*Notes on Set Theory*
by Yiannis N. Moschovakis
Springer-Verlag, 1994, $272 + ix$ pages
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How much set theory do you need to know? Should you read this book? To help you answer these questions, we partition, rather arbitrarily, basic set theory into "elementary", "intermediate", and "advanced", and we touch on the relevance of set theory to philosophy and computer science. Then we comment on what Moschovakis includes, and we conclude with some additional remarks on the exposition.

*Elementary*: Children learn rather quickly how to count the cookies in a jar:

$$0, 1, 2, 3, \ldots \quad .$$

By high school, students know that they could call the jar a "set", and they know some basic facts about unions and intersections, and how these relate to the sizes (cardinalities) of sets. Related to this are some facts from discrete math, such as what a function is, and what it means for a relation on a set to be a total ordering, reflexive, etc. This much set theory will take students through the basic college courses in calculus and abstract algebra, which, after all, cover material primarily discovered before Cantor. It will also take them through elementary courses in computer programming, where they learn how to represent functions, relations, and finite sets in Basic or C.

*Intermediate*: Eventually, it becomes important to know something about post-Cantorian set theory. Every math major should learn that the set of reals, $\mathbb{R}$, cannot be covered by a countable sequence of points (Cantor), or even by a countable sequence of nowhere dense sets (Baire). Undergraduates often learn, in addition, how to recite Zorn's lemma and how to use it to prove, for example, that every vector space has a basis. This material is often covered in introductory courses in real analysis, topology, or algebra, and is all the set theory that most research mathematicians ever need to know.

*Advanced*: In some areas of mathematics, one needs to know about well-orderings and ordinals, and how to count the cookies in an infinite jar:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \ldots, \omega + \omega, \ldots, \omega_1, \ldots, \omega_2, \ldots \quad .$$

If the jar is $\mathbb{R}$, it has size $\omega_\alpha$ (or $\aleph_\alpha$) for some $\alpha > 0$; the Continuum Hypothesis says that $\alpha = 1$. These topics are covered in detail in an undergraduate course in set theory. Unfortunately, most math majors never take such a course. As a result, even graduate level texts avoid using these notions, although a number of topics would be made somewhat less obscure by a knowledge of ordinals. For example, Zorn's lemma seems rather mystical unless you understand how to prove it using ordinals; and, if you understand that proof, you no longer need Zorn's lemma; you just pick a basis for an infinite dimensional vector space inductively, exactly the same way you pick a basis for a finite dimensional one. Or, every book on measure theory states that the Borel sets form the least $\sigma$-algebra containing the open sets, but constructing the Borel sets inductively in $\omega_1$ steps (open sets, $G_\delta$ sets, $G_{\delta\sigma}$ sets, ...) gives you a much a clearer picture of what they are than does producing them by intersecting a family of $\sigma$-algebras.

*Philosophy*: Mathematicians should have some understanding of the foundational underpinnings of their art. Although the axiomatic method in geometry goes back to Euclid, the modern view is that all of mathematics can be developed within the unified framework of axiomatic set theory. One does not introduce a whole new collection of axioms for each mathematics course. One starts with something like the Zermelo-Fraenkel axioms ($ZF$), and proceeds both to develop general abstract facts about sets and functions, as well as to define important specific sets, such as the natural numbers, the rational numbers, and the real numbers, and then the Euclidean plane ($\mathbb{R} \times \mathbb{R}$). So, geometry, like everything else, is a branch of set theory. This axiomatization also reveals the role of the Axiom of Choice ($AC$). Whether or not one admits $AC$ as a basic principle, one should understand which results from elementary mathematics require $AC$.

*Computer Science*: Of course, anything that is *done* on the computer is finite, and can be understood using just elementary set theory, but more advanced methods come in when trying to understand the theory behind what the computer is doing. Thus, books on denotational semantics for programming languages ([5]) use the kind of set-theoretic techniques usually associated with general topology. Ordinals crop up in books on logic programming semantics ([8]) and implementations of constructive mathematics ([2]).

This book starts off in the beginning of the intermediate level, which is ideal for an undergraduate text. The first chapter quickly reviews elementary facts about sets and functions, primarily to establish the notation to be used. The second chapter explains Cantor's basic ideas, covering countable and uncountable sets, and Cantor's diagonal argument. The explanatory remarks and accompanying figures should be very helpful for readers who haven't seen these things before.

The third chapter points out that naive manipulation with sets can lead to contradictions, such as Russell's paradox with $\{x : x \notin x\}$, hence the need for some axiomatic framework. Moschovakis explains what is meant in general by an axiomatic system, and then describes Zermelo's axioms for set theory.

By the end of Chapter 5, the reader sees how, based on Zermelo's axioms, one can develop elementary discrete math (sets, functions, relations), as well as the natural numbers, $\mathbb{N}$. In particular, Moschovakis stresses that the existence of $\mathbb{N}$, along with the basic facts about $\mathbb{N}$ (such as induction and recursion), are all theorems within the framework of axiomatic set theory. At this point, one has the machinery to go on and develop the rational numbers and the real numbers, but this requires some knowledge of algebra, and is put off until Appendix A, where it is done in detail, using both Dedekind cuts and Cauchy sequences.

Of course, every book on set theory would have to tell you this much; the only serious design decision is whether to develop $\mathbb{N}$ first and then cover the transfinite ordinals later, as a more advanced topic, or whether to plunge right in to the general theory of ordinals, obtaining $\mathbb{N}$ as the set of the first $\omega$ ordinals. Moschovakis chooses the first option, which is slightly redundant, but which makes the material more accessible for undergraduates approaching this abstract subject for the first time.

Chapter 6 is an important departure from tradition. It is centered around the notion of an *inductive poset*; this is a partially ordered set, $P$, such that every chain (totally ordered subset) has a least upper bound. The basic result here is that every monotonic mapping on an inductive poset has a least fixed point. It is rather unusual to see this in an undergraduate mathematics text, since the most well-known applications are in computer science. The exercises give a hint of how this is applied in programming language semantics. The text gives the following very concrete application: In most programming languages, you can define a function $f$ *recursively*, defining $f(x)$ by any expression which involves $f$ itself. In general, such a computation might fail to terminate for some (maybe all) values of $x$; the fixed point theorem shows that every such definition uniquely determines $f$ as a *partial* function. If the input and output to $f$ are natural numbers, then the relevant $P$ is the set of all partial functions on $\mathbb{N}$, ordered by subset (if we identify each partial function with its graph).

Besides its interest in computer science, fixed points and inductive posets are a nice way of introducing the mathematics topics which follow. Chapter 7 introduces well-orderings. Fixed point theory yields a motivation for the study of well-orderings, since you actually need well-orderings to prove the fixed-point theorem (Chapter 6 only proves a weakened version of the theorem). Then, Chapter 8 introduces $AC$. It proves the standard equivalents to $AC$ (such as the well-ordering principle and Zorn's lemma), but the proofs, using the poset and fixed point terminology, are a good deal more elegant than the standard ones.

Chapter 9 uses $AC$ to develop some further material, such as the theory of cofinalities, and König's Theorem. For example, although Gödel and Cohen tell us that the continuum, $2^{\aleph_0}$, could be $\aleph_1$ or $\aleph_5$ or $\aleph_{\omega+1}$, it cannot be $\aleph_\omega$ or $\aleph_{\omega+\omega}$ by König's Theorem.

Chapter 10 covers basic descriptive set theory. It is another important departure from tradition to do this in an undergraduate text. This material is definitely learnable on an undergraduate level, and is something which every mathematician should know, but often doesn't. A typical result here is that the Continuum Hypothesis is simply a *theorem* for Borel sets. That is, every Borel subset of $\mathbb{R}$ is either countable or contains a perfect subset, and hence has size $2^{\aleph_0}$. Chapter 10 also presents the construction of a Bernstein set (an uncountable $X \subset \mathbb{R}$ such that neither $X$ nor $\mathbb{R} \backslash X$ contains a perfect subset); this is not descriptive set theory, but is a nice application of well-ordering and transfinite induction to elementary real analysis.

Chapter 11 introduces the Axioms of Replacement and Foundation, which are part of $ZF$, but not part of Zermelo's original axioms. Chapter 12, finally, tells you about von Neumann ordinals and cardinals. Here, you learn what the ordinals really are; for example, $0 = \emptyset$; $3 = \{0, 1, 2\}$, and $\omega = \mathbb{N}$. Appendix B is an introduction to models of set theory and consistency proofs; in particular, it describes models where Foundation is true, and other models where Foundation is false.

In general, the text's conversational style makes it easy to read, and its content is instructive. Moschovakis often introduces results with remarks on their importance, then guides the reader through the proofs, helping students not only to follow the steps, but also to learn common proof techniques. For example, in Chapter 5, Moschovakis talks the reader through a proof of the Recursion Theorem, showing the reader how to obtain a function from the finite partial functions which approximate it; this material forms a good introduction to the more sophisticated kinds of recursion covered in Chapters 6 and 7. In addition, he is careful to point out key ingredients to various proofs. For example, after proving the uncountability of the set of real numbers, he stresses the role the completeness property of the reals plays in the proof, noting "the rest of Cantor's construction relies solely on arithmetical properties of numbers which are also true of the rationals".

In relating some of the history of set theory, Moschovakis gives the reader insight into the roots of the subject. He points out that Cantor's approach was rather vague and intuitive, so that mathematicians of the day were naturally suspicious of his methods. His discussion helps the student understand how the paradoxes led to Zermelo's formulation in 1908 of a precise set of axioms. These axioms spelled out exactly what is being assumed, and seemed to be free of contradictions. To be fair to Cantor though, Moschovakis should have noted how close Russell's Paradox is to what Cantor already knew. Chapter 2 presents Cantor's diagonal argument that there is no map $\pi$ from a set $A$ onto its power set, $\mathcal{P}(A)$: the assumption that $B = \{x \in A : x \notin \pi(x)\}$ is in the range of $\pi$ leads to a contradiction. Cantor knew that there was a problem if $A$ is the universal set, $V$, since then $\mathcal{P}(V)$ is a subset of $V$, not bigger than $V$. More succinctly, if $\pi$ is the identity map and $A = V$, we get an outright contradiction from $B = \{x \in V : x \notin \pi(x)\} = \{x : x \notin x\}$. Of course, Cantor left it to Russell to put the paradox this succinctly, and then to popularize it. But, it is misleading to state in Chapter 3 that the paradoxes before Russell were "technical and affected only the most advanced parts of Cantor's theory". It would be more accurate to say that Cantor's methods were informal and intuitive, and that he just intuitively avoided what he called "inkonsistenten Mengen" (see [1] for further discussion).

The descriptive set theory in Chapter 10 focuses on *Baire space*, $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, a countable product of countable discrete spaces. It will be a little difficult for most readers to see that the results also apply to more familiar spaces, such as $\mathbb{R}$. Moschovakis does mention that "there is such a tight connection between $\mathcal{N}, \mathcal{C}$ [the Cantor set], and $\mathbb{R}$ that practically every interesting property of one of these spaces translates immediately to a related, interesting property of the others." But, the details are a bit patchy, and must be ferreted out of the problems (x10.11,x10.12) and Appendix A. Neither in the problems nor in Appendix A does he point out that $\mathcal{N}$ is homeomorphic to the space of irrational numbers, although the classical proof of this result maps $\mathcal{N}$ to the irrationals by a simple use of continued fractions, a topic of numerous *Monthly* articles and one definitely accessible at the undergraduate level.

Moschovakis does not introduce ordinal notation until the last chapter (12), which means that we can't really count our cookies in the standard way until the end of the book, where we finally see $\omega$, $\omega + 1$, $\omega + 2, \ldots$. As he says in the Preface, most courses will not get this far. This is unfortunate, since this method of counting is frequently used when a transfinite sequence is listed. Its roots go back to Cantor's original theory; even in his paper of 1880, Cantor employed such "infinite symbols" to advance the theory of derived sets, three years before his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* presented the transfinite numbers as "the simplest, most appropriate and natural extension [of the concept of number]" ([3]). There is a formal justification for Moschovakis' order of presentation: to develop the modern (von Neumann) theory of ordinals (as opposed to Cantor's intuitive presentation), one needs the Replacement Axiom, which is not introduced until Chapter 11, and it is of some formal interest (to specialists) to see how much set theory can be developed without Replacement. However, in a number of places in the text, especially in Chapters 7 and 9, the explanations would have been much simpler and more natural if ordinal notation were available. For example, without ordinals, the discussion of cofinalities in Chapter 9 is a bit awkward, and the motivation for studying König's Theorem is a bit obscure; since the notation $\aleph_\alpha$ could not be defined yet, he was actually not able to state simply that $2^{\aleph_0} \neq \aleph_{\omega+\omega}$, as we did above.

Finally, the book is only 272 pages long, and cannot cover everything. Overall, the author has made an excellent choice of what to include, and he says just enough about omitted topics to whet the reader's appetite for more. So, the reader may be disappointed to find no references to the literature beyond the two historical sources cited in the Preface. For example, the author mentions logic, including the fact that the Gödel incompleteness theorems apply to systems such as $ZF$; why not suggest an undergraduate logic text (e.g. [4]) where the reader might pursue the subject further? Or, the book tells us that by results of Gödel and Cohen, $CH$ is true in some models of $ZFC$ and false in others, but there are no references to texts such as [6] or [7], where these models are constructed. Given the relevance of Chapter 6 to programming language semantics, it seems strange not to refer the reader to a basic text ([5]) on the subject. Stranger still, many of the missing references were written by the author himself, who is one of the leading contributors to many of the topics highlighted in the book, such as fixed point theory ([9]), descriptive set theory ([10]), and applications of logic to computer science ([11]).

## References


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