

First Countable Continua and Proper Forcing*

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Abstract

Assuming the Continuum Hypothesis, there is a compact first countable connected space of weight \aleph_1 with no totally disconnected perfect subsets. Each such space, however, may be destroyed by some proper forcing order which does not add reals.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As in [12],

Definition 1.1 *A space X is weird iff X is compact and not scattered, and no perfect subset of X is totally disconnected.*

A subset P of X is *perfect* iff P is closed and has no isolated points. As usual, \mathfrak{c} denotes the (von Neumann) cardinal 2^{\aleph_0} . Big weird spaces (of size $2^{\mathfrak{c}}$) were produced from CH in Fedorchuk, Ivanov, and van Mill [10]. Small weird spaces (of size \aleph_1) were constructed from \diamond in [12], which proved:

Theorem 1.2 *Assuming \diamond , there is a connected weird space which is hereditarily separable and hereditarily Lindelöf.*

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The weird spaces of [12], [10], and the earlier Fedorchuk [9] are all separable spaces of weight \aleph_1 . Our \diamond example is also first countable, because it is compact and hereditarily Lindelöf. In contrast, the CH weird spaces of [10, 9] have no convergent ω -sequences. We do not know whether CH can replace \diamond in Theorem 1.2, but weakening hereditarily Lindelöf to first countable we do get:

Theorem 1.3 *Assuming CH, there is a separable first countable connected weird space of weight \aleph_1 .*

This theorem cannot be proved by a *classical* CH construction. Classical CH arguments build the item of interest directly from an enumeration in type ω_1 of some natural set of size \mathfrak{c} (e.g., \mathbb{R} , $\mathbb{R}^{<\omega_1}$, etc.). The result, then, is preserved by any forcing which does not add reals. These arguments include any CH proof found in Sierpiński's text [15], as well as most CH proofs in the current literature, including the constructions of the big weird spaces of [9, 10]. In contrast, every space satisfying Theorem 1.3 is destroyed by some proper forcing order which does not add reals.

Our proof of Theorem 1.3 uses classical CH arguments to make X weird, but then, to make X first countable, we adapt the method of Gregory [11] and Devlin and Shelah [2]. The methods of [11] and [2] are, as Hellsten, Hyttinen, and Shelah [13] pointed out, essentially the same. We review the method in Section 2, and use it to prove Theorem 1.3 in Section 4. Although [11] and [2] derive results from $2^{\aleph_0} < 2^{\aleph_1}$, for Theorem 1.3, we need CH; Section 5 explains why.

In Section 3, we show that each space satisfying Theorem 1.3 can be destroyed by a proper forcing which does not add reals; in $V[G]$, we add a point of uncountable character. More precisely, if X is a compactum in V , then in each generic extension $V[G]$, we still have the same set X with the natural topology obtained by using the open sets from V as a base. If X is first countable in V , then it must remain first countable in $V[G]$, but X need not be compact in $V[G]$. We get the point of uncountable character in the natural corresponding *compact* space \tilde{X} in $V[G]$. This compact space determined by X was described by Bandlow [1] (and later in [3, 4, 6]), and can be defined as follows:

Definition 1.4 *If X is a compactum in V and $V[G]$ is a forcing extension of V , then in $V[G]$ the corresponding compactum \tilde{X} is characterized by:*

1. X is dense in \tilde{X} .
2. Every $f \in C(X, [0, 1]) \cap V$ extends to an $\tilde{f} \in C(\tilde{X}, [0, 1])$ in $V[G]$.
3. The functions \tilde{f} (for $f \in V$) separate the points of \tilde{X} .

In forcing, \hat{X} denotes the \tilde{X} of $V[G]$, while \check{X} denotes the X of $V[G]$.

For example, if X is the $[0, 1]$ of V , then \tilde{X} will be the unit interval of $V[G]$; note that in statement (2), asserted in $V[G]$, the “[0, 1]” really refers to the unit interval of $V[G]$. If in V , we have $X \subseteq [0, 1]^\kappa$, then \tilde{X} is simply the closure of X in the $[0, 1]^\kappa$ of $V[G]$. If in V , X is the Stone space of a boolean algebra \mathcal{B} , then \tilde{X} will be the Stone space, computed in $V[G]$, of the same \mathcal{B} . In general, the weights of X and \tilde{X} will be the same (assuming that cardinals are not collapsed), but their characters need not be.

Following Eisworth and Roitman [8, 7], we call a partial order \mathbb{P} *totally proper* iff \mathbb{P} is proper and forcing with it does not add reals.

Theorem 1.5 *If X is compact, connected, and infinite, and X does not have a Cantor subset, then for some totally proper \mathbb{P} : $\mathbb{1}_{\mathbb{P}} \Vdash “\tilde{X}$ is not first countable”.*

The proof is in Section 3. Observe the importance of connectivity here. Suppose in V that X is the double arrow space, obtained from $[0, 1]$ by doubling the points of $(0, 1)$. Then in any $V[G]$, \tilde{X} is the compactum obtained from $[0, 1]$ by doubling the points of $(0, 1) \cap V$, and is hence first countable.

2 Predictors

In the following, λ^{ω_α} denotes the set of functions from ω_α into λ . Something like the next definition and theorem is implicit in both of [11, 2]:

Definition 2.1 *Let κ, λ be any cardinals and $\Psi : \kappa^{<\omega_1} \rightarrow \lambda$. If $f \in \kappa^{\omega_1}$, $g \in \lambda^{\omega_1}$, and $C \subseteq \omega_1$, then Ψ, f predict g on C iff $g(\xi) = \Psi(f \upharpoonright \xi)$ for all $\xi \in C$. Ψ is a (κ, λ) -predictor iff for all $g \in \lambda^{\omega_1}$ there is an $f \in \kappa^{\omega_1}$ and a club C such that Ψ, f predict g on C .*

Theorem 2.2 *The following are equivalent whenever $2 \leq \kappa \leq \mathfrak{c}$ and $2 \leq \lambda \leq \mathfrak{c}$:*

1. *There is a (κ, λ) -predictor.*
2. *There is a $(\mathfrak{c}, \mathfrak{c})$ -predictor.*
3. $2^{\aleph_0} = 2^{\aleph_1}$.

Proof. (3) \rightarrow (1): Let $C = \omega_1 \setminus \omega$. List λ^{ω_1} as $\{g_\alpha : \alpha < \mathfrak{c}\}$, and choose $f_\alpha \in \kappa^{\omega_1}$ so that the $f_\alpha \upharpoonright \omega$, for $\alpha < \mathfrak{c}$, are all distinct. Then we can define $\Psi : \kappa^{<\omega_1} \rightarrow \lambda$ so that $\Psi(f_\alpha \upharpoonright \xi) = g_\alpha(\xi)$ for all $\xi \in C$.

(1) \rightarrow (2): Fix a (κ, λ) -predictor $\Psi : \kappa^{<\omega_1} \rightarrow \lambda$. We shall define $\Phi : (\kappa^\omega)^{<\omega_1} \rightarrow (\lambda^\omega)$ so that it is a $(\kappa^\omega, \lambda^\omega)$ -predictor in the sense of Definition 2.1. For $p \in (\kappa^\omega)^\xi$

and $n \in \omega$, define $p_{(n)} \in \kappa^\xi$ by: $p_{(n)}(\mu) = (p(\mu))(n) \in \kappa$. Then, for $p \in (\kappa^\omega)^{<\omega_1}$, define $\Phi(p) = \langle \Psi(p_{(n)}) : n \in \omega \rangle \in \lambda^\omega$.

(2) \rightarrow (3): Fix a $(\mathfrak{c}, \mathfrak{c})$ -predictor $\Psi : \mathfrak{c}^{<\omega_1} \rightarrow \mathfrak{c}$. Let $\Gamma : \mathfrak{c}^{<\omega_1} \times \mathfrak{c}^{<\omega_1} \rightarrow \mathfrak{c}$ be any 1-1 function. If $K \subseteq \omega_1$ is unbounded and $\xi < \omega_1$, let $next(\xi, K)$ be the least element of K which is greater than ξ .

For each $B \in \mathfrak{c}^{\omega_1}$, choose $G(n, B), F(n, B) \in \mathfrak{c}^{\omega_1}$ and clubs $C(n, B) \subseteq \omega_1$ for $n \in \omega$ as follows: Let $G(0, B) = B$. Given $G(n, B)$, let $C(n, B)$ be club of limit ordinals and let $F(n, B) \in \mathfrak{c}^{\omega_1}$ be such that $(G(n, B))(\xi) = \Psi((F(n, B)) \upharpoonright \xi)$ for all $\xi \in C(n, B)$. Then define $G(n+1, B)$ so that

$$(G(n+1, B))(\xi) = \Gamma(F(n, B) \upharpoonright next(\xi, C(n, B)), G(n, B) \upharpoonright next(\xi, C(n, B)))$$

for each ξ .

Now, fix $B, B' \in \mathfrak{c}^{\omega_1}$, and consider the statement:

$$\forall n \in \omega [G(n, B) \upharpoonright \xi = G(n, B') \upharpoonright \xi] \quad (\star(\xi))$$

So, $\star(0)$ is true trivially, and $\star(\xi)$ implies $\star(\zeta)$ whenever $\zeta < \xi$. We shall prove inductively that $\star(1)$ implies $\star(\eta)$ for all $\eta < \omega_1$. If we do this, then $\star(1)$ will imply $B = B'$, so we shall have $2^{\aleph_0} = 2^{\aleph_1}$, since there are 2^{\aleph_1} possible values for B but only 2^{\aleph_0} possible values for the sequence $\langle (G(n, B))(0) : n \in \omega \rangle$.

The induction is trivial at limits, so it is sufficient to fix η with $1 \leq \eta < \omega_1$, assume $\star(\eta)$, and prove $\star(\eta+1)$ — that is, $(G(n, B))(\eta) = (G(n, B'))(\eta)$ for all n . Fix n . For $\xi < \eta$, we have $(G(n+1, B))(\xi) = (G(n+1, B'))(\xi)$, which implies:

- a. $next(\xi, C(n, B)) = next(\xi, C(n, B'))$; call this γ_ξ .
- b. $F(n, B) \upharpoonright \gamma_\xi = F(n, B') \upharpoonright \gamma_\xi$.
- c. $G(n, B) \upharpoonright \gamma_\xi = G(n, B') \upharpoonright \gamma_\xi$.

Applying (a) for all $\xi < \eta$: $\eta \in C(n, B)$ iff $\eta \in C(n, B')$. If $\eta \notin C(n, B), C(n, B')$, then fix ξ with $\xi < \eta < \gamma_\xi$; now (c) implies $(G(n, B))(\eta) = (G(n, B'))(\eta)$. If $\eta \in C(n, B), C(n, B')$, then η is a limit ordinal and (b) implies $F(n, B) \upharpoonright \eta = F(n, B') \upharpoonright \eta$; now $(G(n, B))(\eta) = (G(n, B'))(\eta) = \Psi((F(n, B)) \upharpoonright \eta)$. \blacksquare

The *non*-existence of a $(2, 2)$ -predictor is the weak version of \diamond discussed by Devlin and Shelah in [2], where they use it to prove that, assuming $2^{\aleph_0} < 2^{\aleph_1}$, every ladder system on ω_1 has a non-uniformizable coloring. By Shelah [14] (p. 196), each such coloring may be uniformized in some totally proper forcing extension.

A direct proof of (3) \rightarrow (2), resembling the above proof of (3) \rightarrow (1), would obtain C fixed at $\omega_1 \setminus \{0\}$, since one may choose the f_α so that the $f_\alpha(0)$, for $\alpha < \mathfrak{c}$, are all distinct. Gregory [11] used the failure of (2), with this specific C , to derive a result about trees under $2^{\aleph_0} < 2^{\aleph_1}$; see Theorem 3.14 below.

3 Some Totally Proper Orders

We consider *forcing posets*, $(\mathbb{P}; \leq, \mathbb{1})$, where \leq is a transitive and reflexive relation on \mathbb{P} and $\mathbb{1}$ is a largest element of \mathbb{P} . As usual, if $p, q \in \mathbb{P}$, then $p \not\leq q$ means that p, q are compatible (that is, have a common extension), and $p \perp q$ means that p, q are incompatible.

Definition 3.1 *Assume that X is compact, connected, and infinite. Let $\mathbb{K} = \mathbb{K}_X$ be the forcing poset consisting of all closed, connected, infinite subsets of X , with $p \leq q$ iff $p \subseteq q$ and $\mathbb{1}_{\mathbb{K}} = X$. In \mathbb{K} , define $p \perp\!\!\!\perp q$ iff $p \cap q = \emptyset$.*

Note that $p \perp q$ iff $p \cap q$ is totally disconnected. The stronger relation $p \perp\!\!\!\perp q$ will be useful in the proof that \mathbb{K} is totally proper whenever X does not have a Cantor subset. First, we verify that \mathbb{K} is separative; this follows easily from the following lemma, which is probably well-known; a proof is in [12]:

Lemma 3.2 *If P is compact, connected, and infinite, and $U \subseteq P$ is a nonempty open set, then there is a closed $R \subseteq U$ such that R is connected and infinite.*

In particular, in \mathbb{K} , if $p \not\leq q$, then we may apply this lemma with $U = p \setminus q$ to get $r \leq p$ with $r \perp q$, proving the following:

Corollary 3.3 *If X is compact, connected, and infinite, then \mathbb{K}_X is separative and atomless.*

We collect some useful properties of the relation $\perp\!\!\!\perp$ on \mathbb{K} in the following:

Definition 3.4 *A binary relation $\not\leq$ on a forcing poset is a strong incompatibility relation iff*

1. $p \not\leq q$ implies $p \perp q$.
2. Whenever $p \perp q$, there are p_1, q_1 with $p_1 \leq p$, $q_1 \leq q$, and $p_1 \not\leq q_1$.
3. $p \not\leq q$ & $p_1 \leq p$ & $q_1 \leq q \rightarrow p_1 \not\leq q_1$.

This definition does not require $\not\leq$ to be symmetric, but note that the relation $p \not\leq q$ & $q \not\leq p$ is symmetric and is also a strong incompatibility relation.

Lemma 3.5 *The relation $\perp\!\!\!\perp$ is a strong incompatibility relation on \mathbb{K}_X .*

Proof. Conditions (1) and (3) are obvious. For (2): Suppose that $p \perp q$. Let $F = p \cap q$, which is totally disconnected. Then by Lemma 3.2 there is an infinite connected $p_1 \subseteq p \setminus F$. Likewise, we get $q_1 \subseteq q \setminus F$. ☕

Definition 3.6 If \mathbb{P} is a forcing poset with a strong incompatibility relation $\not\leq$, then a strong Cantor tree in \mathbb{P} (with respect to $\not\leq$) is a subset $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ such that each $p_{s \smallfrown \mu} < p_s$ for $\mu = 0, 1$, and each $p_{s \smallfrown 0} \not\leq p_{s \smallfrown 1}$. Then, \mathbb{P} has the weak Cantor tree property (WCTP) (with respect to $\not\leq$) iff whenever $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ is a strong Cantor tree, there is at least one $f \in 2^\omega$ such that \mathbb{P} contains some $q = q_f$ with $q \leq p_{f \upharpoonright n}$ for each $n \in \omega$.

Note that if \mathbb{P} has the WCTP, then the set of f for which q_f is defined must meet every perfect subset of the Cantor set 2^ω , since otherwise we could find a subtree of the given Cantor tree which contradicts the WCTP.

Lemma 3.7 If X is compact, connected, and infinite, and X does not have a Cantor subset, then \mathbb{K}_X has the WCTP.

Definition 3.8 \mathbb{P} has the Cantor tree property (CTP) iff \mathbb{P} has the WCTP with respect to the usual \perp relation.

\mathbb{K}_X need not have the CTP (see Theorem 5.4). A countably closed \mathbb{P} clearly has the CTP. In the case of trees, the CTP was also discussed in [13] (where it was called “ \aleph_0 fan closed”) and in [12]. The following modifies Lemma 3 of [13] and Lemma 5.5 of [12]:

Lemma 3.9 If \mathbb{P} has the WCTP, then \mathbb{P} is totally proper.

Proof. Define $q \leq' p$ iff there is no r such that $r \leq q$ and $r \perp p$. When \mathbb{P} is separative, this is equivalent to $q \leq p$.

Fix a suitably large regular cardinal θ , and let $M \prec H(\theta)$ be countable with $(\mathbb{P}; \leq, \perp, \not\leq) \in M$, and fix $p \in \mathbb{P} \cap M$. It suffices (see [8]) to find a $q \leq p$ such that whenever $A \subseteq \mathbb{P}$ is a maximal antichain and $A \in M$, there is an $r \in A \cap M$ with $q \leq' r$. If \mathbb{P} has an atom $q \leq p$ such that $q \in M$, then we are done. Otherwise, then since $M \prec H(\theta)$, \mathbb{P} must be atomless below p . Let $\{A_n : n \in \omega\}$ list all the maximal antichains which are in M . Build a strong Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P} \cap M$ such that, $p_\emptyset \leq p$, and such that, when $n \in \omega$ and $s \in 2^n$, p_s extends some element of $A_n \cap M$. Then choose $f \in 2^\omega$ such that there is some $q \in \mathbb{P}$ with $q \leq p_{f \upharpoonright n}$ for each $n \in \omega$. ☕

Proof of Theorem 1.5. Let $\mathbb{P} = \mathbb{K}_X$. Working in $V[G]$, let $G' = \{\tilde{p} : p \in G\}$; then $\bigcap G' = \{y\}$ for some $y \in \tilde{X} \setminus X$. Since \mathbb{P} does not add ω -sequences, $\bigcap E \not\supseteq \{y\}$ whenever E is a countable subset of G' . Thus, $\chi(y, \tilde{X})$ is uncountable. ☕

These totally proper partial orders yield natural weakenings of PFA:

Definition 3.10 *If \mathfrak{P} is a class of forcing posets, then $\text{MA}_{\mathfrak{P}}(\aleph_1)$ is the statement that whenever $\mathbb{P} \in \mathfrak{P}$ and \mathcal{D} is a family of $\leq \aleph_1$ dense subsets of \mathbb{P} , then there is a filter on \mathbb{P} meeting each $D \in \mathcal{D}$.*

Trivially, $\text{PFA} \rightarrow \text{MA}_{\text{WCTP}}(\aleph_1) \rightarrow \text{MA}_{\text{CTP}}(\aleph_1)$, but in fact $\text{MA}_{\text{WCTP}}(\aleph_1) \leftrightarrow \text{MA}_{\text{CTP}}(\aleph_1)$ (see Lemma 3.13). Also, $\text{MA}_{\text{CTP}}(\aleph_1) \rightarrow 2^{\aleph_0} = 2^{\aleph_1}$ (see Corollary 3.15), so, the natural iteration of (totally proper) CTP orders with countable supports must introduce reals at limit stages. By the proof of Theorem 5.9 in [12], PFA does not follow from $\text{MA}_{\text{CTP}}(\aleph_1) + \text{MA}(\aleph_1) + 2^{\aleph_0} = \aleph_2$, which in fact can be obtained by ccc forcing over L .

We now consider some CTP trees.

Definition 3.11 *Order $\lambda^{<\omega_1}$ by: $p \leq q$ iff $p \supseteq q$. Let $\mathbb{1} = \emptyset$, the empty sequence.*

So, $\lambda^{<\omega_1}$ is a tree, with the root $\mathbb{1}$ at the top. Viewed as a forcing order, it is equivalent to countable partial functions from ω_1 to λ . We often view $p \in \lambda^{<\omega_1}$ as a countable sequence and let $\text{lh}(p) = \text{dom}(p)$. Then $\text{lh}(\mathbb{1}) = 0$.

Kurepa showed that SH is equivalent to the non-existence of Suslin trees. A similar proof shows that $\text{MA}_{\text{CTP}}(\aleph_1)$ is equivalent to the non-existence of Gregory trees:

Definition 3.12 *A Gregory tree is a forcing poset \mathbb{P} which is a subtree of $\mathfrak{c}^{<\omega_1}$ and satisfies:*

1. \mathbb{P} has the CTP.
2. \mathbb{P} is atomless.
3. \mathbb{P} has no uncountable chains.

It is easily seen that if any of conditions (1)(2)(3) are dropped, such trees may be constructed in ZFC. However:

Lemma 3.13 *The following are equivalent:*

1. $\text{MA}_{\text{CTP}}(\aleph_1)$.
2. $\text{MA}_{\text{WCTP}}(\aleph_1)$.
3. There are no Gregory trees.

Proof. (1) \rightarrow (3): Let \mathbb{P} be a Gregory tree. As with Suslin trees under $\text{MA}(\aleph_1)$, a filter G meeting the sets $D_\xi := \{p \in \mathbb{P} : \text{lh}(p) \geq \xi\}$ yields an uncountable chain, and hence a contradiction, but to apply $\text{MA}_{\text{CTP}}(\aleph_1)$, we must prove that each D_ξ is dense in \mathbb{P} . To do this, induct on ξ . The case $\xi = 0$ is trivial. For the successor stages, use the fact that \mathbb{P} is atomless. For the limit stages, use the CTP.

(3) \rightarrow (2): Fix \mathbb{P} with the WCTP and dense sets $D_\xi \subseteq \mathbb{P}$ for $\xi < \omega_1$. We need to produce a filter $G \subseteq \mathbb{P}$ meeting each D_ξ . This is trivial if \mathbb{P} has an atom, so assume that \mathbb{P} is atomless.

Inductively define a subtree T of $2^{<\omega_1}$ together with a function $F : T \rightarrow \mathbb{P}$ as follows: $F(\mathbf{1}) = \mathbf{1}_{\mathbb{P}}$. If $t \in T$ and $\text{lh}(t) = \xi$, then $t \frown 0 \in T$ and $t \frown 1 \in T$, and $F(t \frown 0), F(t \frown 1)$ are extensions of $F(t)$ such that each $F(t \frown i) \in D_\xi$ and $F(t \frown 0) \not\leq F(t \frown 1)$; to accomplish this, given t and $F(t)$: first choose two \perp extensions of $F(t)$, then extend these to be $\not\leq$, and then extend these to be in D_ξ . If $\eta < \omega_1$ is a limit ordinal and $\text{lh}(t) = \eta$, then $t \in T$ iff $\forall \xi < \eta [t \upharpoonright \xi \in T]$ and $\exists q \in \mathbb{P} \forall \xi < \eta [q \leq F(t \upharpoonright \xi)]$; then choose $F(t)$ to be some such q .

T is clearly atomless, and T has the CTP because \mathbb{P} has the WCTP. If there are no Gregory trees, then T has an uncountable chain, so fix $g \in 2^{\omega_1}$ such that $g \upharpoonright \xi \in T$ for all $\xi < \omega_1$, and let $G = \{y \in \mathbb{P} : \exists \xi < \omega_1 [F(g \upharpoonright \xi) \leq y]\}$. ☕

Theorem 3.14 (Gregory [11]) *If $2^{\aleph_0} < 2^{\aleph_1}$ then there is a Gregory tree.*

Corollary 3.15 $\text{MA}_{\text{CTP}}(\aleph_1)$ *implies that $2^{\aleph_0} = 2^{\aleph_1}$.*

4 A Weird Space

We now prove Theorem 1.3. The basic construction is an inverse limit in ω_1 steps, and we follow approximately the terminology in [5, 12]. We build a compact space $X_{\omega_1} \subseteq [0, 1]^{\omega_1}$ by constructing inductively $X_\alpha \subseteq [0, 1]^{1+\alpha} \cong [0, 1] \times [0, 1]^\alpha$. Usually, one has $X_\alpha \subseteq [0, 1]^\alpha$ in these constructions, but for finite α , the notation will be slightly simpler if we start at stage 0 with $X_0 = [0, 1] = [0, 1]^1$; of course, $1 + \alpha = \alpha$ for infinite α .

Definition 4.1 $\pi_\alpha^\beta : [0, 1]^{1+\beta} \rightarrow [0, 1]^{1+\alpha}$ *is the natural projection.*

As usual, $\pi : X \rightarrow Y$ means that π is a *continuous* map from X onto Y . These constructions always have $\pi_\alpha^\beta(X_\beta) = X_\alpha$ whenever $0 \leq \alpha \leq \beta \leq \omega_1$. This determines X_γ for limit γ , so the meat of the construction involves describing how to build $X_{\alpha+1}$ given X_α .

A classical CH argument can ensure that X_{ω_1} is weird, but by Theorem 1.5, such an argument cannot make X_{ω_1} first countable. However, the same classical argument will let us construct a binary tree of spaces, resulting in a weird space $X_g \subseteq [0, 1]^{\omega_1}$ for each $g \in 2^{\omega_1}$. We shall show that if no X_g were first countable, then there would be a $(\mathfrak{c}, 2)$ -predictor $\Psi : [0, 1]^{<\omega_1} \rightarrow 2$; so CH ensures that some X_g is first countable.

Our tree will give us an X_p for each $p \in 2^{<\omega_1}$. We now list requirements (R1)(R2)(R3) \cdots (R17) on the construction; a proof that all the requirements can be satisfied, and that they yield a weird space, concludes this section. We begin with the requirements involving the inverse limit:

- R1. $X_{\mathbb{1}} = [0, 1]$, where $\mathbb{1}$ is the empty sequence.
- R2. X_p is an infinite closed connected subspace of $[0, 1]^{1+\text{lh}(p)}$.
- R3. $\pi_\alpha^\beta \upharpoonright X_p : X_p \rightarrow X_{p \upharpoonright \alpha}$, and is irreducible, whenever $\beta = \text{lh}(p) \geq \alpha$.

When $\gamma = \text{lh}(p) \leq \omega_1$ is a limit, (R2)(R3) force:

$$X_p = \{x \in [0, 1]^\gamma : \forall \alpha < \gamma [\pi_\alpha^\gamma(x) \in X_{p \upharpoonright \alpha}]\} \quad . \quad (\otimes)$$

To simplify notation for the restricted projection maps, we shall use:

Definition 4.2 *If $\beta = \text{lh}(p) \geq \alpha$ and $r = p \upharpoonright \alpha$, define $\pi_r^p = \pi_\alpha^\beta \upharpoonright X_p : X_p \rightarrow X_r$.*

As in [12], each of $X_{p \frown 0}$ and $X_{p \frown 1}$ is obtained from X_p as the graph of a “ $\sin(1/x)$ ” curve. We choose h_q, u_q , and v_q^n for $n < \omega$ and $q \in 2^{<\omega_1}$ of successor length, satisfying, for $i = 0, 1$:

- R4. $u_{p \frown i} \in X_p$ and $h_{p \frown i} \in C(X_p \setminus \{u_{p \frown i}\}, [0, 1])$ and $X_{p \frown i} = \overline{h_{p \frown i}}$.
- R5. $v_{p \frown i}^n \in X_p \setminus \{u_{p \frown i}\}$, and $\langle v_{p \frown i}^n : n \in \omega \rangle \rightarrow u_{p \frown i}$, and all points of $[0, 1]$ are limit points of $\langle h_{p \frown i}(v_{p \frown i}^n) : n \in \omega \rangle$.

As usual, we identify $h_{p \frown i}$ with its graph. So, if $\alpha = \text{lh}(p)$, then $X_{p \frown i}$ is a subset of $[0, 1]^{1+\alpha} \times [0, 1]$, which we identify with $[0, 1]^{1+\alpha+1}$. We shall say that the point $u_{p \frown i}$ gets *expanded* in the passage from X_p to $X_{p \frown i}$; the other points get *fixed*. (R3) follows from (R4) plus (\otimes) . Also, if $\delta < \alpha$, then $\pi_{p \upharpoonright \delta}^p : X_p \rightarrow X_{p \upharpoonright \delta}$, and $(\pi_{p \upharpoonright \delta}^p)^{-1}\{x\}$ is a singleton unless x is in the countable set $\{\pi_{p \upharpoonright \delta}^{p \upharpoonright \xi}(u_{p \upharpoonright (\xi+1)}) : \delta \leq \xi < \alpha\}$.

We now explain how points in $X_g \subset [0, 1]^{\omega_1}$ can predict g , in the sense of Definition 2.1. We shall get A_q and B_q for $q \in 2^{<\omega_1}$ of successor length, satisfying:

R6. For $i = 0, 1$: $A_{p \frown i}, B_{p \frown i} \subseteq X_p$ and $A_{p \frown i} = X_p \setminus B_{p \frown i}$.

R7. For $i = 0, 1$ and $\xi < \text{lh}(p)$: $A_{p \frown i} \supseteq (\pi_{p \upharpoonright \xi}^p)^{-1}(A_{p \upharpoonright (\xi+1)})$.

R8. $B_{p \frown 0} \cap B_{p \frown 1} = \emptyset$

R9. For $i = 0, 1$: $u_{p \frown i} \in B_{p \frown i}$.

Observe that some care must be exercised here in the inductive construction; otherwise, at some stage (R7) might imply that $A_{p \frown i} = X_p$, so that $B_{p \frown i} = \emptyset$, making (R9) impossible.

(R6)(R7)(R9) imply that points in $A_{p \frown i}$ are forever fixed in the passage from X_p to any future X_q with $q \leq p \frown i$; only points in $B_{p \frown i}$ can get expanded. Points which are forever fixed must wind up having countable character, and (R8) lets us use a point of uncountable character in X_g to predict g :

Lemma 4.3 *Assume that we have (R1 – R9), and assume that $2^{\aleph_0} < 2^{\aleph_1}$. Then X_g is first countable for some $g \in 2^{\omega_1}$.*

Proof. We shall define $\Psi : [0, 1]^{<\omega_1} \rightarrow 2$, and prove that Ψ is a $(\mathfrak{c}, 2)$ -predictor if every X_g contains a point of uncountable character.

Say $\text{lh}(p) = \alpha < \omega_1$ and $\delta < \alpha$. If $x \in B_{p \frown i} \subseteq X_p$ then, by (R6)(R7), $\pi_\delta^\alpha(x) \in B_{p \upharpoonright (\delta+1)} \subseteq X_{p \upharpoonright \delta}$. Applying (R8), if $x \in [0, 1]^{1+\alpha}$ and $x \in B_{p \frown i} \cap B_{r \frown j}$, then $p = r$ and $i = j$; to prove this, consider the least $\delta < \alpha$ such that $p \upharpoonright \delta \neq r \upharpoonright \delta$.

Set $\Psi(x) = 0$ if $\text{lh}(x) < \omega$. Now, say $x \in [0, 1]^\alpha$, where $\omega \leq \alpha < \omega_1$ (so $1 + \alpha = \alpha$). If there exist $p \in 2^\alpha$ and $i \in 2$ such that $x \in B_{p \frown i}$, then these p, i are unique, and set $\Psi(x) = i$. If there are no such p, i , then set $\Psi(x) = 0$.

Now, assume that for each g , we can find $z = z_g \in X_g$ with $\chi(z, X_g) = \aleph_1$. Let $C = \omega_1 \setminus \omega$. We shall show that Ψ, z predict g on C . For $\xi \in C$, let $p \frown i = g \upharpoonright (\xi+1)$. Then $z \upharpoonright \xi = \pi_p^g(z) \in X_p$, and $z \upharpoonright \xi$ must be in $B_{p \frown i}$, since if it were in $A_{p \frown i}$, then $(\pi_p^g)^{-1}(\pi_p^g(z)) = \{z\}$, so that $\chi(z, X_g) = \aleph_0$. Thus, $\Psi(z \upharpoonright \xi) = i = g(\xi)$. ☕

Since every X_g clearly has weight \aleph_1 , we are done if we can make every X_g weird. Since points in $A_{p \frown i}$ are forever fixed, we must make sure that $A_{p \frown i}$ has no Cantor subsets. Conditions (R6)(R8) say that $A_{p \frown 0} \cup A_{p \frown 1} = X_p$, so $A_{p \frown 0}$ and $A_{p \frown 1}$ must be Bernstein sets. Note that Condition (R7) may present a problem at limit stages. When $\text{lh}(p) = \alpha$ we have $A_{p \frown i} \supseteq \bigcup_{\xi < \alpha} (\pi_{p \upharpoonright \xi}^p)^{-1}(A_{p \upharpoonright (\xi+1)})$. Points in $A_{p \upharpoonright (\xi+1)}$ are forever fixed, so each $(\pi_{p \upharpoonright \xi}^p)^{-1}(A_{p \upharpoonright (\xi+1)})$ will have no Cantor subsets. Without further requirements, though, $\bigcup_{\xi < \alpha} (\pi_{p \upharpoonright \xi}^p)^{-1}(A_{p \upharpoonright (\xi+1)})$ may contain a Cantor subset. So, we make sure each such union is disjoint from some set in a tree of Bernstein sets:

Definition 4.4 For any topological space Y and $p \in 2^{<\omega_1}$, a Bernstein tree in Y rooted in p is a family of subsets of Y , $\{D^q : q \leq p\}$, satisfying:

1. For each q , neither D^q nor $Y \setminus D^q$ contains a Cantor subset.
2. Each $D^{q \frown 0} \cap D^{q \frown 1} = \emptyset$.
3. If $r \leq q$ then $D^r \subseteq D^q$.

Note that if Y itself does not contain a Cantor subset, then (1) is trivial, and we may take all $D^q = \emptyset$ to satisfy (2) and (3).

Now, in our construction, we also build D_p^q for $q \leq p \in 2^{<\omega_1}$ satisfying:

- R10. For each $p \in 2^{<\omega_1}$: $\{D_p^q : q \leq p\}$ is a Bernstein tree in X_p rooted in p .
- R11. If $q \leq p \leq r$ and $\pi = \pi_r^p : X_p \twoheadrightarrow X_r$ and $x \in X_p$ with $\pi^{-1}(\pi(x)) = \{x\}$, then $x \in D_p^q$ iff $\pi(x) \in D_r^q$.
- R12. For each $p \in 2^{<\omega_1}$ and $i \in 2$: $B_{p \frown i} = D_p^{p \frown i}$ and $A_{p \frown i} = X_p \setminus D_p^{p \frown i}$.

Of course, (R12) simply defines $A_{p \frown i}$ in terms of the D_p^q , and then (R10) guarantees that no $A_{p \frown i}$ has a Cantor subset, but we need to verify that the conditions (R1 – R12) can indeed be satisfied. First, three easy lemmas about Bernstein trees. A standard inductive construction in \mathfrak{c} steps shows:

Lemma 4.5 *If Y is a separable metric space, then there is a Bernstein tree in Y rooted in $\mathbb{1}$.*

Using the fact that every uncountable Borel subset of the Cantor set contains a perfect subset, we get:

Lemma 4.6 *Assume that Y is any topological space, Z is a Borel subset of Y , and $\{D^q : q \leq p\}$ is a family of subsets of Y satisfying (2)(3) of Definition 4.4. Then $\{D^q : q \leq p\}$ is a Bernstein tree in Y iff both $\{D^q \cap Z : q \leq p\}$ is a Bernstein tree in Z and $\{D^q \setminus Z : q \leq p\}$ is a Bernstein tree in $Y \setminus Z$.*

Combining these two lemmas:

Lemma 4.7 *If Y is a separable metric space, Z is a Borel subset of Y , and $\{E^q : q \leq p\}$ is a Bernstein tree in Z rooted in p , then there is a Bernstein tree $\{D^q : q \leq p\}$ in Y rooted in p such that each $D^q \cap Z = E^q$.*

Returning to the construction:

Lemma 4.8 *There exist X_p for $p \in 2^{<\omega_1}$ satisfying Conditions (R1 – R12).*

Proof. We start with $X_{\mathbb{1}} = [0, 1]$, and we obtain the $D_{\mathbb{1}}^q$ by applying Lemma 4.5.

If $\alpha = \text{lh}(p) > 0$ and we have done the construction for $p \upharpoonright \xi$ for all $\xi < \text{lh}(p)$, then X_p is determined either by (R4) when $\text{lh}(p)$ is a successor or by (\clubsuit) when $\text{lh}(p)$ is a limit. If $\alpha < \omega_1$, we construct the D_p^q to satisfy (R10)(R11) as follows: For $\xi < \alpha$, use π_ξ for $\pi_{p \upharpoonright \xi}^p$. Let $Z_\xi = \{x \in X_p : \pi_\xi^{-1}(\pi_\xi(x)) = \{x\}\}$, and let $Z = \bigcup_{\xi < \alpha} Z_\xi$. Observe that Z and all the Z_ξ are Borel sets. Let $\{E_p^q : q \leq p\}$ be the Bernstein tree in Z rooted in p defined by saying that for $x \in Z_\xi$: $x \in E_p^q$ iff $\pi_\xi(x) \in D_{p \upharpoonright \xi}^q$. Note that, by (R11) applied inductively, this is independent of which ξ is used. To obtain the D_p^q from the E_p^q , apply Lemma 4.7. Note that, by (R11) applied inductively once again, these D_p^q work for X_p .

The $A_{p \frown i}$ and $B_{p \frown i}$ (for $i = 0, 1$) are now defined by (R12), and we must verify that this definition satisfies (R7): Assume that $\xi < \text{lh}(p) = \alpha$ and $x \in X_p$ and $\pi_\xi(x) \in A_{p \upharpoonright (\xi+1)}$. We must show that $x \in A_{p \frown i}$; equivalently, by (R12), that $x \notin D_p^{p \frown i}$. Now $\pi_\xi(x) \in A_{p \upharpoonright (\xi+1)}$ implies that $\pi_\xi^{-1}(\pi_\xi(x)) = \{x\}$ (using (R4 – R9) inductively), so that $x \notin D_p^{p \frown i}$ iff $\pi_\xi(x) \notin D_{p \upharpoonright \xi}^{p \frown i}$. By (R10) for $p \upharpoonright \xi$ and Definition 4.4(3), $D_{p \upharpoonright \xi}^{p \frown i} \subseteq D_{p \upharpoonright \xi}^{p \upharpoonright (\xi+1)}$. So $A_{p \upharpoonright (\xi+1)} = X_{p \upharpoonright \xi} \setminus D_{p \upharpoonright \xi}^{p \upharpoonright (\xi+1)}$ gives us (R7).

Since the $B_{p \frown i}$ are nonempty, there is no problem choosing the $u_{p \frown i}$, $v_{p \frown i}^n$, and $h_{p \frown i}$ to satisfy (R4)(R5)(R9), and then the $X_{p \frown i}$ are defined by (R4). \clubsuit

Finally, we must make each X_g weird. Observe:

Lemma 4.9 *Conditions (R1 – R5) imply that if $F \subseteq X_p$ is closed and connected then $(\pi_p^q)^{-1}(F)$ is connected for all $q \leq p$.*

Now, we shall make sure that whenever F is a perfect subset of X_g , there is some $\alpha < \omega_1$ such that $(\pi_{g \upharpoonright (\alpha+1)}^g)^{-1}(\{u_{g \upharpoonright (\alpha+1)}\} \times [0, 1]) \subseteq F$ (recall that our construction gave us $\{u_{g \upharpoonright (\alpha+1)}\} \times [0, 1] \subset X_{g \upharpoonright (\alpha+1)} \subset X_{g \upharpoonright \alpha} \times [0, 1]$). By Lemma 4.9, this implies that F is not totally disconnected. The argument in [12] obtained this α by using \diamond to capture F . Here, we replace this use of \diamond by a classical CH argument. First, as in [12], construct \mathcal{F}_p for $p \in 2^{<\omega_1}$ so that:

R13. \mathcal{F}_p is a countable family of uncountable closed subsets of X_p .

R14. If $F \in \mathcal{F}_p$ and $q \leq p$ then $(\pi_p^q)^{-1}(F) \in \mathcal{F}_q$.

R15. For each $F \in \mathcal{F}_p$, either $u_{p \frown i} \notin F$, or $u_{p \frown i} \in F$ and $v_{p \frown i}^n \in F$ for all but finitely many n .

R16. $\{u_{p \frown i}\} \times [0, 1] \in \mathcal{F}_{p \frown i}$.

We may satisfy (R13)(R14)(R16) simply by defining

$$\mathcal{F}_p = \{(\pi_p^p|_\xi)^{-1}\{u_{p \upharpoonright (\xi+1)}\} : \xi < \text{lh}(p)\} .$$

Requirements (R4)(R14)(R15) imply:

Lemma 4.10 $\pi_p^q : (\pi_p^q)^{-1}(F) \rightarrow F$ is irreducible whenever $F \in \mathcal{F}_p$ and $q \leq p$.

Then, we use CH rather than \diamond to get:

R17. Whenever $p \in 2^{<\omega_1}$ and F is an uncountable closed subset of X_p , there is a β with $\text{lh}(p) < \beta < \omega_1$ such that for all $q < p$ with $\text{lh}(q) = \beta$ and for each $x \in \{u_{q \frown 0}, u_{q \frown 1}\} \cup \{v_{q \frown i}^n : n \in \omega \ \& \ i \in 2\}$, the projections $\pi = \pi_p^q$ satisfy $\pi(x) \in F$ and $|\pi^{-1}(\pi(x))| = 1$.

Proof of Theorem 1.3. Assuming that we can obtain (R1 – R17), note that each X_g is separable, because each $\pi_{\mathbb{1}}^g : X_g \rightarrow X_{\mathbb{1}}$ is irreducible. Then, to finish, by Lemma 4.3, it suffices to show that each X_g is weird. Fix a perfect $H \subseteq X_g$; we shall show that it is not totally disconnected. First, fix $\alpha < \omega_1$ such that, if we set $p = g \upharpoonright \alpha$ and $F = \pi_p^g(H)$, then F is perfect (the set of all such α form a club). Then, fix $\beta > \alpha$ as in (R17), let $q = g \upharpoonright \beta$, and let $i = g(\beta)$, so that $q \frown i = g \upharpoonright (\beta + 1)$. Let $K = (\pi_p^q)^{-1}(F)$. Then $\pi_q^g(H) \subseteq K$, and this inclusion may well be proper. However, $u_{q \frown i} \in \pi_q^g(H)$ and $v_{q \frown i}^n \in \pi_q^g(H)$ for each $n \in \omega$ because $\pi(u_{q \frown i}) \in F$ and $\pi(v_{q \frown i}^n) \in F$ and $|\pi^{-1}(\pi(u_{q \frown i}))| = |\pi^{-1}(\pi(v_{q \frown i}^n))| = 1$. It follows (using (R5)) that $E := \{u_{q \frown i}\} \times [0, 1] \subseteq \pi_q^g(H)$. Since $E \in \mathcal{F}_{q \frown i}$ by (R16) and H maps onto E , Lemma 4.10 implies that $(\pi_q^g)^{-1}(E) \subseteq H$. Since $(\pi_q^g)^{-1}(E)$ is connected by Lemma 4.9, H cannot be totally disconnected.

Next, to obtain conditions (R1 – R17), we must augment the proof of Lemma 4.8: Fix in advance a map ψ from $\omega_1 \setminus \{0\}$ onto $\omega_1 \times \omega_1$, such that $\alpha < \beta$ whenever $\psi(\beta) = (\alpha, \xi)$. Now, given X_p , use CH and let $\{F_\xi^p : \xi < \omega_1\}$ be a listing of all uncountable closed subsets of X_p . Whenever $0 < \beta < \omega_1$ and $\psi(\beta) = (\alpha, \xi)$ and $\text{lh}(q) = \beta$, we may set $p = q \upharpoonright \alpha$ and $F = F_\xi^p \subseteq X_p$. It is sufficient to show how to accomplish (R17) with these specific α, β, p, q, F .

Choose a perfect $K \subset F$ which is disjoint from $\{\pi_p^{(q \upharpoonright \zeta)}(u_{q \upharpoonright (\zeta+1)}) : \alpha \leq \zeta < \beta\}$. Then π_p^q is 1-1 on $(\pi_p^q)^{-1}(K)$, so choosing all $u_{q \frown i}$ and $v_{q \frown i}^n$ in $(\pi_p^q)^{-1}(K)$ will ensure (R17). Now fix $i \in 2$, and write u and v^n for $u_{q \frown i}$ and $v_{q \frown i}^n$. To ensure (R15) and (R9), we modify the argument of [12]. Let $\{Q^n : n \in \omega\}$ list \mathcal{F}_q . Let d be a metric on $(\pi_p^q)^{-1}(K)$. For each $s \in 2^{<\omega}$, choose a perfect $L_s \subseteq (\pi_p^q)^{-1}(K)$. Make these into a tree, in the sense that each $L_{s \frown 0} \cap L_{s \frown 1} = \emptyset$, each $\text{diam}(L_s) \leq 2^{-\text{lh}(s)}$, and

$L_{s \smallfrown 0} \subseteq L_s$ and $L_{s \smallfrown 1} \subseteq L_s$. Also make sure that whenever $\text{lh}(s) = n + 1$ we have either $L_s \subseteq Q^n$ or $L_s \cap Q^n = \emptyset$. Let $v[s \smallfrown \ell]$ be any point in $L_{s \smallfrown \ell} \setminus L_s$. For $f \in 2^\omega$, let $\{u[f]\} = \bigcap_n L_{f \upharpoonright n}$. For any $f \in 2^\omega$, if we set $u = u[f]$ and $v^n = v[f \upharpoonright (n + 1)]$, then (R15) will hold. Now, (R9) requires $u \in B_{q \smallfrown i}$. Since $B_{q \smallfrown i}$ is a Bernstein set and $\{u[f] : f \in 2^\omega\}$ is a Cantor set, we may choose f so that $u[f] \in B_{q \smallfrown i}$. ☕

If $H \subseteq X_g$ is closed and for some initial segment $p = g \upharpoonright \alpha$ the projection $\pi_p^g(H) \in \mathcal{F}_p$, then, by irreducibility, $H = (\pi_p^g)^{-1}(\pi_p^g(H))$, so that H is a G_δ . To make X_g hereditarily Lindelöf, it suffices to capture projections for each closed $H \subseteq X_g$ this way, but it is not clear whether this can be done without using \diamond .

5 Remarks and Examples

One cannot replace “CH” by “ $2^{\aleph_0} < 2^{\aleph_1}$ ” in the statement of Theorem 1.3, since by Proposition 5.3, it is consistent with any cardinal arithmetic that every non-scattered compactum of weight less than \mathfrak{c} contains a copy of the Cantor set. As usual, define

Definition 5.1 $\text{cov}(\mathcal{M})$ is the least κ such that \mathbb{R} is the union of κ meager sets.

Note that $\text{cov}(\mathcal{M})$ is the least κ such that MA(κ) for countable partial orders fails. Using this, we easily see:

Lemma 5.2 If $\kappa < \text{cov}(\mathcal{M})$ and $E_\alpha \subset [0, 1]$ is meager for each $\alpha < \kappa$, then $[0, 1] \setminus \bigcup_{\alpha < \kappa} E_\alpha$ contains a copy of the Cantor set.

Proposition 5.3 If X is compact and not scattered, and $w(X) < \text{cov}(\mathcal{M})$, then X contains a copy of the Cantor set.

Proof. Replacing X by a subspace, we may assume that we have an irreducible map $\pi : X \rightarrow [0, 1]$. Let \mathcal{B} be an open base for X with $|\mathcal{B}| < \text{cov}(\mathcal{M})$ and $\emptyset \neq \mathcal{B}$.

Whenever $U, V \in \mathcal{B}$ with $\overline{U} \cap \overline{V} = \emptyset$, let $E_{U,V} = \pi(\overline{U}) \cap \pi(\overline{V})$. Then $E_{U,V} \subset [0, 1]$ is nowhere dense because π is irreducible. Applying Lemma 5.2, let $K \subset [0, 1]$ be a copy of the Cantor set disjoint from all the $E_{U,V}$. Note that $|\pi^{-1}\{y\}| = 1$ for all $y \in K$. Thus, $\pi^{-1}(K)$ is homeomorphic to K . ☕

Note that one can force “ $\text{cov}(\mathcal{M}) = \mathfrak{c}$ ” by adding \mathfrak{c} Cohen reals, which does not change cardinal arithmetic, but in the statement of Proposition 5.3, “ $\text{cov}(\mathcal{M})$ ” cannot be replaced by “ \mathfrak{c} ”. If CH holds in V , then one may force \mathfrak{c} to be arbitrarily large by adding random reals, and any random real extension $V[G]$ will have

a compact non-scattered space of weight \aleph_1 which does not contain a Cantor subset. In fact, Dow and Fremlin [4] show that if X is a compact F-space in V , then in a random real extension $V[G]$, the corresponding compact space \tilde{X} has no convergent ω -sequences, and hence no Cantor subsets.

The weird space constructed in [12] also failed to satisfy the CSWP (the complex version of the Stone–Weierstrass Theorem). Using the method there, we can modify the proof of Theorem 1.3 to get:

Theorem 5.4 *Assuming CH, there is a separable first countable connected weird space X of weight \aleph_1 such that X fails the CSWP and \mathbb{K}_X fails the CTP.*

Proof. First, in the proof of Theorem 1.3, replace $[0, 1]$ by \overline{D} , the closed unit disc in the complex plane, so that we may view X as a subspace of the \aleph_1 -dimensional polydisc. Then, as in [12], by carefully choosing the functions $h_{p \frown i}$, one can ensure that the restriction to X of the natural analog of the disc algebra refutes the CSWP of X . To refute the CTP, construct in \overline{D} a Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{K}_{\overline{D}}$ such that each p_s is a wedge of the disc with center 0 and radius $2^{-\text{lh}(s)}$; then each $\bigcap_{n \in \omega} p_{f \upharpoonright n} = \{0\}$. Then, since we may assume the point 0 is not expanded in the construction of X , the inverse images of the p_s yield a counterexample to the CTP of \mathbb{K}_X . ☕

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