Orthogonal Continuous Functions

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Abstract

We consider the question of whether there is an orthonormal basis for $L^2$ consisting of continuous functions.

1 Introduction

In elementary analysis, the typical orthonormal bases for $L^2[0,1]$ (trig functions, orthogonal polynomials, etc.) frequently consist of continuous functions. It is natural to ask whether such orthonormal bases must exist if $[0,1]$ is replaced by a more general space and measure. One commonly studied generalization of $[0,1]$ is:

Definition 1.1 $(X,\nu)$ is a nice measure space iff $X$ is a compact Hausdorff space and $\nu$ is a regular Borel probability measure on $X$ which is strictly positive (i.e., all non-empty open sets have positive measure).

The assumption that $\nu$ is strictly positive is mainly for notational convenience. In general, one can simply delete the union of all open null sets to obtain a strictly positive measure.

Since $\nu$ is strictly positive, distinct elements of $C(X)$ do not become equivalent in $L^2$, so we may regard $C(X)$ as contained in $L^2(X,\nu)$. There

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are then two well-known situations where there is an $\mathcal{F} \subseteq C(X)$ which forms an orthonormal basis for $L^2(X, \nu)$. The first is whenever $L^2(X, \nu)$ is separable (by Gram-Schmidt). The second is when $X$ is a compact group and $\nu$ is Haar measure (by the Peter-Weyl Theorem; see, e.g., Folland [1]). However, there need not be such an $\mathcal{F}$ in general, since Theorem 3.7 provides an example where $L^2(X, \nu)$ is not separable but any orthogonal family from $C(X)$ is countable.

In the example of Theorem 3.7, $X$ is actually a topological group, since it is a product of two-element spaces, and $\nu$ looks a bit like the product measure, which in this case would be Haar measure. Nevertheless, by Theorem 2.5, no such $\nu$ can be absolutely continuous with respect to Haar measure.

The proof for the specific example of Theorem 3.7 works equally well whether one considers the scalar field to be $\mathbb{R}$ or $\mathbb{C}$. However, if one starts with an arbitrary nice $(X, \nu)$, it is reasonable to ask whether the properties discussed here can depend on the scalar field. They do not, as we show in Corollary 2.2. Of course, any orthogonal family of real-valued functions remains orthogonal when viewed as members of $L^2(X, \nu, \mathbb{C})$, but Corollary 2.2 explains how to replace orthogonal complex-valued functions by real-valued ones. The familiar method from Fourier series replaces $\varphi$ and $\overline{\varphi}$ by $(\varphi + \overline{\varphi})/\sqrt{2}$ and $(\varphi - \overline{\varphi})/(i\sqrt{2})$, but this requires assuming that $\varphi \in \mathcal{F} \iff \overline{\varphi} \in \mathcal{F}$.

One might study the following property of $X$: For every finite regular Borel measure $\nu$ on $X$, there is an $\mathcal{F} \subseteq C(X)$ which forms an orthonormal basis for $L^2(X, \nu)$. We do not know whether this is equivalent to some interesting topological property of $X$. Note that every compact F-space and every compact metric space has this property.

2 Basics

Throughout, when discussing $C(X)$ and $L^2(X, \nu)$ and general Hilbert spaces, we always presume that the scalar field is the complex numbers. We shall show that we can convert a family of orthogonal continuous functions to a family of real-valued orthogonal continuous functions with the same span. To do this, we use the following lemma about Hilbert spaces, which gives us a uniform way to transform an “almost orthogonal” family to an orthogonal one:
Lemma 2.1 Suppose that $\mathcal{H}$ is a Hilbert space and $\mathcal{E} \subseteq \mathcal{H}$ is such that the closed linear span of $\mathcal{E}$ is all of $\mathcal{H}$ and $\{g \in \mathcal{E} : (g,f) \neq 0\}$ is countable for all $f \in \mathcal{E}$. Then there is an orthonormal basis $\mathcal{F}$ for $\mathcal{H}$ such that every element of $\mathcal{F}$ is a finite linear combination of elements of $\mathcal{E}$. Furthermore, the coefficients in these linear combinations will all be real whenever the $(g,f)$, for $g,f \in \mathcal{E}$, are all real.

Proof. On $\mathcal{E}$, let $\sim$ be the smallest equivalence relation such that $g \sim f$ whenever $(g,f) \neq 0$. Let $\mathcal{E}_j$, for $j \in J$, list all the $\sim$ equivalence classes. Then the $\mathcal{E}_j$ are all countable, and are pairwise orthogonal. For each $j$, apply Gram-Schmidt to obtain an orthonormal family $\mathcal{F}_j$ with the same linear span, such that the elements of $\mathcal{F}_j$ are linear combinations of elements of $\mathcal{E}_j$. Then, let $\mathcal{F} = \bigcup_j \mathcal{E}_j$. 

Corollary 2.2 Suppose that $(X,\nu)$ is a nice measure space and $\mathcal{G} \subseteq C(X)$ is an orthonormal family. Then there is an orthonormal family $\mathcal{F} \subseteq C(X)$, consisting of real-valued functions, such that the closed linear span of $\mathcal{F}$ contains the closed linear span of $\mathcal{G}$.

Proof. As usual, write each $G \in \mathcal{G}$ as $G = \Re(G) + i\Im(G)$, where $\Re(G)$ and $\Im(G)$ are real-valued functions. Let $\mathcal{E} = \{\Re(G) : G \in \mathcal{G}\} \cup \{\Im(G) : G \in \mathcal{G}\}$. Then the closed linear span $\mathcal{H}$ of $\mathcal{E}$ contains $\mathcal{G}$, so Lemma 2.1 will apply if we can verify that $\{g \in \mathcal{E} : (g,f) \neq 0\}$, for any $f \in \mathcal{E}$, is countable. To see this, apply Bessel’s inequality: $\sum_{G \in \mathcal{G}} |(G,f)|^2 \leq \|f\|^2$. Since $f$ is real-valued, $|(G,f)|^2 = (\Re(G), f)^2 + (\Im(G), f)^2$, so that $(\Re(G), f) = (\Im(G), f) = 0$ for all but countably many $G \in \mathcal{G}$. 

In particular, if $\mathcal{G}$ is an orthonormal basis, we may replace $\mathcal{G}$ by a real-valued orthonormal basis $\mathcal{F}$. Or, if $\mathcal{G}$ is an uncountable orthonormal family, then $\mathcal{F}$ will be a real-valued uncountable orthonormal family. So, the properties of $(X,\nu)$ considered in this paper do not depend on the scalar field.

The next definition and lemma give us a way of ensuring that there are no uncountable orthonormal families within $C(X)$:

Definition 2.3 We say $\mathcal{F} \subseteq C(X)$ is maximal orthogonal iff $\mathcal{F}$ is orthogonal in $L^2(X,\nu)$ and there is no orthogonal $\mathcal{G}$ with $\mathcal{F} \subsetneq \mathcal{G} \subseteq C(X)$.

Observe that even in $L^2([0,1])$, a maximal orthogonal $\mathcal{F} \subseteq C([0,1])$ need not be an orthogonal basis for $L^2([0,1])$; for example, its closed linear span may be the orthogonal complement of a step function. Nevertheless,
Lemma 2.4 Suppose \((X, \nu)\) is a nice measure space, and assume that there is a maximal orthogonal \(\mathcal{F} \subseteq C(X)\) which is countable. Then every orthogonal \(\mathcal{G} \subseteq C(X)\) is countable.

Proof. Let \(\mathcal{F}\) and \(\mathcal{G}\) be any two orthogonal families contained in \(C(X)\). For each fixed \(f \in \mathcal{F}\), Bessel’s Inequality implies that \(g \perp f\) for all but countably many \(g \in \mathcal{G}\). Hence, if \(\mathcal{F}\) is countable and maximal, then \(\mathcal{G}\) must be countable also. \(\circ\)

Now, the existence of an uncountable orthogonal family contained in \(C(X)\) depends on \(\nu\), not just \(X\), as the example in Section 3 shows. However,

Theorem 2.5 Suppose that \((X, \nu)\) and \((X, \mu)\) are nice measure spaces with \(\mu \ll \nu\). Suppose that \(\mathcal{G} \subseteq C(X)\) is an orthonormal basis for \(L^2(X, \nu)\). Then there is an \(\mathcal{F} \subseteq C(X)\) which is an orthonormal basis for \(L^2(X, \mu)\).

Proof. Fix a Baire-measurable \(\varphi : X \to [0, \infty)\) such that \(\mu(E) = \int_E \varphi(x) \, d\nu(x)\) for all Borel sets \(E\). Then \(\int \varphi \, d\nu = 1\), but \(\varphi\) need not be bounded, in which case \(\mathcal{G}\) might fail to span \(L^2(X, \mu)\).

Choose closed \(G_n\) sets \(K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots\) such that \(\varphi(x) \leq n\) for all \(x \in K_n\) and \(\nu(X \setminus \bigcup_n K_n) = 0\). For each \(n\), choose \(\psi_n \in C(X, [0, 1])\) such that \(K_n = \psi_n^{-1}\{1\}\), and note that the sequence of functions \((\psi_n)^m\) converges pointwise to \(\chi_{K_n}\) as \(m \to \infty\).

Let \(\mathcal{E}\) be the set of all functions of the form \(g \cdot (\psi_n)^m\), where \(g \in \mathcal{G}\) and \(m, n \in \mathbb{N}\). Then \(\mathcal{E} \subseteq C(X) \subseteq L^2(X, \mu)\). Let \(\mathcal{H}\) be the closed linear span of \(\mathcal{E}\) in \(L^2(X, \mu)\). Then \(\mathcal{H} = L^2(X, \mu)\): To see this, first note that \(g \cdot \chi_{K_n} \in \mathcal{H}\) for \(g \in \mathcal{G}\). Then, if \(h \in C(X)\), each \(h \cdot \chi_{K_n} \in \mathcal{H}\) (since \(\varphi\) is bounded on \(K_n\), but this implies that \(h \in \mathcal{H}\). Now, use the fact that \(\mathcal{C}(X)\) is dense in \(L^2(X)\).

The result will now follow by Lemma 2.1 if we can verify, for each \(f \in \mathcal{G}\) and each \(m, n, p, q \in \mathbb{N}\), \(\{g \in \mathcal{G} : (g(\psi_n)^m, f(\psi_p)^q)_\mu \neq 0\}\) is countable. Now for each \(r \in \mathbb{N}\), Bessel’s Inequality (applied in \(L^2(X, \nu)\)) implies that \(\int g(\psi_n)^m f(\psi_p)^q \chi_{K_r} \varphi \, d\nu = 0\) for all but countably many \(g \in \mathcal{G}\), since the function \((\psi_n)^m f(\psi_p)^q \chi_{K_r} \varphi\) is in \(L^2(X, \nu)\). It follows that \((g(\psi_n)^m, f(\psi_p)^q)_\mu = \int g(\psi_n)^m f(\psi_p)^q \varphi \, d\nu = 0\) for all but countably many \(g \in \mathcal{G}\). \(\circ\)

3 Small Orthogonal Families

We shall build a large nice \((X, \nu)\) in which every orthogonal family of continuous functions is countable. In order to do this, we apply Lemma 2.4; it
is enough to obtain some countable maximal \( \mathcal{F} \subseteq C(X) \). Again, we shall, for definiteness, assume that the scalar field is \( \mathbb{C} \). \( \mathcal{F} \) will be obtained by projecting \( X \) onto a small space \( M \), for which we use the following notation:

**Definition 3.1** \((X, \nu, \Gamma, M)\) is a nice quadruple iff \((X, \nu)\) is a nice measure space and \( \Gamma \) is a continuous map onto the compact Hausdorff space \( M \). In this case, let \( \mu = \nu \Gamma^{-1} \) be the induced measure on \( M \). We regard \( L^2(M, \mu) \) as contained in \( L^2(X, \nu) \) via the inclusion \( \Gamma^* \) (where \( \Gamma^*(g) = g \circ \Gamma \)). Let \( \Pi_\Gamma \) be the orthogonal projection from \( L^2(X, \nu) \) onto \( L^2(M, \mu) \). If \( f \in L^2(X, \nu) \), we say \( f \perp L^2(M, \mu) \) iff \( \Pi_\Gamma(f) = 0 \).

**Lemma 3.2** In the notation of Definition 3.1, if \( f \in L^2(X, \nu) \) then the following are equivalent:

1. \( f \perp L^2(M, \mu) \).
2. \( \int_{\Gamma^{-1}(K)} f(x) \, d\nu(x) = 0 \) for all closed \( K \subseteq M \).

**Definition 3.3** The nice quadruple \((X, \nu, \Gamma, M)\) is injective iff \( \Pi_\Gamma \) is 1-1 on \( C(X) \).

**Lemma 3.4** In the notation of Definition 3.1, the following are equivalent:

1. \((X, \nu, \Gamma, M)\) is injective.
2. For all \( f \in C(X) \), if \( f \perp L^2(M, \mu) \), then \( f \equiv 0 \).

**Lemma 3.5** Let \((X, \nu)\) be a nice measure space. Then the following are equivalent:

1. Every orthogonal subfamily of \( C(X) \) is countable.
2. There is a continuous map \( \Gamma \) onto a compact second countable space \( M \) such that \((X, \nu, \Gamma, M)\) is injective.

**Proof.** (2) \( \Rightarrow \) (1): Assuming (2), let \( \mathcal{F} \subseteq C(M) \) be an orthonormal basis for \( L^2(M) \). Then \( \Gamma^*(\mathcal{F}) \cup \{0\} \subseteq C(X) \), and is maximal orthogonal, so (1) follows by Lemma 2.4.

(1) \( \Rightarrow \) (2): Again by Lemma 2.4, let \( \{f_n : n \in \mathbb{N}\} \subseteq C(X) \) be maximal orthogonal. Let \( \Gamma : X \to \mathbb{C}^N \) be the product map: \((\Gamma(x))_n = f_n(x) \). Let \( M \) be the range of \( \Gamma \). Observe that a non-zero \( g \in C(X) \) with \( \Pi_\Gamma(g) = 0 \) would contradict maximality.

The next lemma explains how we obtain the situation of Lemma 3.5.2:
Lemma 3.6 Let \((X, \nu, \Gamma, M)\) be a nice quadruple. Assume, for some fixed \(\epsilon > 0\), we have: Whenever \(W \subseteq X\) is open and non-empty, there is a closed \(K \subseteq M\) such that \(\mu(K) > 0\) and \(\nu(\Gamma^{-1}(K) \cap W) \geq (\frac{1}{2} + \epsilon)\mu(K)\). Then \((X, \nu, \Gamma, M)\) is injective.

Proof. Suppose \(f \in C(X)\) is non-zero and satisfies \(f \perp L^2(M, \mu)\). We may assume that \(\|f\|_{\text{sup}} = 1\), and that some \(f(x) = 1\). For any \(\delta > 0\), we may choose a non-empty open \(W \subseteq X\) such that \(|f(x) - 1| \leq \delta\) for all \(x \in W\), and then choose \(K\) as above. Applying \(f \perp L^2(M, \mu)\) to the characteristic function of \(K\), we have \(\int_{\Gamma^{-1}K} f(x) \, d\nu(x) = 0\), so that \(|\int_{\Gamma^{-1}K \cap W} f| = |\int_{\Gamma^{-1}K \setminus W} f|\).

Note that \(\mu(K) = \nu(\Gamma^{-1}K)\), so that \(\nu(\Gamma^{-1}K \setminus W) \leq (\frac{1}{2} - \epsilon)\mu(K)\). So, we have:

\[
\begin{align*}
|\int_{\Gamma^{-1}K \cap W} f| & \geq \nu(\Gamma^{-1}K \cap W)(1 - \delta) \geq (\frac{1}{2} + \epsilon)\mu(K)(1 - \delta) \\
|\int_{\Gamma^{-1}K \setminus W} f| & \leq \nu(\Gamma^{-1}K \setminus W) \leq (\frac{1}{2} - \epsilon)\mu(K)
\end{align*}
\]

So, \((\frac{1}{2} + \epsilon)(1 - \delta) \leq (\frac{1}{2} - \epsilon)\). Letting \(\delta \searrow 0\), we have a contradiction. ☹️

Note that if \(\epsilon = 0\), the lemma could fail; consider \(X = M \times 2\), with the product measure.

In general, the Maharam dimension of a measure \(\nu\) is the cardinality of an orthonormal basis for \(L^2(\nu)\); \(\nu\) is called Maharam-homogeneous iff there is no set \(K\) of positive measure such that the dimension of \(\nu\) restricted to \(K\) is less than the dimension of \(\nu\). As usual, \(\mathfrak{c} = 2^{\aleph_0}\).

Theorem 3.7 There is a strictly positive regular Borel probability measure \(\nu\) on \(2^\mathfrak{c}\) (i.e., \(\{0, 1\}^\mathfrak{c}\), with the usual product topology) such that

1. \(\nu\) is Maharam-homogeneous of dimension \(\mathfrak{c}\).
2. \(L^2(2^\mathfrak{c}, \nu)\) contains no uncountable orthogonal family of continuous functions.

Proof. Let \(M = 2^{\aleph_0}\), with \(\mu\) the usual product measure. Let \(X = M \times 2^\mathfrak{c}\), and let \(\Gamma : X \rightarrow M\) be projection. We shall build \(\nu\) on \(X\), which is homeomorphic to \(2^\mathfrak{c}\).
Let \( \{d_m : m \in \mathbb{N}\} \) be dense in \((0, 1)^c\). For each \( m \), let \( \lambda_m \) be the product measure on \( 2^c \) obtained by flipping unfair coins with bias \( d_m \). That is, let 
\[
d_m^1(\alpha) = d_m(\alpha) \quad \text{and} \quad d_m^0(\alpha) = 1 - d_m(\alpha).
\]
If 
\[
B = \{v \in 2^c : v(\alpha_1) = \ell_1 \land \cdots \land v(\alpha_r) = \ell_r\}
\]
is a basic clopen set, then \( \lambda_m(B) = \prod_{j=1}^r d_m^\ell_j(\alpha_j) \).

List all non-empty clopen subsets of \( M \) as \( \{U_n : n \in \mathbb{N}\} \). Then, choose closed nowhere dense \( K_{m,n} \subseteq U_n \) so that the \( K_{m,n} \) for \( m, n \in \mathbb{N} \) are all disjoint, each \( \mu(K_{m,n}) > 0 \), and \( \sum_{m,n} \mu(K_{m,n}) = 1 \). Finally, let \( \nu \) on \( M \times 2^c \) be the sum of the product measures \( \mu \mid K_{m,n} \times \lambda_m \), so that for Borel \( E \subseteq M \times 2^c \),
\[
\nu(E) = \sum_{m,n} \int_{K_{m,n}} \lambda_m(E_x) \, d\mu(x).
\]
We are now done if we can verify the hypotheses of Lemma 3.6. We actually show that whenever \( W \subseteq X \) is open and non-empty and \( \epsilon > 0 \), there is a closed \( K \subseteq M \) such that \( \mu(K) > 0 \) and \( \nu(\Gamma^{-1}(K) \cap W) \geq (1 - \epsilon)\mu(K) \). To do this, we may assume that \( W = U_n \times B \), where \( B \) is as in (1) above. \( K \) will be \( K_{m,n} \) for a suitable \( m \). Then \( \nu(\Gamma^{-1}(K) \cap W) = \nu(K_{m,n} \times B) = \mu(K_{m,n}) \prod_{j=1}^r d_m^\ell_j(\alpha_j) \). We thus only need choose \( m \) so that \( \prod_{j=1}^r d_m^\ell_j(\alpha_j) \geq (1 - \epsilon) \), which is certainly possible since \( \{d_m : m \in \mathbb{N}\} \) is dense in \((0, 1)^c\). \( \square \)

Finally, we remark that this example is as large as possible, since if \( |C(X)| > c \), then there is an uncountable orthogonal family by Lemma 3.5. (Note that whenever \( X \) is an infinite compact Hausdorff space, \( |C(X)| = w(X)^{w_0} \), where \( w(X) \) is the weight of \( X \) (the least size of a base for the topology)). However, one can construct arbitrarily large examples with no continuous orthonormal bases by applying:

**Theorem 3.8** Suppose that \((X, \nu)\) and \((Y, \rho)\) are both nice measure spaces, and there is an orthonormal basis for \( L^2(X \times Y, \nu \times \rho) \) contained in \( C(X \times Y) \). Then there are orthonormal bases for \( L^2(X, \nu) \), \( L^2(Y, \rho) \) contained in \( C(X), C(Y) \), respectively.

**Proof.** Let \( \mathcal{G} \subseteq C(X \times Y) \) be an orthonormal basis for \( L^2(X \times Y, \nu \times \rho) \). To produce a basis for \( L^2(X, \nu) \), let \( \Gamma : X \times Y \to X \) be projection, and apply Lemma 2.1, with \( \mathcal{E} = \Pi_{\Gamma}(\mathcal{G}) \subseteq \mathcal{H} = L^2(X, \nu) \) (regarding \( L^2(X) \) as contained in \( L^2(X \times Y) \), as in Definition 3.1).
First, note that the closed linear span of \( \mathcal{E} \) will be all of \( L^2(X) \), because the closed linear span of \( \mathcal{G} \) is \( L^2(X \times Y) \) and \( \Pi \Gamma \) is orthogonal projection.

Next, observe that for each \( G \in \mathcal{G} \), \( \Pi \Gamma(G) = g \), where \( g(x) = \int G(x, y) \, dy \).

To see this, note that since \( G \) is continuous, \( g \in C(X) \subseteq L^2(X) \).

Also, for each \( f \in L^2(X) \),

\[
(g, f) = \int g(x) \overline{f}(x) \, dx = \int \int G(x, y) \overline{f}(x) \, dx \, dy = (G, f) .
\]

So \( \Pi \Gamma(G) = g \) follows from the uniqueness of orthogonal projections.

In particular, \( \mathcal{E} \subseteq C(X) \), so that Lemma 2.1 will produce an orthonormal base contained in \( C(X) \).

Finally, countability of \( \mathcal{E}_f = \{ g \in \mathcal{E} : (g, f) \neq 0 \} \), for any \( f \in \mathcal{E} \), follows from Bessel’s inequality: For each \( g = \Pi \Gamma(G) \in \mathcal{E} \), since \( (g, f) = (G, f) \), we have

\[
\sum \{(G, f)^2 : G \in \mathcal{G} \} \leq \|f\|^2.
\]

For example, let \( \kappa \) be any infinite cardinal such that \( \kappa^{\aleph_0} = \kappa \). We may then obtain a nice \( (Z, \mu) \) such that \( |C(Z)| = \kappa \) and there is no orthonormal basis for \( L^2(Z, \mu) \) contained in \( C(Z) \); we just start with an \( X \) as in Theorem 3.7, and then \( Z = X \times Y \) for a suitable \( Y \) (applying Theorem 3.8). However, assuming also that \( 2^\lambda < \kappa \) for all \( \lambda < \kappa \) (for example, \( \kappa \) could be \( \beth_1 \), or \( \kappa \) could be strongly inaccessible), every maximal orthogonal family \( \mathcal{F} \subseteq C(Z) \) must have size \( \kappa \): If \( |\mathcal{F}| = \lambda < \kappa \), we could always find distinct \( g, h \in C(Z) \) such that \( (g, f) = (h, f) \) for all \( f \in \mathcal{F} \) (since there are only \( 2^\lambda < \kappa = |C(Z)| \) possibilities for \( \{(g, f) : f \in \mathcal{F}\} \)). Then \( (g-h) \perp \mathcal{F} \), so \( \mathcal{F} \) cannot be maximal.

References