamma < \omega_1\}$ and $W = \mathrm{int}(\mathrm{cl}(\{x_\xi : \xi \in I_0\})$. Then $I := \{\xi \in I_0 : x_\xi \in W\}$ still has size $\aleph_1$ because X is Luzin.

To prove the second part, fix $\xi \in I$. Then x_ξ is in W. The denseness of $\{x_\eta : \eta \in I\}$ in W gives us $\eta_1 \in I\backslash\{\xi\}$ such that $x_{\eta_1} \in \vec{U}_\xi$, because $\vec{U}_\xi$ is a product of $\widehat{\mathcal{T}}$ *basic* open neighborhoods, making $\vec{U}_\xi\backslash\{x_\xi\}$ open in the euclidean topology of $(\mathbb{R}^2)^m$. Now, $\widehat{\mathcal{T}}$ is a nice refinement of $\mathcal{T}$ only on $\mathbb{R}^2$, but in $\mathbb{R}^2$ the set of projections $S = \{x^1_\eta : \eta \in I\}$ is dense in the open $\pi_1(W)$, where $\pi_1 : (\mathbb{R}^2)^m \to \mathbb{R}^2$ denotes the projection map. Also, the projection $\pi_1(\vec{U}_\xi)$ is a $\widehat{\mathcal{T}}$ basic open neighborhood of x^1_ξ, and hence (because $\widehat{\mathcal{T}}$ is nice) the set S of projections meets $\mathbb{R}^2\backslash\pi_1(\vec{U}_\xi)$. So there is $\eta_0 \in I\backslash\{\xi\}$ such that $x^1_{\eta_0} \in \mathbb{R}^2\backslash\pi_1(\vec{U}_\xi)$, and then $x_{\eta_0} \in X\backslash\vec{U}_\xi$. ☕

Lemma 2.11 *If $X \subseteq \mathbb{R}^2$ is a Luzin set, and $\widehat{\mathcal{T}}$ is any nice symmetric butterfly refinement of the euclidean topology $\mathcal{T}$, then $(X, \widehat{\mathcal{T}})$ is HG and not suHG (or suHC).*

Note that by symmetry, suHG and suHC are the same here.

Proof. To prove HG, start with a uniform assignment $\mathcal{U} = \langle (x_\xi, U_\xi^n) : \xi < \omega_1 \rangle$, where the $x_\xi \in X$ are all different. Then apply Lemma 2.10 with $\kappa = J = \omega_1$ and $m = 1$ to get $I \in [\omega_1]^{\aleph_1}$ and a nonempty $\mathcal{T}$-open set $W \subseteq \mathbb{R}^2$ such that $\{x_\xi : \xi \in I\}$ is a $\mathcal{T}$-dense subset of W. Then fix any $\xi \in I$ and apply the second part of the lemma to get $\eta \in I \backslash \{\xi\}$ such that $x_\eta \in U_\xi^n$. The symmetry of $\widehat{\mathcal{T}}$ implies $x_\xi \in U_\eta^n$.

To refute suHG, suppose that $J \subset \omega_1$ with $x_\eta \in U_\xi^n$ and $x_\xi \in U_\eta^n$ for all $\xi, \eta \in J$. If $|J| = \aleph_1$, apply Lemma 2.10 again to get $I \in [J]^{\aleph_1}$ and $W \subseteq \mathbb{R}^2$ with $\{x_\xi : \xi \in I\}$ a $\mathcal{T}$-dense subset of W, and fix any $\xi \in I$. Apply the second part of the lemma to get $\eta \in I \backslash \{\xi\}$ such that $x_\eta \notin U_\xi^n$, which then yields a contradiction.

This actually proves that suHG fails for *every* assignment with the x_ξ all different.

Next, consider a *super Luzin set* $X \subset \mathbb{R}^2$ in the sense of [11] V.6.40: X is uncountable and for all $m \in \omega \backslash \{0\}$, no uncountable *spaced* subset of X^m is nowhere dense in $(\mathbb{R}^2)^m$. Here, $Z \subseteq X^m$ is *spaced* iff for all $\vec{x}, \vec{y} \in Z$ and $i, j < m$ $x_i \neq y_j$ unless $\vec{x} = \vec{y}$ and $i = j$. Equivalently, an uncountable X is super Luzin iff all uncountable spaced subsets of each X^m are Luzin sets in $(\mathbb{R}^2)^m$. Super Luzin sets are Luzin sets, since every subset of X^1 is spaced. Super Luzin sets are called strongly Luzin sets in [15].

Note the importance of "spaced" here: Even if $E \subseteq \mathbb{R}^2$ is Luzin, the sets $\{(x, x) : x \in E\}$ and $\{(x, c) : x \in E\}$ are nowhere dense in $(\mathbb{R}^2)^2$, so E^2 is not Luzin in $(\mathbb{R}^2)^2$ (and likewise, E^m is not Luzin in $(\mathbb{R}^2)^m$ for any $m \geq 2$).

Super Luzin sets of size $2^{\aleph_0}$ exist under CH (see [11] Exercise V.6.41) and in Cohen forcing extensions (see Lemma 2.14 below).

Lemma 2.12 *If $X \subseteq \mathbb{R}^2$ is a super Luzin set, and $\widehat{\mathcal{T}}$ is any nice symmetric butterfly refinement of the euclidean topology $\mathcal{T}$, then $(X, \widehat{\mathcal{T}})$ is stHG.*

Proof. We prove that X^m is HG by induction on $m \in \omega \backslash \{0\}$. For $m = 1$, we just use Lemma 2.11, so assume that $m \geq 2$ and X^r is HG for all $r < m$.

Start with an assignment $\mathcal{U} = \langle (x_\xi, \vec{U}_\xi) : \xi < \omega_1 \rangle$, where each x_ξ is an m-tuple $(x_\xi^1, \ldots, x_\xi^m) \in X^m$ and each $\vec{U}_\xi$ is a product $U_\xi^1 \times \cdots \times U_\xi^m \subseteq X^m$, where each U_ξ^i is a $\widehat{\mathcal{T}}$ basic open neighborhood of x_ξ^i.

We now thin the sequence several times:

First, passing to a subsequence, we may assume that either for each ξ, the $x_\xi^1, \ldots, x_\xi^m$ are all different, or for some $r \neq s$, $x_\xi^r = x_\xi^s$ for all $\xi \in \omega_1$. If the second alternative holds, then we can prove this instance of HG from the inductive assumption that X^{m-1} is HG. So, we may assume that each $\{x_\xi^1, \ldots, x_\xi^m\}$ is an m–element set.

Again passing to a subsequence, assume that the family of *sets* $\mathcal{A} = \{\{x_\xi^1, \ldots, x_\xi^m\} : \xi \in \omega_1\}$ forms a delta system with a root of size r, where $0 \leq r < m$.

If $r = 0$, then the sets are pairwise disjoint, so that the set of m–tuples $(x_\xi^1, \ldots, x_\xi^m)$ is spaced and forms a Luzin set; so we can apply Lemma 2.10 with $\kappa = J = \omega_1$ to get $\xi, \eta \in \omega_1$ with $\xi \neq \eta$ and $x_\xi \in \vec{U}_\eta$ and $x_\eta \in \vec{U}_\xi$, verifying that in this case X^m is HG.

If $0 < r < m$, then passing once again to a subsequence, assume for all $i \in \{1, \ldots r\}$ the x_ξ^i for $\xi < \omega_1$ are all the same. This instance of HG for X^m follows from the inductive hypothesis that X^{m-r} is HG.

The following computes the net weight of our X:

Lemma 2.13 *If $X \subseteq \mathbb{R}^2$ is a Luzin set of size $\aleph_2$ and $\widehat{\mathcal{T}}$ is any nice symmetric butterfly refinement of the euclidean topology, then $nw(X, \widehat{\mathcal{T}}) = \aleph_2$.*

Proof. Clearly, $nw(X, \widehat{\mathcal{T}}) \leq |X| = \aleph_2$; so it suffices to derive a contradiction from the assumption that $nw(X, \widehat{\mathcal{T}}) \leq \aleph_1$. To do this, we shall show that any space X with $nw(X) \leq \aleph_1$ has the ω_2-suHG, while our X does not have the ω_2-suHG.

Here, by *definition*, the ω_2-suHG for X means that given an *ω_2–assignment* $\mathcal{U} = \langle (x_\mu, U_\mu) : \mu < \omega_2 \rangle$ for X : $\exists S \in [\omega_2]^{\aleph_2}\ \forall \mu, \nu \in S\ [x_\nu \in U_\mu\ \&\ x_\mu \in U_\nu]$. As in Definition 1.1, "assignment" means that each U_μ is open in X and each $x_\mu \in U_\mu$. Thus, for any X having a network $\{N_\xi : \xi < \omega_1\}$, for each $\mu \in \omega_2$ we can choose $\xi_\mu \in \omega_1$ such that $x_\mu \in N_{\xi_\mu} \subseteq U_\mu$. Then, fix $S \in [\omega_2]^{\aleph_2}$ and $\xi < \omega_1$ such that $\xi_\mu = \xi$ for all $\mu \in S$. Then for $\mu, \nu \in S$, $x_\mu \in N_\xi \subseteq U_\mu$ and $x_\nu \in N_\xi \subseteq U_\nu$, so that $x_\mu, x_\nu \in N_\xi \subseteq U_\mu \cap U_\nu$.

To show that $(X, \widehat{\mathcal{T}})$ does not have the ω_2-suHG, start with a uniform assignment $\mathcal{U} = \langle (x_\xi, U_\xi^n) : \xi < \omega_2 \rangle$, where the $x_\xi \in X$ *are all different.* Then, assuming the ω_2-suHG, fix $S \in [\omega_2]^{\aleph_1}$ such that $\forall \mu, \nu \in S\ [x_\nu \in U_\mu\ \&\ x_\mu \in U_\nu]$ (here, we only need $|S|$ to be $\aleph_1$, not $\aleph_2$). Apply Lemma 2.10 with $\kappa = \omega_2$, $J = S$ and $m = 1$ to get $I \in [S]^{\aleph_1}$ and a nonempty $\mathcal{T}$-open set $W \subseteq \mathbb{R}^2$ such that $\{x_\xi : \xi \in I\}$ is a $\mathcal{T}$-dense subset of W. Fix any $\xi \in I$, and apply the second part of that lemma to get $\eta \in I \backslash \{\xi\}$ such that $x_\eta \in X \backslash U_\xi^n$, which yields a contradiction.

In view of these last three lemmas, the following lemma completes Step 2 of the outline in Section 1. We remark that the fact that Cohen real forcing adds a Luzin set is well-known; we generalize this to show that this set is super Luzin.

Lemma 2.14 *In V, let κ be any uncountable cardinal, and let $\mathbb{Q}$ be the standard forcing for adding κ Cohen reals. Then, in the generic extension $V[H]$, there is a super Luzin subset of $\mathbb{R}^2$ of size κ.*

Proof. First, we shall get our super Luzin set in ω^ω, rather than in $\mathbb{R}^2$. This will avoid the issue of coding pairs of real numbers by Cohen reals. View the κ Cohen real forcing as $\mathbb{Q} = \mathrm{Fn}(\kappa \times \omega, \omega)$. When H is $\mathbb{Q}$–generic, we have $\bigcup H : \kappa \times \omega \to \omega$, and it codes our super Luzin set simply as $X^H := \{x_\alpha^H : \alpha < \kappa\} \subset \omega^\omega$, where $x_\alpha^H(n) := (\bigcup H)(\alpha, n)$. Note that $\mathbb{1}$ forces that all the x_α^H are different, so $|X^H| = \kappa$ in $V[H]$, and $\mathbb{1}$ forces that X^H is dense in ω^ω.

We can transfer X^H to $\mathbb{R}^2$ by using a homeomorphism of ω^ω onto a co-meager subset of $\mathbb{R}^2$ (such as all pairs of irrationals) to get the super Luzin set in $\mathbb{R}^2$.

Of course, now we must prove that in $V[H]$, X^H is super Luzin in ω^ω. Our proof is along the lines of the well-known proof that the Cohen reals form a Luzin set (see [10]), which in turn is a consequence of the fact that each Cohen real is not in any closed nowhere dense Borel set whose Borel code is in the ground model (see [6] Lemma 26.4).

To introduce our notation, we first give the proof that X^H is Luzin in ω^ω. Recall that $X \subseteq \omega^\omega$ is Luzin iff X is uncountable and $|X \cap B| \leq \aleph_0$ for all closed nowhere dense $B \subset \omega^\omega$.

Each closed $B \subset \omega^\omega$ has a Borel code $\mathfrak{C}$. The details of this coding vary with the presentation, but the key idea is that a Borel code is a hereditarily countable set $\mathfrak{C}$ that describes how the Borel set $B = \mathcal{B}_\mathfrak{C}$ is to be constructed from basic open sets. One specific presentation is described in Jech's text [6]. In any case, we have that $B = \mathcal{B}_\mathfrak{C}$ is determined by a countable object. We need to show, for each closed nowhere dense $B = \mathcal{B}_\mathfrak{C}$, that in $V[H]$, $|X \cap B| \leq \aleph_0$; that is, $x^H_\alpha \notin B$ for all but at most countably many α.

Since $\mathfrak{C}$ is countable, there is a countable $J \subset \kappa$ in V such that $\mathfrak{C}$ is in the submodel $V[H \cap \mathrm{Fn}(J \times \omega, \omega)]$. We may view $V[H]$ as the iterated Cohen extension $V[H] = V[H \cap \mathrm{Fn}(J \times \omega, \omega)][H \cap \mathrm{Fn}((\kappa \backslash J) \times \omega, \omega)]$. Then for each $\alpha \in \kappa \backslash J$, x^H_α is the Cohen generic real added by the extension by $\mathrm{Fn}((\kappa \backslash J) \times \omega, \omega)$ over $V[H \cap \mathrm{Fn}(J \times \omega, \omega)]$, and J was chosen so that the Borel code for B is in the "ground model" $V[H \cap \mathrm{Fn}(J \times \omega, \omega)]$; thus $x^H_\alpha \notin B$.

So far, we have that in $V[H]$, $\{x_\alpha : \alpha < \kappa\}$ is Luzin, and hence so are all its uncountable subsets. In particular, in $V[H]$ for each subsequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ of κ, with all the α_ξ different, the set $\{x_{\alpha_\xi} : \xi < \omega_1\}$ is Luzin in $V[H]$; the subsequences of κ are in $V[H]$, but *need not* be in V.

We now prove that X^H is super Luzin in ω^ω. In $V[H]$ we have $m \geq 1$ and $\{\vec{x}_{\sigma_\xi} : \xi < \omega_1\} \subset (\omega^\omega)^m$, where $\vec{\sigma}_\xi$ is an m–tuple $(\alpha^\xi_0, \ldots, \alpha^\xi_{m-1})$ and $\vec{x}_{\sigma_\xi}$ abbreviates $(x_{\alpha^\xi_0}, \ldots, x_{\alpha^\xi_{m-1}}) \in (\omega^\omega)^m$. We do *not* assume that $\langle \vec{\sigma_\xi} : \xi < \omega_1 \rangle \in V$. But we do assume, for all $\xi, \eta < \omega_1$ and $i, j < m$:

$$1.\ i \neq j \rightarrow [\alpha^\xi_i \neq \alpha^\xi_j] \qquad 2.\ \xi \neq \eta \rightarrow [\alpha^\xi_i \neq \alpha^\eta_j] \quad .$$

This is equivalent to $(\alpha^\xi_i = \alpha^\eta_j) \rightarrow (\xi = \eta \;\&\; i = j)$; equivalently, the sequence is spaced. Splitting "spaced" into (2) + (1) emphasizes the two separate roles of (2) and (1) in the following paragraphs.

To finish, we prove that in $V[H]$ the spaced subsets of X^m are Luzin. In $V[H]$, let $B \subset (\omega^\omega)^m$ be a closed nowhere dense Borel set with Borel code $\mathfrak{C}$. Then for some $J \in [\kappa]^{\aleph_0} \cap V$ we have $\mathfrak{C} \in V[H \cap \mathrm{Fn}(J \times \omega, \omega)]$. Again, we have $V[H] = V[H \cap \mathrm{Fn}(J \times \omega, \omega)][H \cap \mathrm{Fn}((\kappa \backslash J) \times \omega, \omega)]$.

In view of (2), in $V[H]$ the set $K = \{\xi < \omega_1 : \exists i < m\ \alpha^\xi_i \in J\}$ is countable. For each $\xi \in \omega_1 \backslash K$, for $i = 0, 1, \ldots, m-1$ all the $\alpha^\xi_i \in \kappa \backslash J$, and, applying (1), they are all different, so that the m–*tuple* $(x_{\alpha_0^\xi}, \ldots, x_{\alpha_{m-1}^\xi})$ is generic over $V[H \cap \mathrm{Fn}(J \times \omega, \omega)]$, and hence $(x_{\alpha_0^\xi} \ldots x_{\alpha_{m-1}^\xi}) \notin B$.

Therefore, for each $m \in \omega$, each uncountable spaced subset of X^m is a Luzin subset of $(\omega^\omega)^m$ in $V[H]$, so that the set $X = \{x_\alpha : \alpha < \kappa\}$ is super Luzin in $V[H]$. ☕

3 More forcing results

We recall here the ccc poset used to prove Theorem [4]4.1 (restated in Section 1), and we show that used as a forcing poset, it preserves the stHG of X.

The following lemma takes us a step closer to completing our proof of Theorem 1.4. Let $\mathsf{MA}(\kappa, X)$ be $\mathsf{MA}(\kappa)$ restricted to those ccc $\mathbb{P}$ such that $\mathbb{1} \Vdash_{\mathbb{P}}$ "X is stHG".

Lemma 3.1 *If X is a stHG space, then when $\kappa \geq \aleph_1$, $\mathsf{MA}(\kappa, X)$ implies the full $\mathsf{MA}(\kappa)$ and also implies that X is suHG.*

Of course, this implies Theorem [4]4.1. The proof of Lemma 3.1 later in this section will use the suHG of X and Lemma 3.5 to reduce $\mathsf{MA}(\kappa)$ to $\mathsf{MA}(\kappa, X)$. Since we shall be tracing a space X through various forcing extensions, we shall be a little pedantic in defining what a space is:

Definition 3.2 *A* space-base *is a pair $(X, \mathcal{B})$, where $X \neq \emptyset$ and $\mathcal{B} \subseteq \mathcal{P}(X) \backslash \{\emptyset\}$ and $\mathcal{B}$ is a base for a T_3 topology on X.*

Note that if $(X, \mathcal{B})$ is a space-base in V, then it remains a space-base in any extension $V[G]$. This would not be true for a *topology*, which must be closed under arbitrary unions. Nevertheless, we shall still use the standard informal "consider the space X in V and in $V[G]$" for "consider the space-base $(X, \mathcal{B}) \cdots\cdots$ ".

Now, we explain how a stHG space X can become suHG in a suitable $V[G]$.

Under $\mathsf{MA}(\aleph_1)$, stHG implies suHG (Theorem [4]4.1). As the proof of Lemma 3.1 will point out, we can also, in ZFC, start with a space X that is stHG and describe a ccc poset $\mathbb{P}$ that forces, for a particular assignment $\mathcal{U}$, *one instance* of suHG in $V[G]$. Lemma 3.4 shows that X remains stHG in $V[G]$, and then we iterate this forcing so that X eventually becomes suHG.

The following describes the forcing:

Definition 3.3 *Given a space X and a κ–assignment $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \kappa \rangle$ for X, the poset $\mathbb{P}(\mathcal{U})$ is the set of all finite $p \subset \kappa$ such that $\forall \alpha, \beta \in p\ [x_\alpha \in U_\beta]$, ordered by $p \leq q$ iff $p \supseteq q$, with $\mathbb{1} = \emptyset$.*

When $\kappa = \omega_1$, this is the poset used to prove Theorem [4]4.1. The proof in [4] notes that for stHG X with assignment $\mathcal{U}$, the poset $\mathbb{P} = \mathbb{P}(\mathcal{U})$ contains all singletons, so that applying $\mathsf{MA}(\aleph_1)$ produces an $S \in [\omega_1]^{\aleph_1}$ such that $\{\{\alpha\} : \alpha \in S\}$ is centered (see [11], Lemma III.3.35). Then S is a set that satisfies $\forall \alpha, \beta \in S\ [x_\alpha \in U_\beta]$.

Given a stHG space X and any assignment $\mathcal{U}$ for X, we can also force with (a suborder $p\!\downarrow$ of) $\mathbb{P}(\mathcal{U})$ to get such an S; see the proof of Lemma 3.1.

Working in ZFC, the next lemma shows that whenever $\mathcal{U}$ is an assignment for a stHG X, the restricted $\mathsf{MA}(\aleph_1, X)$ applies to $\mathbb{P}(\mathcal{U})$:

Lemma 3.4 *If X is stHG and $\mathcal{U}$ is an assignment for X, then $\mathbb{P} = \mathbb{P}(\mathcal{U})$ is ccc and $\mathbb{1} \Vdash_{\mathbb{P}} X$ is stHG.*

Before giving the proof, we remark that a *different* stHG space Y may fail to be stHG in the $\mathbb{P}$-generic extension $V[G]$. As an example, we adapt Example 3.6 of [12], letting $Z \subseteq \mathbb{R}^2$ be super Luzin: Let $X = Y = Z$, but give X a skinny 'horizontal bow-tie' topology as in our Example 2.4 (Type II), and give Y the analogously defined skinny 'vertical bow-tie' topology. Then, as in [12], $\{(x,x) : x \in Z\}$ is an uncountable discrete subset of $X \times Y$, and hence we may let $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ be an assignment for X and $\mathcal{V} = \langle (x_\alpha, V_\alpha) : \alpha < \omega_1 \rangle$ be an assignment for Y such that $\alpha \neq \beta \to (x_\alpha, x_\alpha) \notin U_\beta \times V_\beta$. If G is $\mathbb{P}(\mathcal{U})$-generic and $V[G]$ contains an $S \in [\omega_1]^{\aleph_1}$ such that $\forall \alpha, \beta \in S\ x_\alpha \in U_\beta$, then we have $\forall \{\alpha, \beta\} \in [S]^2\ x_\alpha \notin V_\beta$, proving Y is not even HC in $V[G]$. In our example, both X and Y are stHG but not suHG, and the proof of Lemma 3.1 details how to get such an S.

On the other hand, if Y is a space such that $X^m \times Y$ is HG (in V) for all $m \in \omega$, then for $\mathbb{P}(\mathcal{U})$-generic G the same remains true in $V[G]$. To prove Lemma 3.4, we shall prove this stronger fact. Applied with $Y = X$ (or $|Y| = 1$), this shows that X remains stHG in $V[G]$. As just remarked, simply assuming that Y is stHG in V does not imply that Y will be HG in $V[G]$.

Proof of Lemma 3.4. Suppose that X is stHG and $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ is an assignment for X.

The ccc of $\mathbb{P} = \mathbb{P}(\mathcal{U})$ was verified in [4], in the proof of Theorem [4]4.1. The stHG of X naturally generates a common ingredient that we use in the proofs of the ccc of $\mathbb{P}$ and again here of the fact that $\mathbb{P}$ preserves the stHG of X: we use open sets from $\mathcal{U}$ to form neighborhoods $V_\mu = U_{\alpha^0_\mu} \cap \cdots \cap U_{\alpha^m_\mu}$ in X with $V_\mu = U_{\alpha^0_\mu} \cap \cdots \cap U_{\alpha^m_\mu} \ni x_{\alpha^i_\mu}$ for $i \leq m$. In the ccc proof, the μ correspond to p_μ in a potential antichain; here, the μ correspond to p_μ that would force failure of stHG.

To prove that forcing with this $\mathbb{P}$ preserves the stHG property for X, suppose that G is $\mathbb{P}(\mathcal{U})$-generic. Suppose that Y is a space such that $X^m \times Y$ is HG (in V) for all $m \in \omega$. We shall prove that each $X^m \times Y$ is HG in $V[G]$; the special case $Y = X$ gives the desired stHG result. It is enough to prove this for $m = 0$: that is, we shall prove that Y is HG in $V[G]$. Then, we can apply the result with the space $X^k \times Y$ in place of Y to see that each $X^m \times Y = X^{m-k} \times (X^k \times Y)$ is also HG in $V[G]$.

So, assume that this fails; that is, we have a $q \in \mathbb{P}$ such that $q \Vdash Y$ is not HG. Then we have names $\mathring{y}_\xi$ and $\mathring{O}_\xi$ for $\xi < \omega_1$ such that q forces: $\mathring{y}_\xi \in Y$, $\mathring{O}_\xi$ is a basic open subset of Y, $\mathring{y}_\xi \in \mathring{O}_\xi$, and for all $\{\xi, \eta\} \in [\omega_1]^2$, $\mathring{y}_\xi \notin \mathring{O}_\eta$ or $\mathring{y}_\eta \notin \mathring{O}_\xi$.

Choose $p_\xi \leq q$ for $\xi < \omega_1$ such that $p_\xi \Vdash \mathring{y}_\xi = \check{y}_\xi$ and $p_\xi \Vdash \mathring{O}_\xi = \check{O}_\xi$, where $y_\xi \in Y$ and O_ξ is a basic open subset of Y.

Thinning the sequence, we may assume that each $p_\xi = \{\alpha^0_\xi, \dots \alpha^m_\xi\}$, listed in increasing order, for some fixed m.

Let $V_\xi = U_{\alpha^0_\xi} \cap \cdots \cap U_{\alpha^m_\xi} \subseteq X$, where the $U_{\alpha^i_\xi}$ are from the assignment $\mathcal{U}$. Then $(x_{\alpha^0_\xi}, \dots, x_{\alpha^m_\xi}, y_\xi) \in (V_\xi)^{m+1} \times O_\xi$. For any $\{\xi, \eta\} \in [\omega_1]^2$, if $p_\xi \not\perp p_\eta$, then $y_\xi \notin O_\eta$ or $y_\eta \notin O_\xi$ by our choice of p_ξ, and if $p_\xi \perp p_\eta$, then there are $\alpha \in p_\xi$ and $\beta \in p_\eta$ such that $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$ by definition of $\mathbb{P}$. So, we have $(x_{\alpha^0_\xi}, \dots, x_{\alpha^m_\xi}, y_\xi) \notin (V_\eta)^{m+1} \times O_\eta$ or $(x_{\alpha^0_\eta}, \dots, x_{\alpha^m_\eta}, y_\eta) \notin (V_\xi)^{m+1} \times O_\xi$. So these neighborhoods $(V_\xi)^{m+1} \times O_\xi$ provide a weak separation, contradicting the assumption that $X^{m+1} \times Y$ is HG.

Another useful lemma reduces $\mathsf{MA}(\kappa)$ to $\mathsf{MA}(\kappa, X)$ for suHG spaces X:

Lemma 3.5 *If X is suHG, then* every *ccc $\mathbb{P}$ satisfies $\mathbb{1} \Vdash_{\mathbb{P}}$ "X is stHG".*

Proof. Let $\mathbb{P}$ be any ccc poset.

Suppose that X is suHG. Then X^n is also suHG for each $n \in \omega$, and so it is enough to show that $\mathbb{1} \Vdash_{\mathbb{P}}$ "X is HG". So, suppose that $p \Vdash_{\mathbb{P}}$ "X is not HG". Then, we have names $\mathring{x}_\alpha$ and $\mathring{U}_\alpha$ such that p forces that $\mathring{x}_\alpha \in X$ and $\mathring{U}_\alpha \in \mathcal{B}$ (an open base for X) and $\mathring{x}_\alpha \in U_\alpha$ and $\mathring{x}_\alpha \notin \mathring{U}_\beta$ or $\mathring{x}_\beta \notin \mathring{U}_\alpha$ whenever $\alpha \neq \beta$. Then, for each α, choose $p_\alpha \leq p$ and $x_\alpha \in X$ and $U_\alpha \in \mathcal{B}$ such that $p_\alpha \Vdash \mathring{x}_\alpha = \check{x}_\alpha$ and $p_\alpha \Vdash \mathring{U}_\alpha = \check{U}_\alpha$. Now, $\mathcal{U} := \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ is an assignment, so applying suHG, fix $S \in [\omega_1]^{\aleph_1}$ such that $\forall \alpha, \beta \in S\ [x_\beta \in U_\alpha \ \&\ x_\alpha \in U_\beta]$. For any $\{\alpha, \beta\} \in [S]^2$, if $q \leq p_\alpha$ and $q \leq p_\beta$, then q forces $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$, which is impossible, so $p_\alpha \perp p_\beta$. But this contradicts the ccc of $\mathbb{P}$.

Using Lemma 3.4, we can prove Lemma 3.1. The proof of suHG uses the fact that every ccc poset $\mathbb{P}$ has $\aleph_1$ as a pre-caliber *if* we assume $\mathsf{MA}_{p\downarrow}(\aleph_1)$ for each $p \in \mathbb{P}$. (This restricted MA result is a corollary to the proof of [11] Lemma III.3.35, whose statement simplifies the assumptions to the full $\mathsf{MA}(\aleph_1)$.) Lemma 3.1, likewise, does not assume the full $\mathsf{MA}(\aleph_1)$; it only uses $\mathsf{MA}_{\mathbb{P}}(\aleph_1)$ for those ccc $\mathbb{P}$ such that $\mathbb{1} \Vdash_{\mathbb{P}}$ "X is stHG".

Proof of Lemma 3.1. Suppose that X is stHG. To prove that $\mathsf{MA}(\kappa, X)$, for $\kappa \geq \aleph_1$, implies that X is suHG, fix any assignment $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ for X. Since X is stHG, Lemma 3.4 implies that $\mathbb{P} = \mathbb{P}(\mathcal{U})$ is ccc and $\mathbb{1} \Vdash_{\mathbb{P}(\mathcal{U})}$ "X is stHG".

Since $\mathbb{1} \Vdash_{\mathbb{P}}$ "X is stHG" implies that $p \Vdash_{\mathbb{P}}$ "X is stHG" for each $p \in \mathbb{P}$, assuming $\mathsf{MA}(\kappa, X)$ does indeed give us $\mathsf{MA}_{p\downarrow}(\aleph_1)$ for each $p \in \mathbb{P}$; so, as in [11] Lemma III.3.35, we get an $S \in [\omega_1]^{\aleph_1}$ such that $\{\{\alpha\} : \alpha \in S\} \subset \mathbb{P}(\mathcal{U})$ is centered. (Concretely, the ccc of $\mathbb{P}(\mathcal{U})$ yields an $s \in \mathbb{P}(\mathcal{U})$ so that for each $\alpha < \omega_1$ $D_\alpha := \{p : \exists \beta \geq \alpha [p \leq \{\beta\}]\}$

is dense below s.) Then for all $\alpha, \beta \in S$, $\{\alpha, \beta\}$ is a forcing condition in $\mathbb{P}(\mathcal{U})$, which means that $x_\alpha \in U_\beta$ and $x_\beta \in U_\alpha$, which proves that X is suHG.

Now, by Lemma 3.5, the full $\mathsf{MA}(\kappa)$ follows from the restricted $\mathsf{MA}(\aleph_1, X)$.

Before considering iterations of this $\mathbb{P}(\mathcal{U})$ forcing, we note that it gives us a proof of Theorem 1.2:

Proof of Theorem 1.2. Let $\mathcal{B}$ be an open base with $|\mathcal{B}| \le \kappa$. Then, let $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \kappa \rangle$ be a κ–assignment that lists *all* pairs (x, U) with $U \in \mathcal{B}$ and $x \in U$. Since $\mathbb{P}(\mathcal{U})$ is ccc (by Lemma 3.4) and of size κ, $\mathsf{MA}(\kappa)$ implies that it is σ–centered (see [11], Lemma III.3.46). So, $\kappa = \bigcup_{n \in \omega} S_n$, where for each n, $\{\{\alpha\} : \alpha \in S_n\}$ is centered. Let $N_n = \{x_\alpha : \alpha \in S_n\}$. Then, as the next paragraph shows, $\{N_n : n \in \omega\}$ is a network for X.

Fix any $V \in \mathcal{B}$ and $y \in V$. We show that $y \in N_n \subseteq V$ for some $n \in \omega$. Since all pairs were listed, $(y, V) = (x_\alpha, U_\alpha)$ for some $\alpha < \kappa$. Then $\alpha \in S_n$ for some n, and then $y = x_\alpha \in N_n$. To see that $N_n \subseteq V$: If $z \in N_n$, then $z = x_\beta$ for some $\beta \in S_n$. Since $\alpha, \beta \in S_n$, the forcing conditions $\{\alpha\}$ and $\{\beta\}$ are compatible, so $\{\alpha, \beta\} \in \mathbb{P}(\mathcal{U})$, and hence $y = x_\alpha \in U_\beta$ and $z = x_\beta \in U_\alpha = V$.

For the proof of Theorem 1.4, step (3) iterates this forcing for a fixed stHG X so that X remains stHG throughout the iteration. Lemma 3.9 helps preserve stHG in the limit stages.

For iterated forcing, the next two definitions and Lemma 3.7 are taken from [11], Section V.3.

Definition 3.6 *For any ordinal α, an α*–stage iterated forcing construction *is a pair of sequences of the form:* $\left(\left\langle (\mathbb{P}_\xi, \le_\xi, \mathbb{1}_\xi) : \xi \le \alpha \right\rangle , \left\langle (\mathring{\mathbb{Q}}_\xi, \mathring{\le}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}) : \xi < \alpha \right\rangle \right)$ *satisfying the following:*

1. *Each $(\mathbb{P}_\xi, \le_\xi, \mathbb{1}_\xi)$ is a forcing poset.*
2. *Each $(\mathring{\mathbb{Q}}_\xi, \mathring{\le}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi})$ is a $(\mathbb{P}_\xi, \le_\xi, \mathbb{1}_\xi)$–name for a forcing poset.*
3. *Each $p \in \mathbb{P}_\xi$ is a sequence of the form $\langle \mathring{q}_\mu : \mu < \xi \rangle$, where each $\mathring{q}_\mu \in \mathrm{dom}(\mathring{\mathbb{Q}}_\mu)$. We use $(p)_\mu$ to denote this $\mathring{q}_\mu$.*
4. *If $\xi < \eta$ and $p \in \mathbb{P}_\eta$ then $p \restriction \xi \in \mathbb{P}_\xi$.*
5. *If $\xi < \eta$ and $p \in \mathbb{P}_\xi$ and p' is the η–sequence such that $p' \restriction \xi = p$ and $(p')_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$ whenever $\xi \le \mu < \eta$, then $p' \in \mathbb{P}_\eta$; $i^\eta_\xi(p)$ denotes this p'.*
6. *$\mathbb{1}_\xi$ is the sequence $\langle \mathring{q}_\mu : \mu < \xi \rangle$, where each $\mathring{q}_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$.*
7. *If $p = \langle \mathring{q}_\mu : \mu < \xi \rangle \in \mathbb{P}_\xi$ and $p' = \langle \mathring{q}'_\mu : \mu < \xi \rangle \in \mathbb{P}_\xi$, then $p \le_\xi p'$ iff $p \restriction \mu \Vdash_{\mathbb{P}_\mu} [\mathring{q}_\mu \le \mathring{q}'_\mu]$ for all $\mu < \xi$.*
8. *If $\xi + 1 \le \alpha$, then $\mathbb{P}_{\xi+1}$ is the set of all $p^\frown \mathring{q}$ such that $p \in \mathbb{P}_\xi$ and $\mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}}_\xi)$ and $p \Vdash_{\mathbb{P}_\xi} \mathring{q} \in \mathring{\mathbb{Q}}_\xi$.*

Lemma 3.7 *With the notation of Definition 3.6, $i^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$ is a complete embedding whenever $\xi \le \eta \le \alpha$, and for each $p \in \mathbb{P}_\eta$, $p \restriction \xi$ is a reduction of p to $\mathbb{P}_\xi$.*

Definition 3.6 does not completely specify how to determine $\mathbb{P}_\eta$ from the earlier $\mathbb{P}_\xi$ when η is a limit ordinal. As with most ccc iterations, we shall use *finite supports*:

Definition 3.8 *With the notation of Definition 3.6, if p is a sequence of length ξ, then its* support $\mathrm{supt}(p)$ *is $\{\mu < \xi : (p)_\mu \neq \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}\}$. Then the iteration is* finitely supported *iff for all limit $\eta \le \alpha$, $\mathbb{P}_\eta$ is the set of all sequences p of length η such that* $\mathrm{supt}(p)$ *is finite and $p \restriction \xi \in \mathbb{P}_\xi$ for all $\xi < \eta$.*

When $\xi < \eta \le \alpha$, the map i^η_ξ is 1-1, so we may think of $\mathbb{P}_\xi$ as a complete suborder of $\mathbb{P}_\eta$. For a fixed α, we can make this "think of" true by replacing each $\mathbb{P}_\xi$ by $\widehat{\mathbb{P}}_\xi := i^\alpha_\xi(\mathbb{P}_\xi)$. Now, we have an increasing chain of posets. For finite supports, we take unions at limit ordinals η — that is, $\widehat{\mathbb{P}}_\eta = \bigcup_{\xi<\eta} \widehat{\mathbb{P}}_\xi$. A delta system argument on the supports is used to prove that finite support iterations preserve the ccc (see [11], Lemma V.3.17). A similar argument proves the following "*graph lemma*" (involving adding uncountable homogeneous sets for undirected graphs), which implies what we need for stHG.

Lemma 3.9 *With the notation of Definition 3.6, assume that all the $\mathbb{P}_\xi$ are ccc, and assume that the iteration is finitely supported. Suppose that in V, we have sets A and $E \subseteq [A]^2$. Let* Hom *be the statement that $\exists J \in [A]^{\aleph_1}\ [J]^2 \subseteq E$. Then* Hom *does not first become true at a limit stage. That is, if η is a limit and some $p \in \mathbb{P}_\eta$ forces* Hom, *then there is a $\xi < \eta$ such that some $p \in \mathbb{P}_\xi$ forces* Hom.

In the HG case, we have a base $\mathcal{B}$ for X and $A = \{(x, U) : x \in U\ \&\ U \in \mathcal{B}\}$ and $\{(x, U_x), (y, U_y)\} \in E$ iff $x \notin U_y$ or $y \notin U_x$. Then, Hom holds iff X is *not* HG, and we can apply the graph lemma to each X^n to show that X does not become non-stHG at a limit stage. So, in our intended iterated forcing, eventually making X suHG, we can prove by induction on η that $\mathbb{1} \Vdash_{\mathbb{P}_\eta} X$ is stHG. The limit stage uses Lemma 3.9 and the successor stage uses Lemma 3.4.

The next lemma makes it easy to finish a proof of Lemma 3.9.

Lemma 3.10 *Let A and E and* Hom *be as in Lemma 3.9. Let $\mathbb{P}$ be any ccc poset. Fix $q \in \mathbb{P}$. Then the following are equivalent:*

1. For some $p \le q$, $p \Vdash$ Hom.

2. There are $p_\mu \le q$ for $\mu < \omega_1$ and distinct *$a_\mu \in A$ for $\mu < \omega_1$ such that for all $\{\mu, \nu\} \in [\omega_1]^2 : p_\mu \perp p_\nu$ or $\{a_\mu, a_\nu\} \in E$; that is, $p_\mu \not\perp p_\nu \to \{a_\mu, a_\nu\} \in E$.*

Proof. (2) $\to$ (1): Let $\mathring{H} = \{(\check{a}_\mu, p_\mu) : \mu < \omega_1\}$. Then $\mathbb{1} \Vdash [\mathring{H}]^2 \subseteq E$. If there is some $p \le q$ such that $p \Vdash |\mathring{H}| = \aleph_1$, then $p \Vdash$ Hom, so we have (1). If there is no such p, then $q \Vdash |\mathring{H}| \leq \aleph_0$, so by the ccc, there is a $\delta < \omega_1$ such that $q \Vdash \mathring{H} \subseteq \{a_\mu : \mu < \delta\}$, which is a contradiction because $p_\delta \Vdash a_\delta \in \mathring{H}$.

(1) $\rightarrow$ (2): Assume that $p \leq q$ and $p \Vdash$ Hom. By the maximal principle, there are names $\mathring{a}_\mu$ for $\mu < \omega_1$ such that p forces $\mathring{a}_\mu \in A$ for each μ and, when $\mu \neq \nu$, p forces "$\mathring{a}_\mu \neq \mathring{a}_\nu$ and $\{\mathring{a}_\mu, \mathring{a}_\nu\} \in E$". For each μ, choose an $a_\mu \in A$ and a $p_\mu \leq p$ such that $p_\mu \Vdash \mathring{a}_\mu = \check{a}_\mu$. Passing to a subsequence, we may assume the a_μ are all different, since by the ccc, no collection of $\aleph_1$ of them can be the same. Also, whenever $p_\mu \not\perp p_\nu$ a common extension of p_μ, p_ν guarantees that $\{a_\mu, a_\nu\} \in E$, so we have (2). ☕

Proof of Lemma 3.9. Assume that γ is a limit ordinal and some $p \in \mathbb{P}_\gamma$ forces Hom. We shall show that (2) of Lemma 3.10 holds for $\mathbb{P}_\alpha$ for some $\alpha < \gamma$.

Applying Lemma 3.10 with $q = \mathbb{1}$, we have $p_\mu \in \mathbb{P}_\gamma$ for $\mu < \omega_1$ and distinct $a_\mu \in A$ for $\mu < \omega_1$ such that for all $\{\mu, \nu\} \in [\omega_1]^2 : p_\mu \perp p_\nu$ or $\{a_\mu, a_\nu\} \in E$. Each support $\mathrm{supt}(p_\mu)$ is a finite subset of γ, so passing to a subsequence, we may assume $\{\mathrm{supt}(p_\mu) : \mu < \omega_1\}$ forms a delta system, whose root, then, is a subset of α for some $\alpha < \gamma$. Then for each $\mu, \nu \in [\omega_1]^2 : p_\mu \perp p_\nu$ iff $p_\mu{\restriction}\alpha \perp p_\nu{\restriction}\alpha$. So, (2) holds for $\mathbb{P}_\alpha$. ☕

Our plan for the proof of our *main* Theorem 1.4 is: We start with an X that is stHG but not suHG, and iterate ccc forcing as in the standard MA construction, but only using posets that preserve the stHG of X. As Lemma 3.1 shows, $\mathsf{MA}(\kappa)$ restricted to these posets implies the full $\mathsf{MA}(\kappa)$.

4 Keeping the net weight big

Our *goal* is to prove Theorem 1.4. We have already obtained, in $V[H]$, a butterfly space X such that $|X| = w(X) = nw(X) = \aleph_2 = 2^{\aleph_0}$ and X is stHG but not suHG. We shall now obtain a further ccc extension $V[H][G]$ satisfying MA plus $2^{\aleph_0} = \aleph_2$ in which $nw(X)$ is still $\aleph_2$ and X becomes suHG.

Even though in $V[H]$ we use a set X that is super Luzin, in our extension $V[H][G]$ the resulting space X is not Luzin, by Lemma 2.11. There are *other* ccc posets of size $\aleph_2$ that force a countable network for X; see [9] or the proof of Theorem 1.2 in Section 3. Note that the cardinal functions weight, density, character, and cardinality cannot change in *any* ccc extension.

The following "Proof of Theorem 1.4" elaborates upon our Section 1 outline, and focuses on the primary remaining task, which is to ensure that the iterated forcing we use makes $nw(X)$ remain $\aleph_2$. See Definitions 3.6 and 3.8 for the notation (from [11], Section V.3) we use in Steps 3 and 4.

Proof of Theorem 1.4. The topology on our X will be one of the Example 2.4 butterfly refinements $\widehat{\mathcal{T}}$ of the euclidean topology $\mathcal{T}$ on $\mathbb{R}^2$. We define our model $V[H][G]$ as follows:

Step 1: Start with $V \models \mathfrak{c} \leq \aleph_2 = 2^{\aleph_1}$.

Step 2: Define $\mathbb{Q} = \mathrm{Fn}(\omega_2 \times \omega, \omega)$. In the $\mathbb{Q}$-generic extension, $V[H]$, we have "reals" $x^H_\alpha \in \omega^\omega$ for $\alpha < \omega_2$ defined by $x_\alpha(n) = (\bigcup H)(\alpha, n)$. This gives us $\tilde{x}^H_\alpha \in \mathbb{R}^2$ obtained from x^H_α by applying, in $V[H]$, some absolutely defined homeomorphism from ω^ω onto the set of pairs of irrationals. Then we define $X = \{\tilde{x}^H_\alpha : \alpha < \omega_2\}$. In $V[H]$, X is a super Luzin set and $nw(X, \widehat{\mathcal{T}}) = \aleph_2$ and $(X, \widehat{\mathcal{T}})$ is stHG, by Lemmas 2.12, 2.13 and the proof of Lemma 2.14.

Step 3: We use the "standard bookkeeping" as in the construction of a model of $\mathsf{MA} + \mathfrak{c} = \aleph_2$ (see [11], Section V.4), to have the $\mathring{\mathbb{Q}}_\xi$ enumerate *all* possible names for ccc posets of size at most $\aleph_1$, *except* that we only use those $\mathring{\mathbb{Q}}_\xi$ that preserve the stHG of X. Then, in our final $V[H][G]$, X will be stHG, and MA will hold for those ccc posets that preserve the stHG of X. But then, by Lemma 3.1, $V[H][G]$ will satisfy the full MA, along with the statement that X is suHG.

Step 4. From the "standard bookkeeping" of Step 3, we have the finite support iteration $\big(\big\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \omega_2\big\rangle\ ,\ \big\langle(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}) : \xi < \omega_2\big\rangle\big)$, where $\mathbb{1} \Vdash_{\mathbb{P}_\xi} |\mathring{\mathbb{Q}}_\xi| < \aleph_2$ for all $\xi < \omega_2$. All the posets are ccc, so cardinals are absolute. As in [11] in its "Proof of Theorem V.4.1", we may assume that the $\mathbb{P}_\xi$-name $\mathring{\mathbb{Q}}_\xi$, as a *set* of ordered pairs of names and forcing conditions has size at most $\aleph_1$ and for cofinally many ξ the domain of $\mathring{\mathbb{Q}}_\xi$ has at least two elements. Then $|\mathbb{P}_\xi| \leq \aleph_1$ for all $\xi < \omega_2$ (to see this, apply (8) of the definition of iterated forcing and the remarks in [11] following its Definition V.3.13 of finite supports), whereas $|\mathbb{P}_{\omega_2}| = \aleph_2$.

Let $\mathbb{P} = \mathbb{P}_{\omega_2}$ and let G be $\mathbb{P}$-generic over $V[H]$. We need to show that $nw(X, \widehat{\mathcal{T}})$ is still $\aleph_2$ in the iterated extension $V[H][G]$. In $V[H][G]$, $|X|$ is still $\aleph_2$, making $nw(X, \widehat{\mathcal{T}}) \leq \aleph_2$ clear; we just need to refute $nw(X, \widehat{\mathcal{T}}) \leq \aleph_1$.

Let $G_\xi = G \cap \mathbb{P}_\xi$ for $\xi \leq \omega_2$; so $G_{\omega_2} = G$.

Case i: $\xi < \omega_2$. We use $|\mathbb{P}_\xi| \leq \aleph_1$ to produce a Luzin set $X_2 \in [X]^{\aleph_2}$; then (by Lemma 2.13) $nw(X_2, \widehat{\mathcal{T}}) = \aleph_2$. To do this, in V we view the $\mathbb{Q} = \mathrm{Fn}(\omega_2 \times \omega, \omega)$ of Step 2 as a convenient $\mathbb{Q}_1 \times \mathbb{Q}_2$: choose $I_1 \in [\omega_2]^{\aleph_1}$ so that with $I_2 = \omega_2 \backslash I_1$ and $\mathbb{Q}_i = \mathrm{Fn}(I_i \times \omega, \omega)$ and $H_i = H \cap \mathbb{Q}_i$ for $i = 1, 2$, we have $(\mathbb{P}_\xi, \leq_{\mathbb{P}_\xi}) \in V[H_1]$. Then $V[H][G_\xi] = V[H_1][H_2][G_\xi] = V[H_1][G_\xi][H_2]$. Also, $X = X_1 \dot{\cup} X_2$, where each X_i is obtained from H_i the way X was obtained from H; that is, $X_i = \{\tilde{x}^{H_i}_\alpha : \alpha \in I_i\}$, where $x^{H_i}_\alpha(n) = (\bigcup H_i)(\alpha, n)$. Now, working in $V[H_1][G_\xi][H_2]$, X_2 is the desired Luzin set with $nw(X_2, \widehat{\mathcal{T}}) = \aleph_2$. Then, in $V[H][G_\xi] = V[H_1][G_\xi][H_2]$, we have $nw(X, \widehat{\mathcal{T}}) = \aleph_2$, because $X \supset X_2$ and $|X|$ is only $\aleph_2$.

Case ii: $\xi = \omega_2$. We need to show that a network of size $\aleph_1$ cannot suddenly appear at stage ω_2. Our proof will use the observation that such a network is essentially an object of size $\aleph_1$. This observation will follow from the fact that $(X, \widehat{\mathcal{T}})$ is HL in $V[H][G]$. This in turn is true because $(X, \widehat{\mathcal{T}})$ remains stHG in our iteration (by using Lemma 3.9, as in Section 3).

Suppose that, in $V[H][G]$, $\{N_\mu : \mu < \omega_1\}$ is a network for $(X, \widehat{\mathcal{T}})$. Since $(X, \widehat{\mathcal{T}})$ is regular, we may assume all the N_μ are closed. Since $(X, \widehat{\mathcal{T}})$ is HL, all closed sets are G_δ sets, and so each N_μ is a countable intersection of open sets. That is, $N_\mu = \bigcap_{i\in\omega} W_\mu^i$, where each $W_\mu^i \in \widehat{\mathcal{T}}$. Since $(X, \widehat{\mathcal{T}})$ is HL, each W_μ^i is a countable union of basic butterflies, which gives us $W_\mu^i = \bigcup_{j\in\omega} B_\mu^{i,j}$, where each $B_\mu^{i,j}$ is a basic butterfly neighborhood of some point; say $B_\mu^{i,j} = U_{x(i,j,\mu)}^{n(i,j,\mu)}$.

Note that $\langle n(i,j,\mu) : i,j \in \omega \ \&\ \mu \in \omega_1\rangle$ and $\langle x(i,j,\mu) : i,j \in \omega \ \&\ \mu \in \omega_1\rangle$ are objects of size $\aleph_1$; so they must occur in $V[H][G_\xi]$ for some $\xi < \omega_2$. But then $\{N_\mu : \mu < \omega_1\}$ is a network in $V[H][G_\xi]$, contradicting the result for case *i*. Thus $nw(X, \widehat{\mathcal{T}})$ is still $\aleph_2$ in the iterated extension $V[H][G]$. ☕

5 Nice finer topologies

This section dissects the two closure conditions built into our definition of *nice* butterfly refinement, and views them in the context of more general spaces and base assignments.

Our Definition 2.2 of *nice* butterfly refinement requires each basic open set and *also* its complement to satisfy closure conditions: If $(X, \widehat{\mathcal{T}})$ is a butterfly refinement determined by local open bases $\widehat{\mathcal{B}}_x = \{U_x^n : n \in \omega\}$ for $x \in X$, then $\widehat{\mathcal{T}}$ is *nice* iff for each $x \in X$ and for each $B_x \in \mathcal{B}_x$ the closure requirement $x \in \mathrm{cl}(\mathrm{int}(A_x, \mathcal{T}), \mathcal{T})$ holds for $A_x = B_x$ and *also* for $A_x = X \backslash B_x$.

We generalize the butterfly's neighborhood base assignments $x \mapsto \{U_x^n : n \in \omega\}$ to apply the closure conditions to more general spaces. Recall that a topology $\widehat{\mathcal{T}}$ is said to be *finer* than $\mathcal{T}$ iff $\mathcal{T} \subseteq \widehat{\mathcal{T}}$.

Definition 5.1 *A* base assignment *for a topology $\mathcal{T}$ on X is a map $x \mapsto \mathcal{B}_x$ for $x \in X$ where $\mathcal{B}_x \subset \mathcal{P}(X)$ is a local open base for $\mathcal{T}$ at x.*

If $\widehat{\mathcal{T}}$ is a finer topology than $\mathcal{T}$ on X, then $\widehat{\mathcal{T}}$ is a nice *finer topology than $\mathcal{T}$, denoted $\mathcal{T} \subseteq_N \widehat{\mathcal{T}}$, iff there* exists *a base assignment $x \mapsto \widehat{\mathcal{B}}_x$ for $\widehat{\mathcal{T}}$ satisfying for each $x \in X$ and for each $W \in \widehat{\mathcal{B}}_x$ the two closure conditions $x \in \mathrm{cl}(\mathrm{int}(A, \mathcal{T}), \mathcal{T})$ for $A = W$ and* also *for $A = X \backslash W$. If we drop the second closure condition (requiring $x \in \mathrm{cl}(\mathrm{int}(A, \mathcal{T}), \mathcal{T})$ for $A = X \backslash W$), then we say that $\widehat{\mathcal{T}}$ is a* mild *finer topology than $\mathcal{T}$, denoted $\mathcal{T} \subseteq_M \widehat{\mathcal{T}}$.*

The second closure condition is explicitly used in the proofs of Lemmas 2.11 and 2.13.

Obviously, $\mathcal{T} \subseteq_N \widehat{\mathcal{T}} \to \mathcal{T} \subseteq_M \widehat{\mathcal{T}}$. Although a butterfly refinement $\widehat{\mathcal{T}}$ of a separable metric space $(X, \mathcal{T})$ is our motivating example for a *nice* finer topology, we introduce the weaker *mild* finer topology because it is "nice enough" to have a fairly simple characterization (see Lemma 5.3).

Although Definition 5.1 is phrased to suggest our butterflies in the base assignment $x \mapsto \widehat{\mathcal{B}}_x$, we could expand $\widehat{\mathcal{B}}_x$ to the set of *all* W satisfying the required property:

Lemma 5.2 *Let $\mathcal{T}$ and $\widehat{\mathcal{T}}$ be topologies on X with $\mathcal{T} \subseteq \widehat{\mathcal{T}}$. Then $\mathcal{T} \subseteq_M \widehat{\mathcal{T}}$ iff for all $x \in X$, $\{W \in \widehat{\mathcal{T}} : x \in W \ \&\ x \in \mathrm{cl}(\mathrm{int}(W, \mathcal{T}), \mathcal{T})\}$ is a local $\widehat{\mathcal{T}}$ base at x. Also, $\mathcal{T} \subseteq_N \widehat{\mathcal{T}}$*

iff $\{W \in \widehat{\mathcal{T}} : x \in W \ \&\ x \in \mathrm{cl}(\mathrm{int}(W, \mathcal{T}), \mathcal{T}) \ \&\ x \in \mathrm{cl}(\mathrm{int}(X \backslash W, \mathcal{T}), \mathcal{T})\}$ *is a local* $\widehat{\mathcal{T}}$ *base at* x *for all* $x \in X$.

Now, focusing on mild pairs, we can avoid mentioning bases by the following:

Lemma 5.3 *Let* $\mathcal{T}$ *and* $\widehat{\mathcal{T}}$ *be topologies on* X *with* $\mathcal{T} \subseteq \widehat{\mathcal{T}}$. *Then the following are equivalent:*

1. $\mathcal{T} \subseteq_M \widehat{\mathcal{T}}$.
2. $\forall x \in X\ \forall V \in \widehat{\mathcal{T}}\ [x \in V \to x \in \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})]$.
3. $\forall V \in \widehat{\mathcal{T}} \backslash \{\emptyset\}\ \ \exists U \in \mathcal{T} \backslash \{\emptyset\}\ \ U \subseteq V$.
4. $\forall E \subseteq X\ [\mathrm{int}(E, \mathcal{T}) = \emptyset \leftrightarrow \mathrm{int}(E, \widehat{\mathcal{T}}) = \emptyset]$.

Proof. First, (1) $\to$ (2) is clear by applying the definition in the case that $V \in \widehat{\mathcal{B}}_x$ and then using the fact that $\widehat{\mathcal{B}}_x$ is a local $\widehat{\mathcal{T}}$ base.

For (2) $\to$ (1), we can assume (2) and *define* $\widehat{\mathcal{B}}_x$ to be the set of all $V \in \widehat{\mathcal{T}}$ such that $x \in V$ (and hence $x \in \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})$). Note that Definition 5.1 says “there *exists* a base assignment $\cdots\cdots$”.

For (2) $\to$ (3): Assume (2), fix $V \in \widehat{\mathcal{T}} \backslash \{\emptyset\}$, and fix any $x \in V$. Then $x \in \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})$, so $\mathrm{int}(V, \mathcal{T}) \neq \emptyset$, so there is a nonempty $U \in \mathcal{T}$ with $U \subseteq V$.

For $\neg(2) \to \neg(3)$: Fix x, V with $x \in V$ and $V \in \widehat{\mathcal{T}}$ and $x \notin \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})]$. Then $x \in V^* := V \setminus \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})$, so (3) would give us a nonempty $U \in \mathcal{T}$ with $U \subseteq V^*$. Then $U \subseteq V$, so $U \subseteq \mathrm{int}(V, \mathcal{T})$, so $U \subseteq \mathrm{cl}(\mathrm{int}(V, \mathcal{T}), \mathcal{T})$, contradicting $U \subseteq V^*$.

For (3) $\iff$ (4): Use the fact that $\mathcal{T} \subseteq \widehat{\mathcal{T}}$ iff $\forall E \subseteq X\ [\mathrm{int}(E, \mathcal{T}) \subseteq \mathrm{int}(E, \widehat{\mathcal{T}})]$. ☕

Question: Is there a lemma similar to Lemma 5.3 that gives a *simple* and *natural* topological equivalent to $\mathcal{T} \subseteq_N \widehat{\mathcal{T}}$, without mentioning bases? We say “natural”, because one can always express artificially the assumption that the family of sets satisfying *both* of the closure conditions in Definition 5.1 forms a base; see Lemma 5.2.

Condition (3) of Lemma 5.3 implies the following:

Corollary 5.4 *The relation* $\subseteq_M$ *is a partial order on the family of topologies on a set* X*; that is, it is transitive and reflexive.*

The stronger $\subseteq_N$ is *ir*reflexive. To see this, consider the second closure condition when $\mathcal{T}$ and $\widehat{\mathcal{T}}$ denote the same topology. If $x \in W$, where W is open, then $x \notin \mathrm{cl}(\mathrm{int}(X \backslash W))$.

Also, the mild relation is preserved by products: if $\mathcal{T}_1 \subseteq_M \widehat{\mathcal{T}_1}$ and $\mathcal{T}_2 \subseteq_M \widehat{\mathcal{T}_2}$, then $\mathcal{T}_1 \times \mathcal{T}_2 \subseteq_M \widehat{\mathcal{T}_1} \times \widehat{\mathcal{T}_2}$. This is easily verified using Lemma 5.3(3).

The $\mathcal{T}$–Luzin set used in Section 2 is also $\widehat{\mathcal{T}}$–Luzin. This follows easily from the fact that a mild pair preserves denseness (using Lemma 5.3(3)). Recall that an uncountable set L is Luzin whenever every $S \in [L]^{\aleph_1}$ is dense in some nonempty open set. Then, we have the following:

Proposition 5.5 *Whenever $\mathcal{T} \subseteq_M \widehat{\mathcal{T}}$:*
If $U \in \mathcal{T}$ and $D \subseteq U$: D is $\mathcal{T}$–dense in U iff D is $\widehat{\mathcal{T}}$–dense in U.
If $L \subseteq X$: L is a $\mathcal{T}$–Luzin set iff L is a $\widehat{\mathcal{T}}$–Luzin set.

Lemma 5.3(3) also implies that cellularity is always the same for $(X, \mathcal{T})$ and $(X, \widehat{\mathcal{T}})$ whenever $\mathcal{T} \subseteq_M \widehat{\mathcal{T}}$. On the other hand, even for $\mathcal{T} \subseteq_N \widehat{\mathcal{T}}$, some cardinal functions, such as weight, net weight, and spread, can be vastly different for $\mathcal{T}$ and $\widehat{\mathcal{T}}$. In $\mathbb{R}^2$, if $\mathcal{T}$ is the usual topology and $\widehat{\mathcal{T}}$ is one of our butterflies from Example 2.4, then $w(\mathbb{R}^2, \mathcal{T}) = nw(\mathbb{R}^2, \mathcal{T}) = s(\mathbb{R}^2, \mathcal{T}) = \aleph_0$, while $w(\mathbb{R}^2, \widehat{\mathcal{T}}) = nw(\mathbb{R}^2, \widehat{\mathcal{T}}) = s(\mathbb{R}^2, \widehat{\mathcal{T}}) = \mathfrak{c}$.

6 On first countability

While $\mathsf{MA}(\aleph_1)$ implies that every stHG space is suHG (Theorem [4]4.1), our "butterfly plus super Luzin set" construction in Section 2 built, in some models of set theory, a space X that is stHG but not suHG (or even suHC). But such an X was already constructed in the paper [4]. Here we contrast the two constructions.

The basic idea, as exemplified by Section 2, is the same in both constructions:

Start with an arbitrary assignment $\mathcal{U} = \langle (x_\xi, U_\xi) : \xi < \omega_1 \rangle$. Then show that $\exists \xi \neq \eta \, [x_\xi \in U_\eta \; \& \; x_\eta \in U_\xi]$; this establishes HG. If we can do the same in all X^n, we establish stHG (see the proof of Lemma 2.12). If we also show that if we choose the U_ξ to be "suitably small" then $\exists \xi \neq \eta \, [x_\xi \notin U_\eta \; \& \; x_\eta \notin U_\xi]$, then we can conclude that X fails to have the suHC (see the proof of Lemma 2.11).

The difference between Section 2 and [4] is that Section 2 obtained a butterfly space (where X was a super Luzin set in the plane), while [4] obtained what we might call a "matrix space". These differ in that our butterfly space must be first countable, whereas our matrix space was not. We explain here why our matrix space was not first countable.

We begin by summarizing briefly the argument from [3, 4] *in simplest form*: We start with a function $\Psi : \omega_1 \times \omega_1 \to \{0, 1\}$ (our papers [3, 4] consider more generally $\Psi : \omega_1 \times \omega_1 \to \omega$; but that's not required here). As in the HFD/HFC spaces of Hajnal and Juhász (see the Juhász [8] survey) and a construction of Roitman (see her [13] survey), we view Ψ as a matrix whose rows encode our space. So, define $f^\Psi_\beta(\alpha) = \Psi(\alpha, \beta)$ and $\mathcal{F}^\Psi = \{f^\Psi_\beta : \beta \in \omega_1\} \subseteq 2^{\omega_1}$. We sometimes drop the superscript Ψ when Ψ is clear from context. Using forcing or CH or the COMA, we can construct Ψ so that $\mathcal{F}^\Psi$ provides useful counterexamples. For example, in [4], we got $\mathcal{F}^\Psi$ to be stHG but not suHC. We shall explain now why this method causes $\mathcal{F}^\Psi$ to be not first countable. We remark that the space is HG and hence HL, so that points are G_δ sets; that is, each point of $\mathcal{F}^\Psi$ will have countable *pseudo*-character.

A word of caution: By Tychonov (essentially), every zero dimensional space of size and weight $\aleph_1$ is homeomorphic to $\mathcal{F}^\Psi$ for some $\Psi : \omega_1 \times \omega_1 \to 2$. This applies in particular to the butterflies constructed in Section 2, since it is easy to modify the construction (replacing $\mathbb{R}^2$ by $(\mathbb{R}\backslash\mathbb{Q})^2$) to make a butterfly example zero dimensional. So, we can only hope to prove that those $\mathcal{F}^\Psi$ constructed *by our methods* in [3, 4] are not first countable.

Our methods entailed building Ψ to satisfy a property that we called the SSD, and then showing that the SSD implies that $\mathcal{F}^\Psi$ is stHG and not suHC (see Section 3 of [4]). A Ψ satisfying the SSD was obtained using the COMA from [3], which showed that the COMA holds under CH and in any model V[one or more random real] and in any model V[one or more Cohen real].

We shall now recall the definition of this SSD, and then show that the SSD implies that $\mathcal{F}^\Psi$ is not first countable. We specialize the discussion in [3] to the case where $\Psi : \omega_1 \times \omega_1 \to 2$.

The next two definitions are from [3], somewhat simplified:

Definition 6.1 *Fix $n \in \omega\backslash\{0\}$. Then $\mathcal{A}$ is a* normalized block pattern *of block size n iff $\mathcal{A} = \langle A_\xi : \xi < \omega_1\rangle$, where all $A_\xi \in [\omega_1]^n$ and the A_ξ are pairwise disjoint and $\xi < \eta \to \max(A_\xi) < \min(A_\eta)$. We shall list each A_ξ in increasing order as $A_\xi = \{\alpha^i_\xi : i < n\}$.*

Definition 6.2 *For each $n \in \omega\backslash\{0\}$, $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the* SSD_n *iff given any normalized block pattern of block size n, and given values $c_{i,j} \in \{0,1\}$ for all $i, j < n$:*

$$\exists \xi < \eta < \omega_1 \, \forall i, j < n \, [\Psi(\alpha^i_\xi, \alpha^j_\eta) = \Psi(\alpha^i_\eta, \alpha^j_\xi) = c_{i,j}] \quad .$$

Ψ satisfies the SSD iff Ψ satisfies the SSD_n *for each $n \in \omega\backslash\{0\}$.*

Our proof that $\mathcal{F}^\Psi$ is not first countable will just use the following weakened version of (and obvious consequence of) the SSD, that we shall call the SSSD (semi-SSD).

Definition 6.3 *For each $n \in \omega\backslash\{0\}$, $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the* SSSD_n *iff given any normalized block pattern of block size n, and given values $c_i \in \{0,1\}$ for all $i < n$:*

$$\exists \xi < \eta < \omega_1 \, \forall i < n \, [\Psi(\alpha^i_\xi, \alpha^0_\eta) = c_i] \quad .$$

Ψ satisfies the SSSD iff Ψ satisfies the SSSD_n *for each $n \in \omega\backslash\{0\}$.*

Theorem 6.4 *If $\Psi : \omega_1 \times \omega_1 \to 2$ satisfies the SSSD, then $\mathcal{F}^\Psi$ is not first countable.*

Proof. Assume that $\mathcal{F}^\Psi$ is first countable. We shall derive a contradiction.

The natural base at each $f_\beta \in \mathcal{F}^\Psi$ is $\{N_s(f_\beta) : s \in [\omega_1]^{<\aleph_0}\}$, where we define $N_s(f_\beta) := \{f_\delta : f_\delta \restriction s = f_\beta \restriction s\}$. Since f_β has countable character, some countable subfamily of the natural base is also a base at f_β. So, fix a strictly increasing function $D : \omega_1 \to \omega_1$ such that for each β: $\beta < D(\beta)$ and $\{N_s(f_\beta) : s \in [D(\beta)]^{<\aleph_0}\}$ is a local base at f_β.

Next, fix a strictly increasing $h : \omega_1 \to \omega_1$ such that $\beta < D(\beta) < h(\beta) < \omega_1$ for all β. Specifically, let $h(\beta) = D(\beta) + \omega$.

Then, for each β, choose an $s_\beta \in [D(\beta)]^{<\aleph_0}$ such that $f_\beta \in N_{s_\beta}(f_\beta) \subseteq N_{\{h(\beta)\}}(f_\beta)$. Then $\forall\delta\, [f_\delta \in N_{s_\beta}(f_\beta) \to f_\delta \in N_{\{h(\beta)\}}(f_\beta)]$;
so $\forall\delta\, [f_\delta{\restriction}s_\beta = f_\beta{\restriction}s_\beta \to f_\delta(h(\beta)) = f_\beta(h(\beta))]$. WLOG, $\beta \in s_\beta$ for each β.

Next, choose a stationary $I \subset \omega_1$ such that for each $\beta \in I$: β is a limit ordinal and $s_\beta = s \mathbin{\dot\cup} t_\beta$, where $\beta = \min(t_\beta) \in t_\beta \subset [\beta, \omega_1)$ and $s \subset \beta$ is fixed (that is, independent of β). Shrinking I, WLOG each $|t_\beta| = k$ for some fixed $k \in \omega\backslash\{0\}$ and all $f_\beta{\restriction}s$ for $\beta \in I$ are the same. Also WLOG, $s \subset \min(I)$. Also WLOG, for all $\beta, \beta' \in I$: $\beta < \beta' \to \beta < h(\beta) < \beta' < h(\beta')$.

List each t_β in increasing order as $\{\tau_\beta^0, \ldots, \tau_\beta^{k-1}\}$; so $\tau_\beta^0 = \beta$.

We now have, applying $f_\beta \in N_{s_\beta}(f_\beta) \subseteq N_{\{h(\beta)\}}(f_\beta)$:

$$\forall\beta \in I\ \forall\delta \in \omega_1\ [f_\delta{\restriction}s_\beta = f_\beta{\restriction}s_\beta \to f_\delta(h(\beta)) = f_\beta(h(\beta))] \ .$$

Then, since the $f_\beta{\restriction}s$ for $\beta \in I$ are all the same, $f_\delta{\restriction}s = f_\beta{\restriction}s$ yields:

$$\forall\beta \in I\ \forall\delta \in I\ [f_\delta{\restriction}t_\beta = f_\beta{\restriction}t_\beta \to f_\delta(h(\beta)) = f_\beta(h(\beta))] \ .$$

Then, rewriting this in terms of Ψ and using $t_\beta = \{\tau_\beta^0, \ldots, \tau_\beta^{k-1}\}$:

$$\forall\beta \in I\ \forall\delta \in I\ [\forall\ell < k[\Psi(\tau_\beta^\ell, \delta) = \Psi(\tau_\beta^\ell, \beta)] \to \Psi(h(\beta), \delta) = \Psi(h(\beta), \beta)] \ .$$

WLOG, shrinking I, we have constants $q, p^\ell \in \{0, 1\}$ for $\ell < k$ such that $\Psi(\tau_\beta^\ell, \beta) = p^\ell$ and $\Psi(h(\beta), \beta) = q$ for all $\beta \in I$. Our implication now becomes:

$$\forall\beta \in I\ \forall\delta \in I\ \left[\forall\ell < k[\Psi(\tau_\beta^\ell, \delta) = p^\ell] \to \Psi(h(\beta), \delta) = q)\right] \ . \qquad (\star)$$

Now, we wish to think of the various ordinals here as part of a block pattern for an application of the SSSD. Focus on the case where $h(\beta) < \delta$ and hence:

$$\beta = \tau_\beta^0 < \tau_\beta^1 < \cdots < \tau_\beta^{k-1} < h(\beta) \quad < \quad \delta = \tau_\delta^0 < \tau_\delta^1 < \cdots < \tau_\delta^{k-1} < h(\delta) \ .$$

We shall derive a contradiction with $(\star)$ by obtaining $h(\beta)$ and τ_β^ℓ for $\ell < k$ in the lower block and δ in the upper block so that each $\Psi(\tau_\beta^\ell, \delta) = p^\ell$, while $\Psi(h(\beta), \delta) = 1 - q$. Here, the block size is $k + 1$, and the lower block is $\{\beta = \tau_\beta^0, \tau_\beta^1, \cdots, \tau_\beta^{k-1}, h(\beta)\}$, and the upper block is $\{\delta = \tau_\delta^0, \tau_\delta^1, \cdots, \tau_\delta^{k-1}, h(\delta)\}$.

So, we build $\mathcal{A}$ to be a normalized block pattern of block size $n := k + 1$ as follows: Each A_ξ is listed in increasing order as $A_\xi = \{\alpha_\xi^i : i < n\} = \{\alpha_\xi^i : i \le k\}$.

For some strictly increasing sequence $\langle \beta_\xi : \xi < \omega_1 \rangle$, we have: Each $\beta_\xi \in I$ and $\alpha_\xi^0 = \beta_\xi = \tau_{\beta_\xi}^0$ and $\alpha_\xi^1 = \tau_{\beta_\xi}^1$ and $\cdots\cdots$ and $\alpha_\xi^{k-1} = \tau_{\beta_\xi}^{k-1}$ and $\alpha_\xi^k = h(\beta_\xi)$.

The following diagram shows block A_0 followed by block A_ξ followed by block A_η, where $0 < \xi < \eta$ and $k \ge 3$:

$$\alpha_0^0 = \beta_0 = \tau_{\beta_0}^0 < \alpha_0^1 = \tau_{\beta_0}^1 < \cdots < \alpha_0^{k-1} = \tau_{\beta_0}^{k-1} < \alpha_0^k = h(\beta_0)$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha_\xi^0 = \beta_\xi = \tau_{\beta_\xi}^0 < \alpha_\xi^1 = \tau_{\beta_\xi}^1 < \cdots < \alpha_\xi^{k-1} = \tau_{\beta_\xi}^{k-1} < \alpha_\xi^k = h(\beta_\xi)$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha_\eta^0 = \beta_\eta = \tau_{\beta_\eta}^0 < \alpha_\eta^1 = \tau_{\beta_\eta}^1 < \cdots < \alpha_\eta^{k-1} = \tau_{\beta_\eta}^{k-1} < \alpha_\eta^k = h(\beta_\eta)$$

Finally, by the SSSD_n, we can fix a $\xi < \eta$ so that each $\Psi(\tau_{\beta_\xi}^\ell, \alpha_\eta^0) = \Psi(\alpha_\xi^\ell, \alpha_\eta^0) = p^\ell$ for each $\ell < k$, while $\Psi(h(\beta_\xi), \alpha_\eta^0) = \Psi(\alpha_\xi^k, \alpha_\eta^0) = 1 - q$.

Converting to the above notation, replace β_ξ by β and replace α_η^0 by δ. Then we have $\Psi(\tau_\beta^\ell, \delta) = p^\ell$ for each $\ell < k$, while $\Psi(h(\beta), \delta) = 1 - q$, and this gives us our contradiction. ☕

Of course, Theorem 6.4 proves our $\mathcal{F}^\Psi$ has $\chi(\mathcal{F}^\Psi) = \aleph_1$, because $\mathcal{F}^\Psi \subseteq 2^{\omega_1}$. The neighborhoods of each $f_\beta \in \mathcal{F}^\Psi$ are uncountable, so that Theorem 6.4 is similar in spirit to statement 3.6 of [8] that $\mathrm{HFC_w}$ spaces have maximum possible character.

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