

Orthogonal Continuous Functions

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Abstract

We consider the question of whether there is an orthonormal basis for L^2 consisting of continuous functions.

1 Introduction

In elementary analysis, the typical orthonormal bases for $L^2[0, 1]$ (trig functions, orthogonal polynomials, etc.) frequently consist of continuous functions. It is natural to ask whether such orthonormal bases must exist if $[0, 1]$ is replaced by a more general space and measure. One commonly studied generalization of $[0, 1]$ is:

Definition 1.1 *(X, ν) is a nice measure space iff X is a compact Hausdorff space and ν is a regular Borel probability measure on X which is strictly positive (i.e., all non-empty open sets have positive measure).*

The assumption that ν is strictly positive is mainly for notational convenience. In general, one can simply delete the union of all open null sets to obtain a strictly positive measure.

Since ν is strictly positive, distinct elements of $C(X)$ do not become equivalent in L^2 , so we may regard $C(X)$ as contained in $L^2(X, \nu)$. There

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are then two well-known situations where there is an $\mathcal{F} \subseteq C(X)$ which forms an orthonormal basis for $L^2(X, \nu)$. The first is whenever $L^2(X, \nu)$ is separable (by Gram-Schmidt). The second is when X is a compact group and ν is Haar measure (by the Peter-Weyl Theorem; see, e.g., Folland [1]). However, there need not be such an \mathcal{F} in general, since Theorem 3.7 provides an example where $L^2(X, \nu)$ is not separable but any orthogonal family from $C(X)$ is countable.

In the example of Theorem 3.7, X is actually a topological group, since it is a product of two-element spaces, and ν looks a bit like the product measure, which in this case would be Haar measure. Nevertheless, by Theorem 2.5, no such ν can be absolutely continuous with respect to Haar measure.

The proof for the specific example of Theorem 3.7 works equally well whether one considers the scalar field to be \mathbb{R} or \mathbb{C} . However, if one starts with an arbitrary nice (X, ν) , it is reasonable to ask whether the properties discussed here can depend on the scalar field. They do not, as we show in Corollary 2.2. Of course, any orthogonal family of real-valued functions remains orthogonal when viewed as members of $L^2(X, \nu, \mathbb{C})$, but Corollary 2.2 explains how to replace orthogonal complex-valued functions by real-valued ones. The familiar method from Fourier series replaces φ and $\bar{\varphi}$ by $(\varphi + \bar{\varphi})/\sqrt{2}$ and $(\varphi - \bar{\varphi})/(i\sqrt{2})$, but this requires assuming that $\varphi \in \mathcal{F} \iff \bar{\varphi} \in \mathcal{F}$.

One might study the following property of X : For every finite regular Borel measure ν on X , there is an $\mathcal{F} \subseteq C(X)$ which forms an orthonormal basis for $L^2(X, \nu)$. We do not know whether this is equivalent to some interesting topological property of X . Note that every compact F-space and every compact metric space has this property.

2 Basics

Throughout, when discussing $C(X)$ and $L^2(X, \nu)$ and general Hilbert spaces, we *always* presume that the scalar field is the complex numbers. We shall show that we can convert a family of orthogonal continuous functions to a family of real-valued orthogonal continuous functions with the same span. To do this, we use the following lemma about Hilbert spaces, which gives us a uniform way to transform an “almost orthogonal” family to an orthogonal one:

Lemma 2.1 *Suppose that \mathcal{H} is a Hilbert space and $\mathcal{E} \subseteq \mathcal{H}$ is such that the closed linear span of \mathcal{E} is all of \mathcal{H} and $\{g \in \mathcal{E} : (g, f) \neq 0\}$ is countable for all $f \in \mathcal{E}$. Then there is an orthonormal basis \mathcal{F} for \mathcal{H} such that every element of \mathcal{F} is a finite linear combination of elements of \mathcal{E} . Furthermore, the coefficients in these linear combinations will all be real whenever the (g, f) , for $g, f \in \mathcal{E}$, are all real.*

Proof. On \mathcal{E} , let \sim be the smallest equivalence relation such that $g \sim f$ whenever $(g, f) \neq 0$. Let \mathcal{E}_j , for $j \in J$, list all the \sim equivalence classes. Then the \mathcal{E}_j are all countable, and are pairwise orthogonal. For each j , apply Gram-Schmidt to obtain an orthonormal family \mathcal{F}_j with the same linear span, such that the elements of \mathcal{F}_j are linear combinations of elements of \mathcal{E}_j . Then, let $\mathcal{F} = \bigcup_j \mathcal{E}_j$. ☺

Corollary 2.2 *Suppose that (X, ν) is a nice measure space and $\mathcal{G} \subseteq C(X)$ is an orthonormal family. Then there is an orthonormal family $\mathcal{F} \subseteq C(X)$, consisting of real-valued functions, such that the closed linear span of \mathcal{F} contains the closed linear span of \mathcal{G} .*

Proof. As usual, write each $G \in \mathcal{G}$ as $G = \Re(G) + i\Im(G)$, where $\Re(G)$ and $\Im(G)$ are real-valued functions. Let $\mathcal{E} = \{\Re(G) : G \in \mathcal{G}\} \cup \{\Im(G) : G \in \mathcal{G}\}$. Then the closed linear span \mathcal{H} of \mathcal{E} contains \mathcal{G} , so Lemma 2.1 will apply if we can verify that $\{g \in \mathcal{E} : (g, f) \neq 0\}$, for any $f \in \mathcal{E}$, is countable. To see this, apply Bessel's inequality: $\sum_{G \in \mathcal{G}} |(G, f)|^2 \leq \|f\|^2$. Since f is real-valued, $|(G, f)|^2 = (\Re(G), f)^2 + (\Im(G), f)^2$, so that $(\Re(G), f) = (\Im(G), f) = 0$ for all but countably many $G \in \mathcal{G}$. ☺

In particular, if \mathcal{G} is an orthonormal basis, we may replace \mathcal{G} by a real-valued orthonormal basis \mathcal{F} . Or, if \mathcal{G} is an uncountable orthonormal family, then \mathcal{F} will be a real-valued uncountable orthonormal family. So, the properties of (X, ν) considered in this paper do not depend on the scalar field.

The next definition and lemma give us a way of ensuring that there are no uncountable orthonormal families within $C(X)$:

Definition 2.3 *We say $\mathcal{F} \subseteq C(X)$ is maximal orthogonal iff \mathcal{F} is orthogonal in $L^2(X, \nu)$ and there is no orthogonal \mathcal{G} with $\mathcal{F} \subsetneq \mathcal{G} \subseteq C(X)$.*

Observe that even in $L^2([0, 1])$, a maximal orthogonal $\mathcal{F} \subseteq C([0, 1])$ need not be an orthogonal basis for $L^2([0, 1])$; for example, its closed linear span may be the orthogonal complement of a step function. Nevertheless,

Lemma 2.4 *Suppose (X, ν) is a nice measure space, and assume that there is a maximal orthogonal $\mathcal{F} \subseteq C(X)$ which is countable. Then every orthogonal $\mathcal{G} \subseteq C(X)$ is countable.*

Proof. Let \mathcal{F} and \mathcal{G} be any two orthogonal families contained in $C(X)$. For each fixed $f \in \mathcal{F}$, Bessel's Inequality implies that $g \perp f$ for all but countably many $g \in \mathcal{G}$. Hence, if \mathcal{F} is countable and maximal, then \mathcal{G} must be countable also. ☺

Now, the existence of an uncountable orthogonal family contained in $C(X)$ depends on ν , not just X , as the example in Section 3 shows. However,

Theorem 2.5 *Suppose that (X, ν) and (X, μ) are nice measure spaces with $\mu \ll \nu$. Suppose that $\mathcal{G} \subseteq C(X)$ is an orthonormal basis for $L^2(X, \nu)$. Then there is an $\mathcal{F} \subseteq C(X)$ which is an orthonormal basis for $L^2(X, \mu)$.*

Proof. Fix a Baire-measurable $\varphi : X \rightarrow [0, \infty)$ such that $\mu(E) = \int_E \varphi(x) d\nu(x)$ for all Borel sets E . Then $\int \varphi d\nu = 1$, but φ need not be bounded, in which case \mathcal{G} might fail to span $L^2(X, \mu)$.

Choose closed G_δ sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ such that $\varphi(x) \leq n$ for all $x \in K_n$ and $\nu(X \setminus \bigcup_n K_n) = 0$. For each n , choose $\psi_n \in C(X, [0, 1])$ such that $K_n = \psi_n^{-1}\{1\}$, and note that the sequence of functions $(\psi_n)^m$ converges pointwise to χ_{K_n} as $m \rightarrow \infty$.

Let \mathcal{E} be the set of all functions of the form $g \cdot (\psi_n)^m$, where $g \in \mathcal{G}$ and $m, n \in \mathbb{N}$. Then $\mathcal{E} \subseteq C(X) \subseteq L^2(X, \mu)$. Let \mathcal{H} be the closed linear span of \mathcal{E} in $L^2(X, \mu)$. Then $\mathcal{H} = L^2(X, \mu)$: To see this, first note that $g \cdot \chi_{K_n} \in \mathcal{H}$ for $g \in \mathcal{G}$. Then, if $h \in C(X)$, each $h \cdot \chi_{K_n} \in \mathcal{H}$ (since φ is bounded on K_n), but this implies that $h \in \mathcal{H}$. Now, use the fact that $C(X)$ is dense in $L^2(X)$.

The result will now follow by Lemma 2.1 if we can verify, for each $f \in \mathcal{G}$ and each $m, n, p, q \in \mathbb{N}$, $\{g \in \mathcal{G} : (g(\psi_n)^m, f(\psi_p)^q)_\mu \neq 0\}$ is countable. Now for each $r \in \mathbb{N}$, Bessel's Inequality (applied in $L^2(X, \nu)$) implies that $\int g(\psi_n)^m \bar{f}(\psi_p)^q \chi_{K_r} \varphi d\nu = 0$ for all but countably many $g \in \mathcal{G}$, since the function $(\psi_n)^m \bar{f}(\psi_p)^q \chi_{K_r} \varphi$ is in $L^2(X, \nu)$. It follows that $(g(\psi_n)^m, f(\psi_p)^q)_\mu = \int g(\psi_n)^m \bar{f}(\psi_p)^q \varphi d\nu = 0$ for all but countably many $g \in \mathcal{G}$. ☺

3 Small Orthogonal Families

We shall build a large nice (X, ν) in which every orthogonal family of continuous functions is countable. In order to do this, we apply Lemma 2.4; it

is enough to obtain some countable maximal $\mathcal{F} \subseteq C(X)$. Again, we shall, for definiteness, assume that the scalar field is \mathbb{C} . \mathcal{F} will be obtained by projecting X onto a small space M , for which we use the following notation:

Definition 3.1 (X, ν, Γ, M) is a nice quadruple iff (X, ν) is a nice measure space and Γ is a continuous map onto the compact Hausdorff space M . In this case, let $\mu = \nu\Gamma^{-1}$ be the induced measure on M . We regard $L^2(M, \mu)$ as contained in $L^2(X, \nu)$ via the inclusion Γ^* (where $\Gamma^*(g) = g \circ \Gamma$). Let Π_Γ be the orthogonal projection from $L^2(X, \nu)$ onto $L^2(M, \mu)$. If $f \in L^2(X, \nu)$, we say $f \perp L^2(M, \mu)$ iff $\Pi_\Gamma(f) = 0$.

Lemma 3.2 In the notation of Definition 3.1, if $f \in L^2(X, \nu)$ then the following are equivalent:

1. $f \perp L^2(M, \mu)$.
2. $\int_{\Gamma^{-1}K} f(x) d\nu(x) = 0$ for all closed $K \subseteq M$.

Definition 3.3 The nice quadruple (X, ν, Γ, M) is injective iff Π_Γ is 1-1 on $C(X)$.

Lemma 3.4 In the notation of Definition 3.1, the following are equivalent:

1. (X, ν, Γ, M) is injective.
2. For all $f \in C(X)$, if $f \perp L^2(M, \mu)$, then $f \equiv 0$.

Lemma 3.5 Let (X, ν) be a nice measure space. Then the following are equivalent:

1. Every orthogonal subfamily of $C(X)$ is countable.
2. There is a continuous map Γ onto a compact second countable space M such that (X, ν, Γ, M) is injective.

Proof. (2) \rightarrow (1): Assuming (2), let $\mathcal{F} \subseteq C(M)$ be an orthonormal basis for $L^2(M)$. Then $\Gamma^*(\mathcal{F}) \cup \{0\} \subseteq C(X)$, and is maximal orthogonal, so (1) follows by Lemma 2.4.

(1) \rightarrow (2): Again by Lemma 2.4, let $\{f_n : n \in \mathbb{N}\} \subseteq C(X)$ be maximal orthogonal. Let $\Gamma : X \rightarrow \mathbb{C}^\mathbb{N}$ be the product map: $(\Gamma(x))_n = f_n(x)$. Let M be the range of Γ . Observe that a non-zero $g \in C(X)$ with $\Pi_\Gamma(g) = 0$ would contradict maximality. ☹

The next lemma explains how we obtain the situation of Lemma 3.5.2:

Lemma 3.6 *Let (X, ν, Γ, M) be a nice quadruple. Assume, for some fixed $\epsilon > 0$, we have: Whenever $W \subseteq X$ is open and non-empty, there is a closed $K \subseteq M$ such that $\mu(K) > 0$ and $\nu(\Gamma^{-1}(K) \cap W) \geq (\frac{1}{2} + \epsilon)\mu(K)$. Then (X, ν, Γ, M) is injective.*

Proof. Suppose $f \in C(X)$ is non-zero and satisfies $f \perp L^2(M, \mu)$. We may assume that $\|f\|_{\text{sup}} = 1$, and that some $f(x) = 1$. For any $\delta > 0$, we may choose a non-empty open $W \subseteq X$ such that $|f(x) - 1| \leq \delta$ for all $x \in W$, and then choose K as above. Applying $f \perp L^2(M, \mu)$ to the characteristic function of K , we have $\int_{\Gamma^{-1}K} f(x) d\nu(x) = 0$, so that $|\int_{\Gamma^{-1}K \cap W} f| = |\int_{\Gamma^{-1}K \setminus W} f|$. Note that $\mu(K) = \nu(\Gamma^{-1}K)$, so that $\nu(\Gamma^{-1}K \setminus W) \leq (\frac{1}{2} - \epsilon)\mu(K)$. So, we have:

$$\begin{aligned} \left| \int_{\Gamma^{-1}K \cap W} f \right| &\geq \nu(\Gamma^{-1}K \cap W)(1 - \delta) \geq \left(\frac{1}{2} + \epsilon\right)\mu(K)(1 - \delta) \\ \left| \int_{\Gamma^{-1}K \setminus W} f \right| &\leq \nu(\Gamma^{-1}K \setminus W) \leq \left(\frac{1}{2} - \epsilon\right)\mu(K) \end{aligned}$$

So, $(\frac{1}{2} + \epsilon)(1 - \delta) \leq (\frac{1}{2} - \epsilon)$. Letting $\delta \searrow 0$, we have a contradiction. $\odot\odot$

Note that if $\epsilon = 0$, the lemma could fail; consider $X = M \times 2$, with the product measure.

In general, the *Maharam dimension* of a measure ν is the cardinality of an orthonormal basis for $L^2(\nu)$; ν is called *Maharam-homogeneous* iff there is no set K of positive measure such that the dimension of ν restricted to K is less than the dimension of ν . As usual, $\mathfrak{c} = 2^{\aleph_0}$.

Theorem 3.7 *There is a strictly positive regular Borel probability measure ν on $2^{\mathfrak{c}}$ (i.e., $\{0, 1\}^{\mathfrak{c}}$, with the usual product topology) such that*

1. ν is Maharam-homogeneous of dimension \mathfrak{c} .
2. $L^2(2^{\mathfrak{c}}, \nu)$ contains no uncountable orthogonal family of continuous functions.

Proof. Let $M = 2^{\aleph}$, with μ the usual product measure. Let $X = M \times 2^{\mathfrak{c}}$, and let $\Gamma : X \rightarrow M$ be projection. We shall build ν on X , which is homeomorphic to $2^{\mathfrak{c}}$.

Let $\{d_m : m \in \mathbb{N}\}$ be dense in $(0, 1)^{\mathfrak{c}}$. For each m , let λ_m be the product measure on $2^{\mathfrak{c}}$ obtained by flipping unfair coins with bias d_m . That is, let $d_m^1(\alpha) = d_m(\alpha)$ and $d_m^0(\alpha) = 1 - d_m(\alpha)$. If

$$B = \{v \in 2^{\mathfrak{c}} : v(\alpha_1) = \ell_1 \& \cdots \& v(\alpha_r) = \ell_r\} \quad (1)$$

is a basic clopen set, then $\lambda_m(B) = \prod_{j=1}^r d_m^{\ell_j}(\alpha_j)$.

List all non-empty clopen subsets of M as $\{U_n : n \in \mathbb{N}\}$. Then, choose closed nowhere dense $K_{m,n} \subseteq U_n$ so that the $K_{m,n}$ for $m, n \in \mathbb{N}$ are all disjoint, each $\mu(K_{m,n}) > 0$, and $\sum_{m,n} \mu(K_{m,n}) = 1$. Finally, let ν on $M \times 2^{\mathfrak{c}}$ be the sum of the product measures $(\mu \upharpoonright K_{m,n}) \times \lambda_m$, so that for Borel $E \subseteq M \times 2^{\mathfrak{c}}$,

$$\nu(E) = \sum_{m,n} \int_{K_{m,n}} \lambda_m(E_x) d\mu(x) \ .$$

We are now done if we can verify the hypotheses of Lemma 3.6 We actually show that whenever $W \subseteq X$ is open and non-empty and $\epsilon > 0$, there is a closed $K \subseteq M$ such that $\mu(K) > 0$ and $\nu(\Gamma^{-1}(K) \cap W) \geq (1 - \epsilon)\mu(K)$. To do this, we may assume that $W = U_n \times B$, where B is as in (1) above. K will be $K_{m,n}$ for a suitable m . Then $\nu(\Gamma^{-1}(K) \cap W) = \nu(K_{m,n} \times B) = \mu(K_{m,n}) \prod_{j=1}^r d_m^{\ell_j}(\alpha_j)$. We thus only need choose m so that $\prod_{j=1}^r d_m^{\ell_j}(\alpha_j) \geq (1 - \epsilon)$, which is certainly possible since $\{d_m : m \in \mathbb{N}\}$ is dense in $(0, 1)^{\mathfrak{c}}$. ☺

Finally, we remark that this example is as large as possible, since if $|C(X)| > \mathfrak{c}$, then there is an uncountable orthogonal family by Lemma 3.5. (Note that whenever X is an infinite compact Hausdorff space, $|C(X)| = w(X)^{\aleph_0}$, where $w(X)$ is the weight of X (the least size of a base for the topology)). However, one can construct arbitrarily large examples with no continuous orthonormal bases by applying:

Theorem 3.8 *Suppose that (X, ν) and (Y, ρ) are both nice measure spaces, and there is an orthonormal basis for $L^2(X \times Y, \nu \times \rho)$ contained in $C(X \times Y)$. Then there are orthonormal bases for $L^2(X, \nu), L^2(Y, \rho)$ contained in $C(X), C(Y)$, respectively.*

Proof. Let $\mathcal{G} \subseteq C(X \times Y)$ be an orthonormal basis for $L^2(X \times Y, \nu \times \rho)$. To produce a basis for $L^2(X, \nu)$, let $\Gamma : X \times Y \rightarrow X$ be projection, and apply Lemma 2.1, with $\mathcal{E} = \Pi_{\Gamma}(\mathcal{G}) \subseteq \mathcal{H} = L^2(X, \nu)$ (regarding $L^2(X)$ as contained in $L^2(X \times Y)$, as in Definition 3.1).

First, note that the closed linear span of \mathcal{E} will be all of $L^2(X)$, because the closed linear span of \mathcal{G} is $L^2(X \times Y)$ and Π_Γ is orthogonal projection.

Next, observe that for each $G \in \mathcal{G}$, $\Pi_\Gamma(G) = g$, where $g(x) = \int G(x, y) dy$. To see this, note that since G is continuous, $g \in C(X) \subseteq L^2(X)$. Also, for each $f \in L^2(X)$,

$$(g, f) = \int g(x)\bar{f}(x) dx = \int \int G(x, y)\bar{f}(x) dx dy = (G, f) .$$

So $\Pi_\Gamma(G) = g$ follows from the uniqueness of orthogonal projections.

In particular, $\mathcal{E} \subseteq C(X)$, so that Lemma 2.1 will produce an orthonormal base contained in $C(X)$.

Finally, countability of $\mathcal{E}_f = \{g \in \mathcal{E} : (g, f) \neq 0\}$, for any $f \in \mathcal{E}$, follows from Bessel's inequality: For each $g = \Pi_\Gamma(G) \in \mathcal{E}$, since $(g, f) = (G, f)$, we have $\sum\{|(G, f)|^2 : G \in \mathcal{G}\} \leq \|f\|^2$. ☺

For example, let κ be any infinite cardinal such that $\kappa^{\aleph_0} = \kappa$. We may then obtain a nice (Z, μ) such that $|C(Z)| = \kappa$ and there is no orthonormal basis for $L^2(Z, \mu)$ contained in $C(Z)$; we just start with an X as in Theorem 3.7, and then $Z = X \times Y$ for a suitable Y (applying Theorem 3.8). However, assuming also that $2^\lambda < \kappa$ for all $\lambda < \kappa$ (for example, κ could be \beth_{ω_1} , or κ could be strongly inaccessible), every maximal orthogonal family $\mathcal{F} \subseteq C(Z)$ must have size κ : If $|\mathcal{F}| = \lambda < \kappa$, we could always find distinct $g, h \in C(Z)$ such that $(g, f) = (h, f)$ for all $f \in \mathcal{F}$ (since there are only $2^\lambda < \kappa = |C(Z)|$ possibilities for $\langle (g, f) : f \in \mathcal{F} \rangle$). Then $(g-h) \perp \mathcal{F}$, so \mathcal{F} cannot be maximal.

References

- [1] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.