Locally Constant Functions

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ABSTRACT

Let X be a compact Hausdorff space and M a metric space. $E_0(X,M)$ is the set of $f \in C(X,M)$ such that there is a dense set of points $x \in X$ with f constant on some neighborhood of x. We describe some general classes of X for which $E_0(X,M)$ is all of $C(X,M)$. These include $\beta\mathbb{N}\backslash\mathbb{N}$, any nowhere separable LOTS, and any X such that forcing with the open subsets of X does not add reals. In the case that M is a Banach space, we discuss the properties of $E_0(X,M)$ as a normed linear space. We also build three first countable Eberlein compact spaces, F, G, H, with various E_0 properties. For all metric M, $E_0(F,M)$ contains only the constant functions, and $E_0(G,M) = C(G,M)$. If M is the Hilbert cube or any infinite dimensional Banach space, $E_0(H,M) \neq C(H,M)$, but $E_0(H,M) = C(H,M)$ whenever $M \subseteq \mathbb{R}^n$ for some finite n.

§**0. Introduction.** If X is a compact Hausdorff space and M is a metric space, let $C(X,M)$ be the space of all continuous functions from X into M. $C(X,M)$ is a metric space under the sup norm. $C(X)$ denotes $C(X,\mathbb{R})$, which is a (real) Banach algebra. Following [5, 6, 7, 13, 14], if $f \in C(X,M)$, let Ω_f be the union of all open $U \subseteq X$ such that f is constant on U. Then, $E_0(X,M)$ is the set of all $f \in C(X,M)$ such that Ω_f is dense in X; these functions are called "locally constant on a dense set". $E_0(X)$ denotes $E_0(X,\mathbb{R})$.

Clearly, $E_0(X)$ is a subalgebra of $C(X)$ and contains all the constant functions. As Bernard and Sidney point out [6, 7, 14], if X is compact metric with no isolated points, then $E_0(X)$ is a proper dense subspace of $C(X)$. In this paper, we study the two extreme situations: where $E_0(X)$ contains only the constant functions, and where $E_0(X) = C(X)$. In §5, we give some justification for studying these two extremes.

A standard example of elementary analysis is a monotonic $f \in C([0,1])$ which does all its growing on a Cantor set; then f is a nonconstant function in $E_0([0,1])$. More generally, for "many" X, $E_0(X)$ separates points in X, and hence (by the Stone-Weierstrass Theorem), is dense in $C(X)$. Specifically,

0.1. Theorem. If X is compact Hausdorff and $E_0(X)$ is *not* dense in $C(X)$, then
a. X has a family of $2^{\aleph_0}$ disjoint nonempty open subsets.
b. X is not locally connected.
c. X is not zero-dimensional.

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Part (c) of the Theorem is obvious. Parts (a) and (b) are due to M. E. Rudin and W. Rudin [13], and generalize earlier results of Bernard and Sidney that if X is compact and second countable, then $E_0(X)$ is dense in $C(X)$.

However, *first* countable is not enough. In §2, we produce a first countable compact X such that $E_0(X)$ contains only the constant functions. A *non* first countable example was constructed in [13]. Our result is patterned after [13]. Roughly, we replace the family of Cantor sets used in their construction by a *disjoint* family. This adds some complexity to the construction. However, we also simplify the geometry of the construction by building the space inside a Hilbert space. Our space will be compact in the weak topology, and hence a uniform Eberlein compact (that is, a weakly compact subspace of a Hilbert space). One may use the approach of §2 to simplify the construction of [13] and to demonstrate that their space is also a uniform Eberlein compact.

In §3, we look at the other extreme; there are many familiar compact X for which $E_0(X)$ is all of $C(X)$, such as $\beta\mathbb{N}\backslash\mathbb{N}$ and a Suslin line. For some classes of spaces, such as compact ordered spaces and compact extremally disconnected spaces, we present simple necessary and sufficient conditions for $E_0(X) = C(X)$.

In §§3,4, we also consider $E_0(X, M)$ for other metric spaces M. It is easy to see that $E_0(X, \mathbb{C}) = C(X, \mathbb{C})$ iff $E_0(X, \mathbb{R}) = C(X, \mathbb{R})$, and $E_0(X, \mathbb{C})$ is dense in $C(X, \mathbb{C})$ iff $E_0(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$, but the situation for general M is a bit more complex. In particular, in §4, we produce a uniform Eberlein compact X such that $E_0(X, \mathbb{R}) = C(X, \mathbb{R})$ but $E_0(X, Q) \neq C(X, Q)$, where Q is the Hilbert cube. In §5, we let M be a Banach space, and consider the properties of $E_0(X, M)$ as a normed linear space.

Also in §5, we show that $E_0(X)$ is a proper dense subspace of $C(X)$ whenever X is a nontrivial infinite product.

In §1, we prove some preliminary results on Cantor sets used in our construction in §2.

Independently of Bernard and Sidney, Bella, Hager, Martinez, Woodward, and Zhou [2, 3, 12] defined the space $E_0(X)$ (they called it $dc(X)$), and showed (in the spirit of Theorem 0.1) that $E_0(X)$ is dense in $C(X)$ in many cases. We comment further on their work at the end of §3.

§1. Cantor Sets. By a *closed interval* we mean any compact space homeomorphic to $[0, 1] \subseteq \mathbb{R}$. By a *Cantor set* we mean any space homeomorphic to the usual Cantor set in $\mathbb{R}$; equivalently, homeomorphic to 2^ω, where $2 = \{0, 1\}$ has the discrete topology. The following lemma was used also in [13].

1.1. Lemma. If J is a closed interval, $f \in C(J)$, and f is not constant, then there is a Cantor set $H \subset J$ such that f is $1-1$ on H.

In our construction, we need a uniform version of this. If H is a subset of a product $X \times J$, we use H_x to denote $\{y \in J : (x, y) \in H\}$.

1.2. Lemma. Suppose J is a closed interval and X is a compact zero-dimensional Hausdorff space, and suppose $f \in C(X \times J)$ is such that for every $x \in X$, $f \restriction (\{x\} \times J)$ is not constant. Then there is a set $H \subset X \times J$ such that:

(1) H_x is a Cantor set for every $x \in X$.
(2) f is $1-1$ on $\{x\} \times H_x$ for every $x \in X$.
(3) There is a continuous $\varphi : H \to 2^\omega$ such that the map $(x,y) \mapsto (x, \varphi(x,y))$ is a homeomorphism from H onto $X \times 2^\omega$.

Some remarks: Lemma 1.1 is the special case of Lemma 1.2 where X is a singleton. If we deleted (3), then 1.2 would be immediate from 1.1, using the Axiom of Choice, without any assumption on X. But (3) says that we can choose the Cantor sets continuously. As stated, the Theorem requires X to be zero-dimensional. For example, suppose $X = J = [0,1]$, and we take f to be constant on the strip $\{(x,y) : |x-y| < \frac{1}{3}\}$. Then H must be disjoint from the strip, which is easily seen to contradict (3). Of course, (1) follows from (3).

Lemma 1.1 may be proved by a binary tree argument, and we prove Lemma 1.2 by showing how to build this tree "uniformly" for all $x \in X$. A simpler proof of Lemma 1.1 in [13] takes advantage of the ordering on $\mathbb{R}$, but this proof does not easily generalize to a proof of Lemma 1.2. Moreover, the tree argument extends to non-ordered spaces. For example, in Lemma 1.2, J could be any compact metric space which is connected and locally connected, and f could be any map into a Hausdorff space.

The following general tree notation will be used here and in §§2-4. If Δ is some index set, then $\Delta^{<\omega}$ denotes the tree of all finite sequences from Δ; this is the complete Δ-ary tree of height ω. For $s \in \Delta^{<\omega}$, let $lh(s) \in \omega$ be its length. We use () to denote the empty sequence. If $i \leq lh(s)$, let $s \restriction i$ be the sequence of length i consisting of the first i elements of s; $t \subseteq s$ iff $t = s \restriction i$ for some $i \leq lh(s)$. Let $t\alpha$ denote the sequence of length $lh(t)+1$ obtained by appending α to t. Note that $\Delta^{<\omega}$, ordered by $\subseteq$, is a tree with root (), and the nodes immediately above s are the $s\alpha$ for $\alpha \in \Delta$. We say $s,t \in \Delta^{<\omega}$ are *compatible* iff $s \subseteq t$ or $t \subseteq s$. We let $s \perp t$ abbreviate the statement that s,t are *incompatible* (not compatible).

A *path* in $\Delta^{<\omega}$ is a chain, P, such that $s\alpha \in P$ implies $s \in P$ for all s and α. A path may be empty or finite or countably infinite. The infinite paths are all of the form $\{\psi \restriction n : n \in \omega\}$ where $\psi : \omega \to \Delta$. In particular, for binary trees, $\Delta = 2 = \{0,1\}$, and the infinite paths through the Cantor tree, $2^{<\omega}$, are associated with the points in the Cantor set, 2^ω.

To prove 1.2, fix a metric on J. For $E \subseteq J$, let $diam(E)$ be the diameter of E with respect to this metric. Call a subset of $X \times J$ *simple* iff it is of the form $\bigcup_{i<k} Q_i \times I_i$, where k is finite, the Q_i for $i<k$ form a disjoint family of clopen sets whose union is X, and each I_i is a closed interval. We prove 1.2 by iterating the following splitting lemma.

1.3. Lemma. Let J, X, f be as in 1.2 and let $\epsilon > 0$. Then there are simple $A_0, A_1 \subset X \times J$ such that the following hold:
(a) $A_0 \cap A_1 = \emptyset$.
(b) For each $x \in X$, $f(\{x\} \times (A_0)_x) \cap f(\{x\} \times (A_1)_x) = \emptyset$.
(c) For each $x \in X$ and $\mu = 0,1$, $diam((A_\mu)_x) \leq \epsilon$.
(d) For each $x \in X$ and $\mu = 0,1$, $f \restriction (\{x\} \times (A_\mu)_x)$ is not constant.

Proof. For each $z \in X$, $f \restriction (\{z\} \times J)$ is a nonconstant map from an interval into an interval, so we may choose disjoint closed intervals $I_0(z), I_1(z) \subset J$ such that $f(\{z\} \times I_0(z)) \cap f(\{z\} \times I_1(z)) = \emptyset$, $diam(I_\mu(z)) \leq \epsilon$, and $f \restriction (\{z\} \times I_\mu(z))$ is not constant

($\mu = 0, 1$). By continuity, there is a neighborhood U_z of z such that for all $x \in U_z$, $f(\{x\} \times I_0(z)) \cap f(\{x\} \times I_1(z)) = \emptyset$ and $f \restriction (\{x\} \times I_\mu(z))$ is not constant. Since X is compact and zero-dimensional, there are a finite k, points $z_i \in X$ ($i < k$), and clopen $Q_i \subseteq U_{z_i}$ such that the Q_i form a partition of X. Then let $A_\mu = \bigcup_{i<k} Q_i \times I_\mu(z_i)$. ☺

Proof of 1.2. For $s \in 2^{<\omega}$, choose simple $A_s \subseteq X \times J$ such that
(a) For each $s \in 2^{<\omega}$, $A_{s0} \cap A_{s1} = \emptyset$.
(b) For each $x \in X$ and $s \in 2^{<\omega}$, $f(\{x\} \times (A_{s0})_x) \cap f(\{x\} \times (A_{s1})_x) = \emptyset$.
(c) For each $x \in X$ and $t \in 2^{<\omega}$, $diam((A_t)_x) \leq 1/lh(t)$.
(d) For each $x \in X$ and $t \in 2^{<\omega}$, $f \restriction (\{x\} \times (A_t)_x)$ is not constant.
We may take $A_{()} = X \times J$; then, for $t = ()$, (c) is vacuous and (d) follows from the hypothesis of 1.2. Given A_s, we obtain A_{s0} and A_{s1} by applying 1.3 to each box making up A_s. Let $H = \bigcap_{n\in\omega} \bigcup\{A_s : lh(s) = n\}$. Let $\varphi(x, y)$ be the (unique) $\psi \in 2^\omega$ such that $(x, y) \in A_{\psi\restriction n}$ for all $n \in \omega$. ☺

§**2. Making $E_0(X)$ Small.** We describe how to construct a first countable compact space L_ω such that $E_0(L_\omega)$ contains only the constant functions. Let $D \subseteq \mathbb{C}$ be the closed unit disk; D will be a subspace of L_ω. We shall first focus on the easier task of constructing a space L_2 such that $D \subset L_2$ and each $f \in E_0(L_2)$ is constant on D. After explaining this, we shall iterate the procedure to produce L_ω.

Before we build L_2, we shall show that every nonconstant function $f \in C(D)$ is $1-1$ on "many" disjoint Cantor sets. Then, by gluing new disks on those Cantor sets to form L_2, we can make sure that no such f can extend to a function in $E_0(L_2)$.

For $\theta \in [0, 2\pi)$, let R_θ denote the *ray* $\{z \in D : z \neq 0 \,\&\, \arg(z) = \theta\}$. Let $\mathfrak{c} = 2^{\aleph_0}$.

2.1. Lemma. If $f \in C(D)$ is nonconstant, then there are $\mathfrak{c}$ distinct θ such that f is nonconstant on R_θ.

Proof. The set of all such θ is open. ☺

We identify $\mathfrak{c}$ with a von Neumann ordinal, so that we may use $\mathfrak{c}$ also as an index set.

2.2. Lemma. There is a *disjoint* family $\{K_\alpha \subset D\backslash\{0\} : \alpha \in \mathfrak{c}\}$ of $\mathfrak{c}$ Cantor sets, with the following property: For each nonconstant $f \in C(D)$, there is a Cantor set $H_f \subset D\backslash\{0\}$ such that f is $1-1$ on H_f and such that $A = \{\alpha \in \mathfrak{c} : K_\alpha \subseteq H_f\}$ has size $\mathfrak{c}$.

Proof. First, applying Lemma 2.1 and transfinite induction, choose, for each nonconstant $f \in C(D)$, a *distinct* $\theta_f \in [0, 2\pi)$ such that f is nonconstant on R_{θ_f}. Then, applying Lemma 1.1, choose a Cantor set $H_f \subset R_{\theta_f}$ such that f is $1-1$ on H_f. Partition each H_f into $\mathfrak{c}$ disjoint Cantor sets. Since the H_f are all disjoint, this gives us the desired family of $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$ Cantor sets. ☺

Informally, we now replace each K_α by a copy of $K_\alpha \times D$, identifying $K_\alpha \times \{0\}$ with the old K_α. For different α, we want the $K_\alpha \times D$ to point in "perpendicular directions". To make the notion of "perpendicular" formal, we simply embed L_2 into a Hilbert space. Since we want each "direction" to be a whole disk, we use a *complex* Hilbert space to simplify the notation. One could use a real Hilbert space instead by replacing each unit

vector in the following proof by a pair of unit vectors. In either case, the following simple criterion can be used to verify first countability.

2.3. Lemma. If $\mathbb{B}$ is a Hilbert space and $X \subset \mathbb{B}$ is compact in the weak topology, then X is first countable in the weak topology iff for each $\vec{x} \in X$, there is a countable (or finite) $C_{\vec{x}} \subset \mathbb{B}$ such that no $\vec{v} \in X \backslash \{\vec{x}\}$ satisfies $\forall \vec{c} \in C_{\vec{x}} (\vec{x} \cdot \vec{c} = \vec{v} \cdot \vec{c})$.

Proof. By definition of the weak topology, the stated condition is equivalent to each $\{\vec{x}\}$ being a G_δ set in X, which is equivalent to first countability in a compact space. ☺

We remark that the condition of Lemma 2.3 need not imply first countability when X is not weakly compact.

2.4. Lemma. There is a first countable uniform Eberlein compact space L_2 such that D is a retract of L_2 and each $f \in E_0(L_2)$ is constant on D.

Proof. Let $\mathbb{B}$ be a complex Hilbert space with an orthonormal basis consisting of $\mathfrak{c}$ unit vectors $\vec{e}_\alpha$, for $\alpha \in \mathfrak{c}$, together with one more, $\vec{e}$. We identify D with its homeomorphic copy, $D' = \{z\vec{e} : |z| \leq 1\} \subset \mathbb{B}$. Let π be the perpendicular projection from $\mathbb{B}$ onto the one-dimensional subspace spanned by $\vec{e}$.

Let the $K_\alpha \subset D' \backslash \{\vec{0}\}$ be as in Lemma 2.2 (replacing the D there by D'). Let L_2 be the set of all $\vec{x} \in \mathbb{B}$ that satisfy (1) - (3):

(1) $|\vec{x} \cdot \vec{e}| \leq 1$, and, for each $\alpha \in \mathfrak{c}$, $|\vec{x} \cdot \vec{e}_\alpha| \leq \frac{1}{2}$.

(2) For all distinct α, β, either $\vec{x} \cdot \vec{e}_\alpha = 0$ or $\vec{x} \cdot \vec{e}_\beta = 0$.

(3) For all α, either $\vec{x} \cdot \vec{e}_\alpha = 0$ or $\pi(\vec{x}) \in K_\alpha$.

So, points of L_2 are either of the form $z\vec{e}$, with $|z| \leq 1$, or of the form $z\vec{e} + w\vec{e}_\alpha$, where $|z| \leq 1$, $|w| \leq \frac{1}{2}$, and $z\vec{e} \in K_\alpha$. In particular, $D' = \pi(L_2) \subset L_2$.

We give L_2 the topology inherited from the *weak* topology on $\mathbb{B}$. Note that L_2 is weakly closed. Since L_2 is also norm bounded, L_2 is compact. To see that L_2 is first countable, apply Lemma 2.3. If $\pi(\vec{x})$ is in no K_α, set $C_{\vec{x}} = \{\vec{e}\}$, while if $\pi(\vec{x})$ is in some K_α, this α is unique (by the disjointness of the K_α), and we set $C_{\vec{x}} = \{\vec{e}, \vec{e}_\alpha\}$.

Let $U_\alpha = L_2 \cap \pi^{-1}(K_\alpha) \backslash K_\alpha = \{\vec{x} \in L_2 : \vec{x} \cdot \vec{e}_\alpha \neq 0\}$. Observe:

i. U_α is an open subset of L_2, but

ii. For each $\vec{x} \in K_\alpha$, $L_2 \cap \pi^{-1}(\{\vec{x}\})$ is nowhere dense in L_2.

Now, suppose $f \in E_0(L_2)$. We show that f is constant on D'. If not, fix a Cantor set $H \subseteq D'$ such that f is $1-1$ on H and such that $A = \{\alpha \in \mathfrak{c} : K_\alpha \subseteq H\}$ has size $\mathfrak{c}$. Since $f \in E_0(L_2)$, we may, for each $\alpha \in A$, choose a nonempty open $W_\alpha \subseteq U_\alpha$ such that f is constant on W_α. Then, applying (ii) above, choose two distinct points $\vec{x}_\alpha$ and $\vec{y}_\alpha$ in W_α such that $\pi(\vec{x}_\alpha) \neq \pi(\vec{y}_\alpha)$.

For each $\alpha \in A$, $(\pi(\vec{x}_\alpha), \pi(\vec{y}_\alpha))$ is a point in $\{(\vec{v}, \vec{w}) \in H \times H : \vec{v} \neq \vec{w}\}$, which is a second countable space. Since A is uncountable, these points have a limit point in the same space, so we may fix *distinct* $\vec{v}, \vec{w} \in H$ and a sequence of *distinct* elements α_n in A ($n \in \omega$) such that the $\pi(\vec{x}_{\alpha_n})$ converge to $\vec{v}$ and the $\pi(\vec{y}_{\alpha_n})$ converge to $\vec{w}$. Hence, in the *weak* topology of $\mathbb{B}$ and L_2, the $\vec{x}_{\alpha_n}$ converge to $\vec{v}$ and the $\vec{y}_{\alpha_n}$ converge to $\vec{w}$. Since $f(\vec{x}_{\alpha_n}) = f(\vec{y}_{\alpha_n})$, we have $f(\vec{v}) = f(\vec{w})$, contradicting that f was $1-1$ on H. ☺

A similar use of Cantor sets occurs in the construction in [13], with the following differences: Their K_α were not disjoint; in fact, in [13] it appears necessary that every

Cantor set gets listed uncountably many times. As a result, the space constructed was not first countable. However, if one does not care about disjointness, there is no advantage to using a disk, so [13] used an interval where we used D. The extra dimension in D lets us prove Lemma 2.2, which is easily seen to be false of $[0,1]$. Actually, when the K_α are disjoint, condition (2) above is redundant, since it follows from (3), but if the K_α are not disjoint, (2) is required to guarantee that L_2 is norm bounded.

By iterating our construction, we now prove the following theorem.

2.5. Theorem. There is a first countable uniform Eberlein compact space L_ω such that every function in $E_0(L_\omega)$ is constant.

Observe that this is not true for the L_2 of Lemma 2.4. For example, let $g \in E_0(D)$ be nonconstant, and define f by $f(\vec{x}) = g(\vec{x} \cdot \vec{e}_\alpha)$. Then $f \in E_0(L_2)$, and is not constant on U_α. To prevent such functions from existing, we shall, for each α: take disjoint Cantor sets $K_{\alpha\beta} \subset U_\alpha$, and, for each β, attach a new disk going off in a new direction, labeled by a unit vector $\vec{e}_{\alpha\beta}$. This would create a space L_3. But now, we must iterate this procedure, to take care of functions on these new disks. Iterating ω times, we have unit vectors $\vec{e}_t$ indexed by finite sequences from $\mathfrak{c}$.

To describe L_ω, we use the same tree notation as in §1, where now $\mathfrak{c}$ is our index set. For the rest of this section, let $\mathbb{B}$ be a complex Hilbert space with an orthonormal basis consisting of unit vectors $\{\vec{e}_s : s \in \mathfrak{c}^{<\omega}\}$. We shall use $\vec{e}$ to abbreviate $\vec{e}_{()}$ and $\vec{e}_\alpha$ to abbreviate $\vec{e}_{(\alpha)}$. Let π_n be the perpendicular projection from $\mathbb{B}$ onto the subspace spanned by $\{\vec{e}_s : lh(s) < n\}$. In particular, $\pi_0(\vec{x}) = \vec{0}$ for all $\vec{x}$, and π_1 is the projection onto the one-dimensional subspace spanned by $\vec{e}$.

If $lh(s) = n$, let D_s be the set of vectors of the form $\sum_{i<n} z_i \vec{e}_{s\restriction i}$, where each $|z_i| \le 2^{-i}$. Since D_s is finite dimensional, the weak and norm topologies agree on D_s, and D_s is homeomorphic to D^{n+1}. In particular, $D_{()} = \{z\vec{e} : |z| \le 1\}$ plays the role of the D' in the proof of Lemma 2.4. Note that if $i \le n$, then $\pi_{i+1}(D_s) = D_{s\restriction i}$.

We begin by enumerating enough of the conditions required of the Cantor sets K_t ($t \in \mathfrak{c}^{<\omega}$) to define L_ω. Then, after defining L_ω, we prove a sequence of lemmas, adding conditions on the K_t as necessary, to show L_ω has the desired properties.

2.6. Basic requirements on the K_t.

(Ra) $K_{()} = \{\vec{0}\}$.

(Rb) For each s, the $K_{s\alpha}$ for $\alpha \in \mathfrak{c}$ are disjoint closed subsets of D_s, and $\vec{x} \cdot \vec{e}_s \neq 0$ for all $\vec{x} \in K_{s\alpha}$.

(Rc) For each s and each β, if $n = lh(s)$, then $\pi_n(K_{s\beta}) \subseteq K_s$.

In particular, for $s = ()$, we have $K_\alpha \subset D_{()}$, as in the proof of Lemma 2.4. Now, we iterate that construction by using the $K_{\alpha\beta}$, $K_{\alpha\beta\gamma}$, etc. The $K_{()} = \{\vec{0}\}$ plays no role in the definition of L_ω, but is included to make some of the notation more uniform. Item (Rc) for $n = 0$ says nothing; for $n = 1$, $\pi_1(K_{\alpha\beta}) \subseteq K_\alpha$ corresponds to the informal idea above that the $K_{\alpha\beta}$ are chosen inside U_α.

We shall need to add conditions (Rd)(Re) to (Ra)(Rb)(Rc) later.

2.7 Definition. L_ω is the set of all $\vec{x} \in \mathbb{B}$ that satisfy (1) - (3):
(1) For each s, $|\vec{x} \cdot \vec{e}_s| \leq 2^{-lh(s)}$.
(2) For all s, t such that $s \perp t$, $\vec{x} \cdot \vec{e}_s = 0$ or $\vec{x} \cdot \vec{e}_t = 0$.
(3) For all t, if $n = lh(t)$, then either $\vec{x} \cdot \vec{e}_t = 0$ or $\pi_n(\vec{x}) \in K_t$.
We give L_ω the weak topology. $L_n = \pi_n(L_\omega)$. For $\vec{x} \in L_\omega$, $P(\vec{x}) = \{s \in \mathfrak{c}^{<\omega} : \vec{x} \cdot \vec{e}_s \neq 0\}$. For $t \in \mathfrak{c}^{<\omega}$ and $n = lh(t)$, set $U_t = L_\omega \cap (\pi_n^{-1}(K_t) \backslash K_t)$.

2.8. Lemma. Each L_n is a closed subset of L_ω and $\bigcup_{n \in \omega} L_n$ is dense in L_ω.

Proof. $L_n \subseteq L_\omega$ holds because each of (1), (2), (3) is preserved under π_n. Density follows because for every $\vec{x} \in \mathbb{B}$, the $\pi_n(\vec{x})$ converge weakly (and in norm) to $\vec{x}$. L_n is closed in L_ω because $\pi_n(\mathbb{B})$ is weakly closed in $\mathbb{B}$. ☺

We think of the L_n as the *levels* in the construction. $L_0 = K_{()}$. $L_1 = D_{()}$. L_2 is exactly the space constructed in the proof of Lemma 2.4. The U_t will play the same role here as the U_α did there. Elements of $L_3 \backslash L_2$ are of the form $r_0\vec{e} + r_1\vec{e}_\alpha + r_2\vec{e}_{\alpha\beta}$, where $0 < |r_i| \leq 2^{-i}$ for each i, $r_0\vec{e} \in K_\alpha$, and $r_0\vec{e} + r_1\vec{e}_\alpha \in K_{\alpha\beta}$.

2.9. Lemma.
i. For each $\vec{x} \in L_\omega$, $P(\vec{x})$ is a path in $\mathfrak{c}^{<\omega}$.
ii. For each $\vec{x} \in L_\omega$, $\|\vec{x}\|^2 \leq \frac{4}{3}$.
iii. L_ω is weakly closed in $\mathbb{B}$.
iv. L_ω is first countable and compact.
v. Each U_t is open in L_ω.

Proof For (i), use items (2),(3) in the definition of L_ω and the fact that $\vec{x} \cdot \vec{e}_s \neq 0$ for all $\vec{x} \in K_{s\alpha}$. Now, (ii) follows by item (1). (iii) is immediate from the definition of L_ω, and compactness of L_ω follows by (iii) and (ii). First countability follows from Lemma 2.3; $C_{\vec{x}} = \{\vec{e}_s : s \in P(\vec{x})\}$, unless $P(\vec{x})$ is finite with maximal element s and $\vec{x} \in K_{s\alpha}$, in which case $C_{\vec{x}} = \{\vec{e}_s : s \in P(\vec{x})\} \cup \{\vec{e}_{s\alpha}\}$. For (v), note that $U_t = \{\vec{x} \in L_\omega : \vec{x} \cdot \vec{e}_t \neq 0\}$. ☺

Applying conditions (Rc) and (Rb) on the K_s, we have the following lemma.

2.10. Lemma.
i. For each t, if $n \leq lh(t)$ and $s = t \restriction n$, then $K_s \supseteq \pi_n(K_t)$.
ii. Each $K_t \subseteq L_{lh(t)}$.

If the K_α are chosen as in the proof of Lemma 2.4, then every $f \in E_0(L_2)$ will be constant on $D_{()}$. We must be careful not to destroy this property in choosing the $K_{\alpha\beta}$ and passing to L_3. In the proof of Lemma 2.4, it was important that each $\pi^{-1}(\{\vec{x}\})$ was nowhere dense. Now, $L_2 \cap \pi^{-1}(\{\vec{x}\})$ will still be nowhere dense in L_2, but depending on how the $K_{\alpha\beta}$ meet this set, $L_3 \cap \pi^{-1}(\{\vec{x}\})$ might have interior points. To handle this, we assume the following product structure on the K_s:

(Rd) For each s of length $n \geq 0$ and each α, there are a nonempty relatively clopen subset $P \subseteq K_s$ and a homeomorphism ψ from $P \times 2^\omega$ onto $K_{s\alpha}$, satisfying $\pi_n(\psi(\vec{x}, y)) = \vec{x}$ for all $\vec{x} \in P$ and all $y \in 2^\omega$.

Note that (Rd) implies that $\pi_n(K_{s\alpha}) = P$. Induction on $lh(s)$ establishes the next lemma.

2.11. Lemma. K_s is a Cantor set whenever $lh(s) > 0$.

2.12. Lemma. Suppose that $m > 0$ and C is a closed subset of L_m such that C is nowhere dense (in the relative topology of L_m) and $C \cap K_s$ is nowhere dense (in the relative topology) in K_s for all s of length m. Then $L_\omega \cap \pi_m^{-1}(C)$ is nowhere dense in L_ω. In particular, $L_\omega \cap \pi_m^{-1}(\{\vec{x}\})$ is nowhere dense in L_ω for all $\vec{x} \in L_m$.

Proof. The "in particular" follows from Lemma 2.11, which implies that $C = \{\vec{x}\}$ satisfies the hypotheses of Lemma 2.12. Now set $C_n = L_n \cap \pi_m^{-1}(C)$ for each $n \geq m$; so $C_m = C$. To prove 2.12, since $\bigcup_{n\in\omega} L_n$ is dense in L_ω, it suffices to prove claim (i) below. To do this, we prove claims (i) and (ii) together, by induction on $n \geq m$.

i. For each $n \geq m$, C_n is nowhere dense in L_n.
ii. Whenever $lh(s) = n$, $C_n \cap K_s$ is nowhere dense in K_s.

Claim (ii) for $n+1$ follows from (ii) for n plus assumption (Rd) on the K_s, and claim (i) for $n+1$ follows from (i) and (ii) for n (just using (Ra),(Rb),(Rc)). ☺

For each $s \in \mathfrak{c}^{<\omega}$, with $lh(s) = n$, let

$$\hat{K}_s = \{\vec{v} + z\vec{e}_s : \vec{v} \in K_s \,\&\, |z| \leq 2^{-n}\} \quad .$$

Note that $\hat{K}_s$ is homeomorphic to $K_s \times D$ and is a subset of L_ω. If $H \subseteq \hat{K}_s$ and $\vec{v} \in K_s$, let $H_{\vec{v}}$ be the "vertical slice", $\{\vec{v} + z\vec{e}_s : |z| \leq 2^{-n}\}$. Call a function f *s-level-constant* iff f only depends on the $\vec{v}$ here; that is, f is constant on each $(\hat{K}_s)_{\vec{v}}$. In particular, f is ()-level-constant iff f is constant on $D_{()}$, and the K_α chosen as in the proof of Lemma 2.4 will ensure that every $f \in E_0(L_\omega)$ is ()-level-constant. Likewise, we shall choose the $K_{s\alpha}$ to ensure that every $f \in E_0(L_\omega)$ is s-level-constant. Note first that if we do this for all s, then f is constant.

2.13. Lemma. If $f \in C(L_\omega)$ is s-level-constant for all $s \in \mathfrak{c}^{<\omega}$, then f is constant.

Proof. By induction on n, f is constant on each L_n. The result follows because $\bigcup_{n\in\omega} L_n$ is dense in L_ω. ☺

Now we list the final condition on the $K_{s\alpha}$:

(Re) For each s of length n and each $f \in C(L_\omega)$: If f is not s-level-constant, then there are a nonempty clopen set $P \subseteq K_s$, a Cantor set $H \subseteq \{\vec{v} + z\vec{e}_s : \vec{v} \in P \,\&\, |z| \leq 2^{-n}\}$, and uncountably many different α such that $K_{s\alpha} \subset H$, and for each $\vec{v} \in P$, f is $1-1$ on $H_{\vec{v}}$.

We must verify that we may choose the K_t to meet all five conditions (Ra), (Rb), (Rc), (Rd), (Re). We choose these by induction on $lh(t)$. Condition (Ra) specifies $K_{()}$, and the K_α will be exactly as in the proof of Lemma 2.4; these were chosen by applying Lemma 2.2. Likewise, given K_s with $lh(s) > 0$, we choose the $K_{s\alpha}$ by applying the next lemma to $\hat{K}_s$. In fact, we modify the proof of Lemma 2.2, replacing Lemma 1.1 by Lemma 1.2, to prove this lemma.

2.14. Lemma. Let $\{E_\delta : \delta \in \mathfrak{c}\}$ be a partition of 2^ω into $\mathfrak{c}$ Cantor sets. If K is a Cantor set, then there is a disjoint family $\{K_\alpha \subset K \times (D\backslash\{0\}) : \alpha \in \mathfrak{c}\}$ of $\mathfrak{c}$ Cantor sets with the following property: For each $f \in C(K \times D)$ with $f \restriction (\{x\} \times D)$ nonconstant for

some $x \in K$, there are a nonempty clopen $P \subseteq K$ and an $H \subset P \times (D \backslash \{0\})$ that satisfy conditions (1) - (4):

(1) H_x is a Cantor set for every $x \in P$.

(2) f is $1-1$ on $\{x\} \times H_x$ for every $x \in P$.

(3) There is a continuous $\varphi : H \to 2^\omega$ such that the map $(x,y) \mapsto (x, \varphi(x,y))$ is a homeomorphism from H onto $P \times 2^\omega$.

(4) For each $\delta \in \mathfrak{c}$, the set $\{(x,y) \in H : \varphi(x,y) \in E_\delta\}$ is one of the K_α.

Proof. First, for each such f, apply continuity to choose a nonempty clopen $P_f \subseteq K$ such that for $\mathfrak{c}$ different $\theta \in [0, 2\pi)$, $f \restriction (\{x\} \times R_\theta)$ fails to be constant for all $x \in P_f$. Then, by transfinite induction, choose a distinct θ_f for each such f such that $f \restriction (\{x\} \times R_{\theta_f})$ is not constant for all $x \in P_f$. Then, choose $H_f \subseteq P_f \times R_{\theta_f}$ such that (1), (2), and (3) hold; this is possible by Lemma 1.2. Of course, $\varphi = \varphi_f$ depends on f. Finally, let the K_α enumerate all the sets $\{(x,y) \in H_f : \varphi_f(x,y) \in E_\delta\}$ as f and δ vary. ☺

Now we complete the proof of Theorem 2.5.

Proof of Theorem 2.5. Construct L_ω as above. Suppose $f \in E_0(L_\omega)$. By Lemma 2.13, it suffices to prove that f is s-level-constant for each s. Suppose not. Fix H, P as in condition (Re) above, so that $A = \{\alpha : K_{s\alpha} \subseteq H\}$ is uncountable. For $\alpha \in A$, choose a nonempty open W_α such that $W_\alpha \subseteq \overline{W}_\alpha \subseteq U_{s\alpha}$ and f is constant on $\overline{W}_\alpha$. Then $\pi_{n+1}(\overline{W}_\alpha) \subseteq K_{s\alpha} \subseteq H$ and $\pi_n(\overline{W}_\alpha) \subseteq \pi_n(K_{s\alpha}) \subseteq P \subseteq K_s \subseteq L_n$. Choose $\vec{x}_\alpha$ and $\vec{y}_\alpha$ in $\overline{W}_\alpha$ such that $\pi_n(\vec{x}_\alpha) = \pi_n(\vec{y}_\alpha)$ but $\pi_{n+1}(\vec{x}_\alpha) \neq \pi_{n+1}(\vec{y}_\alpha)$; this is possible because $\pi_{n+1}(\overline{W}_\alpha)$ is closed in $K_{s\alpha}$ and, by Lemma 2.12, is not nowhere dense in $K_{s\alpha}$. As in the proof of Lemma 2.4, there are distinct $\vec{v}, \vec{w} \in H$ and a sequence of distinct elements α_k in A ($k \in \omega$) such that the $\pi_{n+1}(\vec{x}_{\alpha_k})$ converge to $\vec{v}$ and the $\pi_{n+1}(\vec{y}_{\alpha_k})$ converge to $\vec{w}$. Then, in the weak topology, the $\vec{x}_{\alpha_k}$ converge to $\vec{v}$ and the $\vec{y}_{\alpha_k}$ converge to $\vec{w}$. So, $f(\vec{v}) = f(\vec{w})$, while $\pi_n(\vec{v}) = \pi_n(\vec{w}) \in \pi_n(H) \subseteq P$, contradicting that f is $1-1$ on $H_{\pi_n(\vec{v})}$. ☺

Finally, we remark on $E_0(X, M)$ for other M.

2.15. Lemma. If X is a compact Hausdorff space and M is any Hausdorff space, then

(1) $E_0(X, \mathbb{R})$ contains only the constant functions

implies

(2) $E_0(X, M)$ contains only the constant functions.

If M contains a closed interval, then (2) implies (1).

Proof. For (1) $\to$ (2), fix $f \in E_0(X, M)$. We may assume $M = f(X)$, whence M is compact. For each $g : M \to [0,1]$, $g \circ f$ is in $E_0(X, \mathbb{R})$ and hence constant, which implies that f is constant. For $\neg(1) \to \neg(2)$, if g maps $\mathbb{R}$ homeomorphically into M and f is a nonconstant function in $E_0(X, \mathbb{R})$, then $g \circ f$ is a nonconstant function in $E_0(X, M)$. ☺

In particular, in making $E(X) = E_0(X, \mathbb{R})$ small, we also make $E_0(X, \mathbb{C})$ small. Note that 2.15 can fail if M does not contain an interval, since then, if X is a closed interval, $E_0(X, M) = C(X, M)$ contains only the constant functions (since every arc contains a simple arc), while $E_0(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$. We do not study 2.15 for such M in detail here, but it seems to involve the geometric-topological properties of X and M.

§**3. Making $E_0(X)$ Big.** Here, we consider spaces X for which $E_0(X,M) = C(X,M)$. This turns out to be an interesting topological property of X. We begin with a simple remark.

The condition $E_0(X,M) = C(X,M)$ is not hereditary to closed subsets of X, but it is, in many cases, hereditary to *regular closed* subsets – that is, to subsets of the form $\overline{U}$, where U is open in X.

3.1. Lemma. Suppose that X is a compact Hausdorff space, $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$, and Y is a regular closed subspace of X. Then $E_0(Y,\mathbb{R}) = C(Y,\mathbb{R})$.

Proof. Say $Y = \overline{U}$, where U is open. Suppose $g \in C(Y,\mathbb{R})$. By the Tietze Extension Theorem, g can be extended to an $f \in C(X,\mathbb{R})$. Then $\Omega_g \cap U = \Omega_f \cap U$. Since $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$, we have that Ω_f is dense in X, so Ω_g is dense in Y. ☺

We remark that in Lemma 3.1, one can replace $\mathbb{R}$ by any Banach space (using a slightly longer proof), but not by an arbitrary metric space M. For a counter-example, let M be a Cantor set and let X be the cone over M. Then $E_0(X,M) = C(X,M)$ contains only constant functions. But X contains a regular closed Y homeomorphic to $M \times [0,1]$, and $E_0(Y,M) \neq C(Y,M)$. Also, even in the simple case $M = \mathbb{R}$, the property $E_0(Y,\mathbb{R}) = C(Y,\mathbb{R})$ holds for *all* closed $Y \subseteq X$ iff X is scattered; if X is not scattered, then X will contain a closed subset Y which is separable with no isolated points, which implies $E_0(Y,\mathbb{R}) \neq C(Y,\mathbb{R})$ (by (2) $\Rightarrow$ (1) of Theorem 3.2 below).

Now, to study the property $E_0(X,M) = C(X,M)$, it is convenient to generalize our notions in two ways.

First, although X will always be compact and M will always be metric, we look at more general functions from X into M. In particular recall that $f : X \to M$ is called *Borel measurable* iff the inverse image of every open set is a Borel subset of X, and *Baire measurable* iff the inverse image of every open set is a Baire subset of X; the Baire sets are the σ-algebra generated by the open F_σ sets. The Baire measurable functions into a separable Banach space form the least class of functions containing the continuous functions and closed under pointwise limits.

Second, we consider also $\widehat{\Omega}_f$, which we define to be the union of all open $U \subseteq X$ such that for some first category set $C \subseteq X$, f is constant on $U \backslash C$. Note that regardless of f, Ω_f (defined in the Introduction) and $\widehat{\Omega}_f$ are open, with $\Omega_f \subseteq \widehat{\Omega}_f$. If f is continuous, then $\Omega_f = \widehat{\Omega}_f$.

The property $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$ is just one of a sequence of related properties:

(1) Every nonempty open subset of X is either nonseparable or contains an isolated point.
(2) $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$.
(3) For all metric spaces M, $E_0(X,M) = C(X,M)$.
(3′) For all separable metric spaces M and all Baire measurable $f : X \to M$, $\widehat{\Omega}_f$ is dense in X.
(3″) For all separable metric spaces M and all Baire measurable $f : X \to M$, Ω_f is dense in X.
(4) For all separable metric spaces M and all Borel measurable $f : X \to M$, $\widehat{\Omega}_f$ is dense in X.
(5) For all separable metric spaces M and all Borel measurable $f : X \to M$, Ω_f is dense in X.
(6) In X, every nonempty G_δ set has a nonempty interior.
$(*)$ In X, every first category set is nowhere dense.

Conditions (1) – (6) are listed in order of increasing strength. Condition $(*)$ does not fit into the sequence, but is relevant by the next Theorem.

3.2. Theorem. Suppose X is compact Hausdorff. Then

$$(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Leftrightarrow (3') \Leftrightarrow (3'') \Rightarrow (2) \Rightarrow (1) \quad .$$

Furthermore (5) is equivalent to $(*)$ plus (4).

Proof. For (2) $\Rightarrow$ (1), assume (1) fails; so there is nonempty open U which is separable and has no isolated points. Let $Y = \overline{U}$. By Lemma 3.1, it is sufficient to produce an $f \in C(Y,\mathbb{R}) \setminus E_0(Y,\mathbb{R})$. Let $\{p_n : n \in \omega\}$ be dense in U, hence in Y. For each distinct m, n, $\{f \in C(Y,\mathbb{R}) : f(p_m) \neq f(p_n)\}$ is dense and open in $C(Y,\mathbb{R})$ (in the usual norm topology), so by the Baire Category Theorem, there is an $f \in C(Y,\mathbb{R})$ such that $f(p_m) \neq f(p_n)$ whenever $m \neq n$. But then for each $r \in \mathbb{R}$, $f^{-1}\{r\}$ contains at most one p_n, and is hence nowhere dense in Y, since Y has no isolated points. Thus, $f \notin E_0(Y,\mathbb{R})$.

Clearly, (3″) $\Rightarrow$ (3′) $\Rightarrow$ (3), so to prove these three are equivalent, we assume (3), fix a Baire measurable $f : X \to M$, and show that Ω_f is dense in X. Since M can be embedded into a separable Banach space, we may assume that M *is* a Banach space; now, we can let $g_n : X \to M$, for $n \in \omega$, be continuous functions such that f can be obtained from the g_n by some transfinite iteration of taking pointwise limits. Define $g : X \to M^\omega$ by: $g(x)_n = g_n(x)$. Then $\Omega_g \subseteq \Omega_f$, and, by (3), Ω_g is dense in X.

To prove (6) $\Rightarrow$ (5), observe that for *any* compact X, if H is a nonempty closed G_δ and f is a Borel measurable map into a second countable space, there is always a nonempty closed G_δ set $K \subseteq H$ such that f is constant on K.

The rest of the chain of implications from (6) down to (1) are now trivial. To see that (5) $\Rightarrow$ $(*)$, let C be first category; then $C \subseteq \bigcup_{n\in\omega} K_n$, where each K_n is closed nowhere dense. Define $f : X \to 2^\omega$ so that $f(x)_n$ is 1 if $x \in K_n$ and 0 if $x \notin K_n$. Then Ω_f is dense and open, and is disjoint from all the K_n, so C is nowhere dense.

To see that $(*)$ plus (4) implies (5), we let f be Borel measurable; to prove Ω_f dense, we fix a nonempty open V and try to find a nonempty open $U \subseteq V$ such that f is constant

on U. By (4), there is a nonempty open $W \subseteq V$ such that f is constant on $W \backslash C$ for some first category C. By $(*)$, C is nowhere dense, so let $U = W \backslash \overline{C}$. ☺

A familiar example of a space satisfying (6) is $\beta\mathbb{N}\backslash\mathbb{N}$.

Conditions (5), (4), (3″), and (3′) involve arbitrary Baire or Borel measurable maps. Each of these conditions is equivalent to the restatement we obtain by replacing M by the Cantor set 2^ω. This is easily seen by translating the condition to one involving an ω-sequence of Borel or Baire sets. For example, (5) is equivalent to the statement that given Borel sets B_n $(n \in \omega)$, the union of all open U such that $\forall n(U \subseteq B_n \textit{ or } U \cap B_n = \emptyset)$ is dense in X.

This is not true for (3), which involves continuous functions. For example, if X is connected, then, trivially, $E_0(X, 2^\omega) = C(X, 2^\omega)$, whereas $E_0(X, \mathbb{R})$ need not be all of $C(X, \mathbb{R})$. If X is zero-dimensional, then $E_0(X, 2^\omega) = C(X, 2^\omega)$ does imply (3). In fact, for zero-dimensional spaces, (3) has a restatement in terms of sequences of clopen sets (see the proof of Theorem 3.3(c) below).

Regarding (3) $\Rightarrow$ (2), if $E_0(X, M) = C(X, M)$ for *any* M containing an interval, then (2) holds. In §4, we show that (2) does not imply (3), although it is easy to see that (2) implies that $E_0(X, \mathbb{R}^n) = C(X, \mathbb{R}^n)$ for each finite n. Counter-examples to the other implications of Theorem 3.2 reversing are provided by some fairly familiar spaces, as we point out below. However, the implications do reverse for certain families of spaces. In particular, we consider the cases when X is *extremally disconnected* (*e.d.*), when X is an *Eberlein compact*, when X is a *LOTS*, and when has the *ccc*. X is called e.d. iff the closure of every open subset of X is clopen. X is an Eberlein compact iff X is homeomorphic to a weakly compact subspace of a Banach space. X is a LOTS iff X is a totally ordered set, given the order topology. X has the ccc iff there is no uncountable family of disjoint open sets in X.

The following theorem summarizes what we know for these and some other simple classes.

3.3. Theorem. Let X be compact Hausdorff.

a. If X is metric, then (1) $\Leftrightarrow$ (5), and (1) – (5) hold iff the isolated points of X are dense in X.
b. If X is e.d., then (2) $\Leftrightarrow$ (4).
c. If X is zero-dimensional, then (2) $\Leftrightarrow$ (3).
d. If X is ccc, then (3) $\Leftrightarrow$ (5).
e. If X is Eberlein compact, then (4) $\Leftrightarrow$ (5), and (4) – (5) hold iff the isolated points of X are dense in X.
f. If X is a LOTS, then (1) $\Leftrightarrow$ (3).

Proof. (a) is immediate from the fact that compact metric spaces are separable.

For (b), assume (2), and let $f : X \to M$ be Borel measurable. Let $\{B_n : n \in \omega\}$ be an open base for M. Since each $f^{-1}(B_n)$ is a Borel set, there are open $U_i \subseteq X$, for $i \in \omega$, such that each $f^{-1}(B_n)$ is in the σ-algebra generated by $\{U_i : i \in \omega\}$. Let $K_i = \overline{U}_i$, which is clopen. Define $g : X \to 2^\omega$ so that $g(x)_i = 1$ iff $x \in K_i$. Since 2^ω is embeddable in $\mathbb{R}$, (2) implies that Ω_g is dense. Since $\bigcup_{i\in\omega}(K_i \backslash U_i)$ is first category, $\Omega_g \subseteq \widehat{\Omega}_f$, so $\widehat{\Omega}_f$ is dense.

For (c), assume (2), and let $f : X \to M$ be continuous. Let the B_n be as in the proof of (b). Since each $f^{-1}(B_n)$ is an open F_σ set, there are clopen sets $K_i \subseteq X$ for $i \in \omega$ such that each $f^{-1}(B_n)$ is a union of some subfamily of the K_i. Now, construct g as in the proof of (b), and note that $\Omega_g \subseteq \Omega_f$.

For (d), assume (3″), and let $f : X \to M$ be Borel measurable. Since X is ccc, there is a Baire measurable $g : X \to M$ and a Baire first category set C such that $f(x) = g(x)$ for all $x \notin C$. Define $h : X \to M \times \{0,1\}$ so that $h(x) = (g(x), 0)$ if $x \notin C$, and $h(x) = (g(x), 1)$ if $x \in C$. Then, applying (3″) to h, Ω_h is dense in X. Since $\Omega_h \subseteq \Omega_g$ and $\Omega_h \cap C = \emptyset$, Ω_f is dense in X.

For (e), assume that X is Eberlein compact and satisfies (4); we prove that the isolated points are dense. By a result of Benyamini, Rudin, and Wage [4], there is a dense G_δ set $Y \subseteq X$ such that Y is metrizable in its relative topology. Fix some metric on Y; then for $E \subseteq Y$, $diam(E)$ denotes the diameter of E with respect to this metric. For each n, let $\mathcal{W}_n$ be a maximal disjoint family of open nonempty subsets of Y of diameter $\leq 2^{-n}$; then $W_n = \bigcup\{W : W \in \mathcal{W}_n\}$ is open and dense. Assume also that each $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n$ in the sense that $\forall W \in \mathcal{W}_{n+1} \exists V \in \mathcal{W}_n (W \subseteq V)$, and for each $V \in \mathcal{W}_n$ which is not a singleton, there are at least two $W \in \mathcal{W}_{n+1}$ such that $W \subseteq V$. Let $Z = \bigcap_n W_n$; then Z is also a dense G_δ subset of X. For each n, let $f_n : Z \to 2$ be any function such that f_n is constant on *every* $W \in \mathcal{W}_{n+1}$ and f_n is constant on *no* $V \in \mathcal{W}_n$ unless V is a singleton. This defines $f : Z \to 2^\omega$ by $f(z)_n = f_n(z)$. Let M be the disjoint sum of 2^ω and a single point, p, and extend f to a function $\tilde{f} : X \to M$ by mapping $X \backslash Z$ to p. Then $\tilde{f}$ is Borel measurable, and every point in $\widehat{\Omega}_{\tilde{f}}$ is isolated in X.

For (f), assume (1), and fix $f \in C(X, M)$; we must show that Ω_f is dense. So, fix a nonempty open interval $(a, b) \subseteq X$. We must produce a nonempty open $W \subseteq (a, b)$ such that f is constant on W. This is trivial if (a, b) contains an isolated point, so assume that (a, b) contains no isolated points, and hence is nonseparable. For each n, there is a finite cover of $[a, b]$ by open intervals, $I_1^n, I_2^n, \ldots$ such that each $diam(f(I_j^n)) \leq 1/n$. Since (a, b) is nonseparable, we can choose $W \subseteq (a, b)$ to be an open interval which contains none of the endpoints of any I_j^n. Then for each n, W is a subset of some I_j^n, so $diam(f(W)) \leq 1/n$. Thus, f is constant on W. ☺

A (compact) Suslin line in which every open interval is nonseparable is a ccc LOTS which satisfies (1), and hence (5), applying (d) and (f) of the Theorem. Of course, the Suslin line does not satisfy (6). The absolute (or projective cover) of a Suslin line is a compact ccc e.d. space which satisfies (5) but not (6). So is $\beta\mathbb{N}$, but this example is "trivial" because the isolated points are dense. Note, however, that it is consistent with the axioms of set theory that there are no Suslin lines, in which case (5) for a ccc space would imply that the isolated points are dense.

In general, for a LOTS, (1) need not imply (4). A simple counter-example is $X = [0,1]^\omega$, ordered lexically; (4) is refuted by $f(x) = \sum_{n \in \omega} x_n \cdot 2^{-n}$. One can replace $[0,1]$ by the Cantor set here to get a zero-dimensional LOTS, providing also a counter-example to (c) extending to (2) $\Leftrightarrow$ (4).

The Stone space of an atomless probability algebra is a compact e.d. space which satisfies (1) but not (3). To refute (3), let the K_i ($i \in \omega$) be clopen independent events of

probability $\frac{1}{2}$, and construct g as in the proof of (b) above. This provides a counter-example to replacing (2) by (1) in either (b) or (c).

Conditions (4), $(*)$, and (5) are equivalent to algebraic conditions on the Boolean algebra of regular open subsets of X (see [9, 15]); in particular, each condition holds for X iff it holds for the absolute of X. Condition (4) is equivalent to the (ω, ω) - distributive law:

$$\bigwedge_{n\in\omega} \bigvee_{i\in\omega} b_{n,i} = \bigvee \{ \bigwedge_{n\in\omega} b_{n,\varphi(n)} : \varphi \in \omega^\omega \} \quad .$$

Condition $(*)$ is equivalent to the weak (ω, ∞) - distributive law; that is, for each cardinal κ,

$$\bigwedge_{n\in\omega} \bigvee_{\alpha\in\kappa} b_{n,\alpha} = \bigvee \{ \bigwedge_{n\in\omega} \bigvee_{\alpha\in\varphi(n)} b_{n,\alpha} : \varphi \in ([\kappa]^{<\omega})^\omega \} \quad .$$

Here, $[\kappa]^{<\omega}$ is the set of finite subsets of κ. Condition (5) is simply (4) plus $(*)$ by Theorem 3.2, which is equivalent to the standard (ω, ∞) - distributive law.

Proceeding completely off the deep end, we may regard the open (or regular open) subsets of X as a forcing order (see a set theory text, such as [10] or [11]). Then (4) is simply the statement that the order adds no reals, while (5) is the stronger statement that the order adds no ω - sequences. Condition $(*)$ is the finite approximation property familiar from random real forcing or Sacks forcing; that is, for each κ and each $\psi : \omega \to \kappa$ in the generic extension, there is a $\varphi : \omega \to [\kappa]^{<\omega}$ in the ground model such that each $\psi(n) \in \varphi(n)$. Prikry forcing at a measurable cardinal (see §37 of [10]) is an example of a forcing order (and hence, by the standard translation, a compact e.d. space) which satisfies (4) but not $(*)$, and hence not (5). Another such example is Namba forcing (see §26 of [10]).

Returning temporarily to Earth, it is natural to ask which of the properties, (1) – (6), $(*)$, are preserved by finite products. Now, (1) and (6) are, trivially. We don't know about (2), but (3) is; to see this, identify $C(X \times Y, M)$ with $C(X, C(Y, M))$, and note that $C(Y, M)$ is another metric space. Finally, $(5) = (4) + (*)$ is refuted by a well known forcing order. Let $S \subset \omega_1$ be stationary and co-stationary. Let $\mathbb{P}, \mathbb{Q}$ be Jensen's forcings for shooting a club through $S, \omega_1 \backslash S$, respectively (see VII.H25 of [11]). Then $\mathbb{P}, \mathbb{Q}$ each satisfy (5), while $\mathbb{P} \times \mathbb{Q}$ collapses ω_1, and hence satisfies neither (4) nor $(*)$. One may now translate $\mathbb{P}, \mathbb{Q}$ into compact e.d. spaces (by the standard translation), or into Corson compacta (using the fact that these partial orders have no decreasing ω_1 chains).

Preservation by infinite products is uninteresting. If X is an infinite product of spaces with more than one point, then all of (2) – (6) fail, as does $(*)$, whereas (1) will hold if, for example, infinitely many of the X_n are nonseparable. See Theorem 5.4 for more about such products.

Some of the results in in this section overlap results of Bella, Hager, Martinez, Woodward, and Zhou [2, 3, 12]. They also defined $E_0(X, \mathbb{R})$ (which they called $dc(X)$), and they considered spaces with our property (2), which they called DC-spaces. With somewhat different terminology, they prove what amounts to the fact that (6) implies (2), and that (1) and (2) are equivalent when X is a LOTS.

§4. **On Eberlein Compacta.** Here we consider the properties (1) – (6) of §3 in the case that X is an Eberlein compact. We already know by Theorem 3.3 that the stronger conditions (4) or (5) can hold only in the trivial case that the isolated points of X are dense in X; it is easy to see that (6) holds iff X is finite. Thus, only (1), (2), and (3) are of interest, and for these, the Eberlein compacta can be tailored to satisfy whatever we want. The one we constructed in §2 satisfies (1), but not (2). We now describe two modifications of this construction, producing Eberlein compacta which satisfy (3) but not (4) (this is easy), and then (2) but not (3) (this requires more work).

For the first example, since we already know that (4) cannot hold unless the isolated points are dense, it suffices to prove the following.

4.1. Theorem. There is a first countable uniform Eberlein compact space X such that X has no isolated points and $E_0(X, M) = C(X, M)$ for all metric spaces M.

Proof. Follow exactly the notation in §2, so that X will be the L_ω there. Choose sets K_t for $t \in \mathfrak{c}^{<\omega}$ so that conditions (Ra)(Rb)(Rc) of §2.6 hold, so that all the lemmas through Lemma 2.10 still apply. But, replace (Rd)(Re) by

(Rf) Each K_s is a singleton, and the $K_{s\alpha}$, for $\alpha \in \mathfrak{c}$, enumerate all the singletons in $\hat{K}_s \backslash K_s$.

As before, $U_t = X \cap (\pi_n^{-1}(K_t) \backslash K_t)$ where $n = lh(t)$.

Now, fix $f \in C(L_\omega, M)$, where M is metric.

Note: f is constant on U_t for all but countably many t. If not, we could find an s and an uncountable $A \subseteq \mathfrak{c}$ such that f is not constant on $U_{s\alpha}$ for all $\alpha \in A$. For $\alpha \in A$, let $K_{s\alpha} = \{\vec{x}_\alpha\}$, and choose $\vec{y}_\alpha \in U_{s\alpha}$ such that $f(\vec{y}_\alpha) \neq f(\vec{x}_\alpha)$. Since the range of f is compact, and hence second countable, we may, as in the last paragraph of the proof of Lemma 2.4, fix *distinct* $p, q \in M$ and distinct $\alpha_n \in A$ ($n \in \omega$) such that the $f(\vec{x}_{\alpha_n})$ converge to p and the $f(\vec{y}_{\alpha_n})$ converge to q. Now, the points $\vec{x}_{\alpha_n}$ are in K_s, which is compact metric, so, by passing to a subsequence, we may assume that the $\vec{x}_{\alpha_n}$ converge to some point $\vec{x} \in K_s$. Hence, in the weak topology, since $\pi_{n+1}(\vec{y}_\alpha) = \vec{x}_\alpha$, the $\vec{y}_{\alpha_n}$ converge to $\vec{x}$ also. Applying f to these sequences, $f(\vec{x}) = p \neq q = f(\vec{x})$, a contradiction.

It follows that Ω_f is dense in X, since every nonempty open set in X contains uncountably many U_t (to see this, apply the above "note" and the fact that the co-zero sets of continuous functions form a basis for X). ☺

We remark that in the above "note", we used the same method to prove $E_0(X)$ big as we used in Lemma 2.4 to prove $E_0(X)$ small; we have simply reversed the roles of f and π.

Also, it is possible to make the space of Theorem 4.1 zero-dimensional by restricting the coordinates to lie in a Cantor set. This would not be possible for the spaces of §2, or the space used for Theorem 4.2(b) below.

Observe that in the proof of Theorem 4.1, the Hilbert space $\mathbb{B}$ can be either complex or real, since unlike in §2, we no longer need the $\vec{e}_s$ direction to be two dimensional. This holds in the next construction as well, although we shall need that the base level L_0 be infinite dimensional.

Also observe that if the $K_{s\alpha}$ were not singletons, the above proof would establish a modified "note": for all but countably many t, $f(\vec{y}) = f(\pi_n(\vec{y}))$ for all $\vec{y} \in U_t$. This is the

key to building a space satisfying (2) but not (3). We shall make sure that $\hat{K}_s$ has "large dimension", so that any real-valued function will be constant on many subsets of $\hat{K}_s$, and these subsets will be the $K_{s\alpha}$; this will ensure that $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$. However, if M itself has "large dimension", then this argument will fail, so that $E_0(X,M) \neq C(X,M)$.

The following definition and theorem pin down precisely for which M we can conclude $E_0(X,M) = C(X,M)$ from $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$. It suffices to consider only compact M, since the range of each continuous map is compact. Let $\mathcal{F}_0$ be the collection of all zero or one point spaces. For an ordinal $\alpha > 0$, let $\mathcal{F}_\alpha$ be the class of all compact metric spaces M such that there is a $\varphi \in C(M,[0,1])$, with $\varphi^{-1}\{r\} \in \bigcup_{\delta<\alpha} \mathcal{F}_\delta$ for every $r \in [0,1]$. So, for example, induction on $n \in \omega$ shows that $[0,1]^n \in \mathcal{F}_n$. Then, if M is the one-point compactification of the disjoint union of the $[0,1]^n$, we may let φ map M to a simple sequence to conclude that $M \in \mathcal{F}_\omega$. Define $\mathcal{F} = \bigcup_{\delta \in ON} \mathcal{F}_\delta$, where ON is the class of ordinals. Actually, since every compact metric space has at most $\mathfrak{c}$ closed subspaces, $\mathcal{F} = \bigcup_{\delta<\mathfrak{c}^+} \mathcal{F}_\delta$.

4.2. Theorem.

a. If X is compact Hausdorff, $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$, and $M \in \mathcal{F}$, then $E_0(X,M) = C(X,M)$.
b. There is a first countable uniform Eberlein compact X such that $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$, but for all compact metric spaces $M \notin \mathcal{F}$, $E_0(X,M) \neq C(X,M)$.

So, if we fix any compact metric space $M \notin \mathcal{F}$, we get an X satisfying condition (2) of §3, but not (3). Of course, we need to know that such an M exists, but that follows by a theorem of Levshenko. There is a class of *strongly infinite dimensional* spaces which includes the Hilbert cube, $[0,1]^\omega$. Levshenko showed that if M is a strongly infinite dimensional compact metric space and $\varphi \in C(M,[0,1])$, then some $\varphi^{-1}\{r\}$ is strongly infinite dimensional (see [1]). This gives us the following lemma.

4.3. Lemma. If $M \in \mathcal{F}$, then M is not strongly infinite dimensional.

Proof. By induction on ordinals α, prove that every $M \in \mathcal{F}_\alpha$ is not strongly infinite dimensional. ☺

The definition of $\mathcal{F}$ gives us the following easy inductive proof of Theorem 4.2(a).

Proof of Theorem 4.2(a). Suppose that $M \in \mathcal{F}_\alpha$, and suppose (inductively) that the result holds for all $M' \in \bigcup_{\delta<\alpha} \mathcal{F}_\delta$. Suppose X is compact Hausdorff and $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$. Fix $f \in C(X,M)$. To prove $f \in E_0(X,M)$, we fix a nonempty open $U \subseteq X$, and we produce a nonempty open $V \subseteq U$ such that f is constant on V. Applying the definition of $\mathcal{F}_\alpha$, fix $\varphi \in C(M,[0,1])$ such that for each $r \in [0,1]$ $\varphi^{-1}\{r\} \in \bigcup_{\delta<\alpha} \mathcal{F}_\delta$. Then $\varphi \circ f \in C(X,\mathbb{R}) = E_0(X,\mathbb{R})$, so fix a nonempty open set $W \subseteq U$ such that $\varphi \circ f$ has some constant value r on $\overline{W}$. Now $\varphi^{-1}\{r\} \in \mathcal{F}_\delta$, for some $\delta < \alpha$, and $E_0(\overline{W},\mathbb{R}) = C(\overline{W},\mathbb{R})$ (by Lemma 3.1). Applying the induction hypothesis, $f \restriction \overline{W} \in C(\overline{W},\varphi^{-1}\{r\}) = E_0(\overline{W},\varphi^{-1}\{r\})$, so we may choose choose a nonempty open subset $V \subseteq W$ such that $f \restriction V$ is constant. ☺

To prove Theorem 4.2(b), we first prove some more lemmas about $\mathcal{F}$. Then, rather than construct a space X which works for every compact metric space $M \notin \mathcal{F}$, we present Lemma 4.8, which allows us to construct a separate X_M for each M. To construct each

X_M, we proceed as in §2; that is, each X_M will be an L_ω, constructed using somewhat modified conditions on the sets K_t. We then glue these X_M together to complete the proof of Theorem 4.2(b).

We begin with the lemmas about $\mathcal{F}$. First, another simple induction yields closure under subsets:

4.4. Lemma. If $M \in \mathcal{F}$ and H is a closed subset of M, then $H \in \mathcal{F}$.

We also get closure under finite unions:

4.5. Lemma. Suppose that M is compact metric and $M = H \cup K$, where H, K are closed subsets of M and $H, K \in \mathcal{F}$. Then $M \in \mathcal{F}$.

Proof. Since H is a closed G_δ, fix $\varphi \in C(M, [0,1])$ such that $\varphi^{-1}\{0\} = H$. Then, $\varphi^{-1}\{0\} \in \mathcal{F}$. For $r > 0$, we have $\varphi^{-1}\{r\} \subseteq K$, so $\varphi^{-1}\{r\} \in \mathcal{F}$ by Lemma 4.4. So, $\varphi^{-1}\{r\} \in \mathcal{F}$ for each $r \in [0,1]$, which implies that $M \in \mathcal{F}$. ☺

Call M *nowhere in* $\mathcal{F}$ iff M is nonempty and for each nonempty open $V \subseteq M$, we have $\overline{V} \notin \mathcal{F}$. Note that such an M has no isolated points, since $\mathcal{F}$ contains all one point spaces.

4.6. Lemma. If M is a compact metric space and $M \notin \mathcal{F}$, then there is a closed set $K \subseteq M$ such that K is nowhere in $\mathcal{F}$.

Proof. Let $\mathcal{U} = \{U \subseteq M : U \text{ is open and } \overline{U} \in \mathcal{F}\}$, and let $K = M \setminus \bigcup \mathcal{U}$.

First, note that K is nonempty: If K were empty, then, by compactness, M would be covered by a finite subfamily of $\mathcal{U}$, which would imply $M \in \mathcal{F}$ by Lemma 4.5.

To prove that K is nowhere in $\mathcal{F}$, it suffices (by Lemma 4.4) to prove that $\overline{B(p,\epsilon)} \cap K \notin \mathcal{F}$ whenever $p \in K$ and $\epsilon > 0$. Note that $B(p,\epsilon)$ and its closure are computed in M, not K. Let $N = \overline{B(p,\epsilon)}$. Fix $\varphi \in C(N, [0,1])$ such that $\varphi^{-1}\{0\} = N \cap K$. Since $B(p,\epsilon) \cap K$ is nonempty, $N \notin \mathcal{F}$, so there must be some $r \in [0,1]$ such that $\varphi^{-1}\{r\} \notin \mathcal{F}$. However, for $r > 0$, $\varphi^{-1}\{r\}$ is compact and disjoint from K, so it is covered by a finite subfamily of $\mathcal{U}$, and hence, as above, is in $\mathcal{F}$. So, r must be 0, so $N \cap K \notin \mathcal{F}$. ☺

Let $\mathcal{N}(K)$ be the family of all compact $H \subseteq K$ such that H is nowhere in $\mathcal{F}$. The following lemma is trivial, given the above results, but we state it to emphasize the abstract properties of our construction.

4.7. Lemma. If K is compact metric and nowhere in $\mathcal{F}$, then

1. $\mathcal{N}(K)$ is a family of nonempty closed subspaces of K.
2. $K \in \mathcal{N}(K)$.
3. For each $H \in \mathcal{N}(K)$ and each nonempty relatively open $U \subseteq H$, there is an $L \in \mathcal{N}(K)$ with $L \subseteq U$.

Most of the proof of Theorem 4.2(b) proceeds using just the conclusion to Lemma 4.7, without any reference to $\mathcal{F}$. Note that if K is a singleton, and $\mathcal{N}(K)$ is redefined to be $\{K\}$, we also have the conclusion to Lemma 4.7, and the proof of 4.2(b) then reproves Theorem 4.1.

Now, as promised earlier, we present Lemma 4.8, which reduces our construction to a modification of that of §2.

4.8. Lemma. If $\{X_\alpha : \alpha < \mathfrak{c}\}$ is a collection of nonempty first countable uniform Eberlein compacta, then there is a first countable uniform Eberlein compact space X, with disjoint clopen subsets J_α homeomorphic to X_α, such that $\bigcup_{\alpha\in\mathfrak{c}} J_\alpha$ is dense in X.

Proof. We may assume that each X_α is a weakly compact subset of the closed unit ball of the Hilbert space $\mathbb{B}_0$, and that $\mathbb{B}_0$ is a closed linear subspace of the Hilbert space $\mathbb{B}$, which contains unit vectors $\vec{m}_\alpha (\alpha \in \mathfrak{c})$ and $\vec{b}$, all orthogonal to each other and to $\mathbb{B}_0$.

Let r_α, for $\alpha \in \mathfrak{c}$, enumerate $(0,1)$. Let $J_\alpha = X_\alpha + r_\alpha\vec{b} + \vec{m}_\alpha$. Then J_α is homeomorphic to X_α (via translation). Let X be the union of the J_α, together with all $r\vec{b}$ for $r \in [0,1]$. Then X is norm bounded (by $\sqrt{3}$), and is weakly closed, since any limit of points in distinct J_α must be of the form $r\vec{b}$; the existence of these limits also shows that the union of the J_α is dense in X. The space X is first countable by Lemma 2.3. To see that the J_α are disjoint and (weakly) clopen in X, project along the $\vec{m}_\alpha$ direction. ☺

We remark that translating along the $\vec{b}$ direction made X first countable. If we just let $J_\alpha = X_\alpha + \vec{m}_\alpha$, and let X be the union of the J_α plus $\{\vec{0}\}$, then X would be simply the one-point compactification of the disjoint union of the X_α. Of course, we could build the one-point compactification even if there are more than $\mathfrak{c}$ X_α, in which case one cannot make X first countable (by Arkhangel'skiĭ's Theorem).

We now construct the space X_M. Applying Lemma 4.6, let K be a closed subset of M which is nowhere in $\mathcal{F}$. Let $\mathbb{B}$ be a *real* Hilbert space with an orthonormal basis consisting of unit vectors $\{\vec{e}_s : s \in \mathfrak{c}^{<\omega}\} \bigcup \{\vec{b}_i : i \in \omega\}$. Let $\mathbb{B}_n$ be the closed linear span of $\{\vec{e}_s : lh(s) < n\} \cup \{\vec{b}_i : i \in \omega\}$. Since $\mathbb{B}_0$ is infinite dimensional, we can embed K in the first level of our space. To do so we replace condition (Ra) of §2.6 by the following:

(Ra′) $K_{()}$ is a weakly compact subset of the closed unit ball of $\mathbb{B}_0$, and $K_{()}$ is homeomorphic to K.

Actually, we could also make $K_{()}$ norm compact, but this is unnecessary.

Let π_n be the perpendicular projection from $\mathbb{B}$ onto $\mathbb{B}_n$. If $lh(s) = n$, let D_s be the set of vectors of the form $\vec{v} + \sum_{i\le n} r_i \vec{e}_{s|i}$, where $\vec{v} \in K_{()}$ and each $|r_i| \le 2^{-i}$. In particular, $D_{()}$ is homeomorphic to $K \times [-1,1]$. As in §2, the product with $[-1,1]$ allows us to make the K_α disjoint subsets of $D_{()}$. As before, if $i \le n$, then $\pi_{i+1}(D_s) = D_{s|i}$.

We will choose the K_t for $t \in \mathfrak{c}^{<\omega}$ so that they satisfy condition **(Ra′)**, along with **(Rb)** and **(Rc)** of §2.6. Now, define $X_M = L_\omega$ to be the set of $\vec{x} \in \mathbb{B}$ satisfying conditions (1), (2), and (3) of §2.7, along with condition (0): $\pi_0(\vec{x}) \in K_{()}$.

As before, for $t \in \mathfrak{c}^{<\omega}$ and $n = lh(t)$, $U_t = L_\omega \cap (\pi_n^{-1}(K_t) \setminus K_t)$. So $U_{()} = L_\omega \setminus K_{()} = \{\vec{x} \in L_\omega : \vec{x}\cdot\vec{e} \ne 0\}$. In this construction, we still have the levels $L_n = \pi_n(L_\omega)$, with $L_0 = K_{()}$ and $L_1 = D_{()}$. Now, elements of level $L_3 \setminus L_2$ are of the form $\vec{v} + r_0\vec{e} + r_1\vec{e}_\alpha + r_2\vec{e}_{\alpha\beta}$, where $0 < |r_i| \le 2^{-i}$ for each i, $\vec{v} \in K_{()}$, $\vec{v} + r_0\vec{e} \in K_\alpha$, and $\vec{v} + r_0\vec{e} + r_1\vec{e}_\alpha \in K_{\alpha\beta}$.

This L_ω still satisfies Lemmas 2.8, 2.9, and 2.10, provided we replace the bound in 2.9(ii) by $\frac{7}{3}$. The proofs are the same, except for the proof of 2.9(iv), where we join $\{\vec{b}_i : i \in \omega\}$ to each $C_{\vec{x}}$.

Now, we utilize $\mathcal{N}(K_{()})$ to choose the K_s. Let $\vec{u}_{s\alpha}$ be of the form $\vec{u}_s + r_{s\alpha}\vec{e}_s$, with $\vec{u}_{()} = \vec{0}$. Choose the K_s so that they satisfy, in addition to **(Ra′)**, **(Rb)**, and **(Rc)**, three more conditions:

(Rg) Each K_s is of the form $H_s + \vec{u}_s$, where $H_s \in \mathcal{N}(K_{()})$.

(Rh) For each s and each $L \in \mathcal{N}(K_{()})$ such that $L \subseteq H_s$, $K_{s\alpha} = L + \vec{u}_{s\alpha}$ for some $\alpha \in \mathfrak{c}$.

(Ri) For each s and each nonempty relatively open $V \subseteq \hat{K}_s$, there are uncountably many α such that $K_{s\alpha} \subseteq V$.

So, (Rg) says that each K_s is a translate of a subset of $K_{()}$. The $\hat{K}_s$ is defined precisely as in §2, so that conditions (Ra′), (Rb), (Rc) already imply that $K_{s\alpha} \subseteq \hat{K}_s$. Condition (Rg) guarantees that, unlike in §2, the projection $\pi_0 : K_t \to K_{()}$ is 1-1 for each t (and its inverse is translation by $\vec{u}_t$). Using Lemma 4.7, it is easy to see that conditions (Ra′), (Rb), (Rc), (Rg), (Rh), (Ri) can all be met.

If f is a function on L_ω and $n = lh(t)$, we shall say that f is *t-extension-constant* iff for all $\vec{x} \in K_t$ and all $\vec{y} \in L_\omega \cap \pi_n^{-1}\{\vec{x}\}$, $f(\vec{y}) = f(\vec{x})$. By repeating the proof of the "note" in the proof of Theorem 4.1, we see the following:

4.9. Lemma. If M is metric and $f \in C(L_\omega, M)$, then f is t-extension-constant for all but countably many t.

In the next lemma, we use condition (Ri) to show that the U_t form a pi-base.

4.10. Lemma. If V is open and nonempty in L_ω, then for some t, $U_t \subseteq V$.

Proof. We may assume that $V = \{\vec{x} \in L_\omega : f(\vec{x}) \neq 0\}$, where $f \in C(L_\omega, \mathbb{R})$. First fix s such that $V \cap \hat{K}_s$ is nonempty, and then apply condition (Ri) plus Lemma 4.9 to set $t = s\alpha$, where α is chosen so that $K_{s\alpha} \subseteq V \cap \hat{K}_s$ and f is $s\alpha$-extension-constant. ☺

In the case of Theorem 4.1, all the K_t were singletons, so "t-extension-constant" meant "constant", and the instance of Lemma 4.10 used there was simple enough that we omitted the proof of it. In general, we cannot improve Lemma 4.9 to conclude that f is constant on any open set. For example, the projection π_0 is 1-1 on each K_t, so cannot be constant on K_t unless K_t is a singleton. Applying Lemma 4.10, we get our last lemma.

4.11. Lemma. If $\mathcal{N}(K_{()})$ contains no singletons, then $\pi_0 \in C(L_\omega, K_{()})$ and $\Omega_{\pi_0} = \emptyset$.

Note, by condition (Ri), however, that $\mathcal{N}(K_{()})$ contains no singletons iff no set in $\mathcal{N}(K_{()})$ has any isolated points. Of course, this is certainly true with $\mathcal{N}$ meaning "nowhere in $\mathcal{F}$". The specific features of this $\mathcal{N}$ appear in the conclusion of our proof.

Proof of Theorem 4.2(b). By Lemma 4.8 and the fact that there are only $\mathfrak{c}$ compact metric spaces (up to homeomorphism), it suffices to fix an $M \notin \mathcal{F}$ and verify that for the space L_ω constructed above, $E_0(L_\omega, \mathbb{R}) = C(L_\omega, \mathbb{R})$, but $E_0(L_\omega, M) \neq C(L_\omega, M)$. Here, L_ω was constructed with $K_{()}$ homeomorphic to a subset of M which was nowhere in $\mathcal{F}$, so that $E_0(L_\omega, M) \neq C(L_\omega, M)$ follows from Lemma 4.11.

Now, fix $f \in C(L_\omega, \mathbb{R})$. In view of Lemma 4.10, to prove that $f \in E_0(L_\omega, \mathbb{R})$, it suffices to fix an s and find a nonempty open $V \subseteq U_s$ on which f is constant. By Lemma 4.10, fix α such that f is $s\alpha$-extension-constant. By condition (Rg), $K_{s\alpha} = H_{s\alpha} + \vec{u}_{s\alpha}$, where $H_{s\alpha} \in \mathcal{N}(K_{()})$. Now, applying the properties of $\mathcal{N}$, we can choose an $L \in \mathcal{N}(K_{()})$ such that $L \subseteq H_{s\alpha}$ and f is constant on $L + \vec{u}_{s\alpha}$. Applying condition (Rh) to $s\alpha$, we can choose a β such that $H_{s\alpha\beta} = L + \vec{u}_{s\alpha\beta}$. So let $V = U_{s\alpha\beta}$. ☺

§**5. On Banach Spaces.** In this section, we make a few remarks on $E_0(X,M)$ in the case that X is an arbitrary compact Hausdorff space and M is a Banach space. For definiteness, we take the scalar field to be $\mathbb{R}$, but all the results are unchanged if we replace $\mathbb{R}$ by $\mathbb{C}$.

First, as we have seen in §3, there are many X for which $E_0(X,M) = C(X,M)$. For a given X, this can depend on M, but in view of §4 and the fact that every infinite dimensional Banach space contains a homeomorphic copy of the Hilbert cube, there are only three possibilities:

1. $E_0(X,M) = C(X,M)$ for all Banach spaces M.
2. $E_0(X,M) = C(X,M)$ for all finite dimensional M, but not for any infinite dimensional M.
3. $E_0(X,M) \neq C(X,M)$ for all Banach spaces M.

Furthermore, there are Eberlein compact X with no isolated points realizing each of these possibilities ((3) is trivial; see §4 for (1) and (2)).

Second, in studying the properties of $E_0(X,M)$ as a normed linear space, we can isolate the two properties which are of fundamental importance. If $f, g_1, g_2 \in C(X,M)$, let us say that f is *refined by* g_1, g_2 iff for all $x, y \in X$, if $g_1(x) = g_1(y)$ and $g_2(x) = g_2(y)$ then $f(x) = f(y)$. A linear subspace $E \subseteq C(X,M)$ has the *refinement property* iff for all $f, g_1, g_2 \in C(X,M)$, if $g_1, g_2 \in E$ and f is refined by g_1, g_2, then $f \in E$. We say that E has the *disjoint summation property* iff whenever $\sum_{i\in\omega} f_i = f$ in $C(X,M)$, each $f_i \in E$, and the sets $\{x : f_i(x) \neq 0\}$, for $i \in \omega$, are all disjoint, then $f \in E$. The set of polynomial functions in $C([0,1],\mathbb{R})$ has the disjoint summation property (trivially) but not the refinement property, while the set of functions which are constant in some neighborhood of $\frac{1}{2}$ has the refinement property but not the disjoint summation property. Let us call E a *nice* subspace of $C(X,M)$ iff E has both properties. Examples of nice E are $E_0(X,M)$, $C(X,M)$, and the space of all constant functions. Or, one may fix any open $U \subseteq X$; then $\{f \in C(X,M) : U \subseteq \overline{\Omega}_f\}$ is nice. Another example is the functions of essentially countable range; that is, let μ be a Baire measure on X, and then let $D(X,M,\mu)$ be the set of $f \in C(X,M)$ such that for some μ-null-set $S \subseteq X$, $f(X\backslash S)$ is countable. Another is the category analog of this – the set of $f \in C(X,M)$ such that for some countable $P \subset M$, $\bigcup\{int(f^{-1}\{p\}) : p \in P\}$ is dense in X (int denotes "interior").

One advantage of studying nice E is that we may restrict our attention to the case where E separates the points of X. In general, given $E \subseteq C(X,M)$, we may define an equivalence relation $\sim$ on X by $x \sim y$ iff $f(x) = f(y)$ for all $f \in E$. Let Y be the quotient, $X/\sim$; then Y is a compact Hausdorff space, and there is a canonical projection, π, from Y onto X. Let $E' = \{g \in C(Y,M) : g \circ \pi \in E\}$. Then E' is isometric to E, and E' separates the points of Y. Further, both the refinement property and the disjoint summation property are preserved here, so if E is nice, then so is E'.

Some examples, when we start with $E = E_0(X,M)$: For the spaces constructed in §2: If $X = L_\omega$, then Y is a singleton. If $X = L_2$, then Y is obtained by collapsing L_1 to a point. In these two cases, $E' = E_0(Y,M)$, but this is not in general true. For example, let Q be any dense subset of $[0,1]$, and form X by attaching a copy of the L_ω of §2 to each $q \in Q$, where each copy goes off in some perpendicular direction. There is then a natural retraction, $r : X \to [0,1]$, and $E_0(X,\mathbb{R})$ consists of all functions of the form $f \circ r$, where

$f \in C([0,1],\mathbb{R})$. So, we may identify Y with $[0,1]$ and π with r, and E' is $C(Y,\mathbb{R})$, not $E_0(Y,\mathbb{R})$.

Third, we remark on some consequences of assuming that $E \subseteq C(X,M)$ has the refinement property. If $\varphi \in C(M,M)$ and $f \in E$, then $\varphi \circ f \in E$ (since $\varphi \circ f$ is refined by f, f). If $M = \mathbb{R}$, and we view $C(X,M)$ as a Banach algebra (under pointwise multiplication), then E is a subalgebra. More generally, if we fix any non-zero vector $\vec{v} \in M$, we may let $\hat{E} \subseteq C(X,\mathbb{R})$ be the set of all $g \in C(X,\mathbb{R})$ such that the map $x \mapsto g(x)\vec{v}$ is in E. Note that this does not depend on the $\vec{v}$ chosen, and if $g \in \hat{E}$ and $f \in E$, then $gf \in E$. Note also that $\hat{E}$ is nice.

It follows that if $E \subseteq C(X,M)$ has the refinement property and separates the points of X, then E is dense in $C(X,M)$. To see this, fix $f \in C(X,M)$. If $M = \mathbb{R}^n$, just apply the Stone-Weierstrass Theorem to f composed with the projections onto n one-dimensional subspaces. Then, for a general M, first approximate f arbitrarily closely by a map into a finite-dimensional subspace.

Actually, one can get more than just what is provided by a simple application of the Stone-Weierstrass Theorem. For example, we can arrange for the approximating function to be identically zero wherever f is zero:

5.1. Lemma. Suppose $E \subseteq C(X,M)$ has the refinement property and separates the points of X. Fix $f \in C(X,M)$ and fix $\epsilon > 0$. Then there is a $g \in E$ with $\|g - f\| \leq \epsilon$, $\|g\| = \|f\|$, and $\|g(x)\| = \|f(x)\|$ for all x such that $\|f(x)\|$ equals either 0 or $\|f\|$.

Proof. Assume $\epsilon < \|f\|$. Fix $h \in E$ with $\|h - f\| \leq \epsilon/2$. Then, let $\varphi : M \to M$ be any continuous map such that for all $\vec{v} \in M$: $\|\varphi(\vec{v}) - \vec{v}\| \leq \epsilon/2$, $\varphi(\vec{v}) = \vec{0}$ when $\|\vec{v}\| \leq \epsilon/2$, and $\varphi(\vec{v}) = \|f\|$ when $|\,\|\vec{v}\| - \|f\|\,| \leq \epsilon/2$ (φ can just move each $\vec{v}$ radially). Then, let $g = \varphi \circ h$. ☺

Fourth, is $E_0(X,M)$ a Banach space? Certainly it is in the extreme cases where it is all of $C(X,M)$ and where it contains only the constant functions. To analyze the general situation, we may, as pointed out above, just consider the case where $E \subseteq C(X,M)$ is nice and separates the points of X. Then, clearly, E a Banach space in the standard norm iff it is all of $C(X,M)$. Furthermore, if E is not all of $C(X,M)$, then, following Bernard and Sidney [6,14], it is not even *Banachizable*; that is, there is no norm which makes E into a Banach space and gives E a topology finer than the one inherited from $C(X,M)$. In fact, every nice E is *barreled*, which is a stronger property. There are a number of equivalents to being barreled, discussed in [14]. One is that for every linear space $\mathbb{L}$ with $E \subseteq \mathbb{L} \subsetneq \overline{E}$, $\mathbb{L}$ is not Banachizable ($\overline{E}$ is the completion of E; here, $\overline{E} = C(X,M)$). Another is the "weak sequential property" for E, which is the conclusion of the next Lemma; this is a convenient way of establishing barreledness. The proof of the next Lemma is very similar in spirit to that of Theorem 2 of [14], but we include it because at first sight, the proof as stated in [14] might appear to require some additional assumptions about E and X. The two examples above of subspaces of $C([0,1],\mathbb{R})$ show that neither of the two components of "nice", "refinement property" and "disjoint summation property", is sufficient here.

5.2. Lemma. Let X be compact and let M be a Banach space. Suppose that E is a nice subspace of $C(X,M)$. Let Λ_n, for $n \in \omega$, be in the dual space, E^*. Assume that for every $g \in E$, $\Lambda_n(g) \to 0$. Then $\sup_n \|\Lambda_n\| < \infty$.

Proof. As pointed out above, we may assume also that E is dense in $C(X,M)$, so we may consider Λ_n to be in $C(X,M)^*$. Note that *if* $E = C(X,M)$, the conclusion is immediate by the Banach-Steinhaus Theorem. In any case, whenever $\mathbb{H}$ is a closed linear subspace of $C(X,M)$ such that $\mathbb{H} \subseteq E$,

$$\sup\{|\Lambda_n(h)| : h \in \mathbb{H} \cap \overline{B}(0,1) \ \& \ n \in \omega\} < \infty \quad . \tag{1}$$

Here, $\overline{B}(0,1)$ is the closed unit ball of $C(X,M)$. Now, assume that $\sup_n \|\Lambda_n\| = \infty$. We shall get a contradiction by applying (1).

For any $f \in C(X,M)$, let $supt(f)$ be the closure of $\{x \in X : f(x) \neq \vec{0}\}$. By compactness of X, we may fix a point p such that for all neighborhoods V of p,

$$\sup\{|\Lambda_n(f)| : f \in \overline{B}(0,1) \ \& \ n \in \omega \ \& \ supt(f) \subseteq V\} = \infty \quad .$$

By Lemma 5.1 (applied to $\hat{E}$ – see above), let $g \in \hat{E}$ be such that $\|g\| = 1$, $g(p) = 1$, and $supt(g) \subseteq V$. Then $\mathbb{H} = \{g\vec{v} : \vec{v} \in M\}$ is a closed linear subspace of $C(X,M)$ (isometric to M) such that $\mathbb{H} \subseteq E$, so we may apply (1) above. It follows, by considering functions of the form $x \mapsto f(x) - g(x)f(p)$, that for all neighborhoods V of p,

$$\sup\{|\Lambda_n(f)| : f \in \overline{B}(0,1) \ \& \ n \in \omega \ \& \ supt(f) \subseteq V \ \& \ f(p) = \vec{0}\} = \infty \quad .$$

Next, we show that for all neighborhoods V of p,

$$\sup\{|\Lambda_n(g)| : g \in \overline{B}(0,1) \cap E \ \& \ n \in \omega \ \& \ supt(g) \subseteq V \backslash \{p\}\} = \infty \quad .$$

To see this, fix $K > 0$, and then fix n and $f \in C(X,M)$ such that $\|f\| \leq 1$, $supt(f) \subseteq V$, $f(p) = \vec{0}$, and $|\Lambda_n(f)| \geq 3K$. Let $f' \in C(X,M)$ be such that $\|f'\| \leq 1$, $supt(f') \subseteq V$, f' vanishes in some neighborhood of p, and $\|f' - f\| \leq K/\|\Lambda_n\|$. Applying Lemma 5.1 to f', let $g \in E$ be such that $\|g\| \leq 1$, $supt(g) \subseteq V$, g vanishes in some neighborhood of p, and $\|g - f'\| \leq K/\|\Lambda_n\|$. Then $|\Lambda_n(g)| \geq K$.

Thus, we may inductively choose open neighborhoods V_j of p, $n_j \in \omega$, and $h_j \in E$ such that each $\overline{V}_{j+1} \subseteq V_j$, $supt(h_j) \subseteq V_j \backslash \overline{V}_{j+1}$, $\|h_j\| = 1$, and $|\Lambda_{n_j}(h_j)| \geq j$. Let $\mathbb{H}$ be the closed linear span in $C(X,M)$ of the h_j. Since the h_j are disjointly supported, $\mathbb{H} \subseteq E$ (and $\mathbb{H}$ is isometric to c_0), so we have a contradiction to (1) above. ☺

Fifth, is $E_0(X,M)$ first category in itself? We ask this because if $E_0(X,M)$ is of second category, then Lemma 5.2 becomes trivial by the Banach-Steinhaus Theorem. Fortunately, $E_0(X,M)$ is first category in many cases; for example, when X contains a nonempty separable open subset with no isolated points (see the proof of (2) $\Rightarrow$ (1) of Theorem 3.2). In fact, as pointed out by Bernard and Sidney, the original interest of $E_0(X)$ was that it provided examples of first category normed linear spaces which satisfy the Banach-Steinhaus Theorem, as well as a number of other results usually proved by category arguments. The following lemma describes some other situations in which $E_0(X,M)$ is of first category.

5.3. Lemma. Let X be compact and let M be a Banach space. Suppose that $E_0(X,M)$ is not a Banach space. Then $E_0(X,M)$ is of first category in itself if either of the following hold:

a. M is infinite dimensional.

b. X is zero-dimensional.

Proof. First, as indicated above, we may pass to a quotient and consider a nice $E \subseteq C(X,M)$ which is dense in $C(X,M)$ but not all of $C(X,M)$; of course, in (b), this quotient operation is trivial. Now, we need only show that E is of first category in $C(X,M)$.

Whenever H and K are closed subsets of X, let $U(H,K) = \{g \in C(X,M) : g(H) \cap g(K) = \emptyset\}$. Note that $U(H,K)$ is always open in $C(X,M)$. If H and K are disjoint, then either (a) or (b) guarantees that $U(H,K)$ is dense in $C(X,M)$.

Fix an $f \in C(X,M)\backslash E$. Since $f(X)$ is second countable, there are closed $H_n, K_n \subseteq X$ for $n \in \omega$ such that each $H_n \cap K_n = \emptyset$, and for all $x, y \in X$, if $f(x) \neq f(y)$, then for some n, $x \in H_n$ and $y \in K_n$. Let $G = \bigcap_{n\in\omega} U(H_n, K_n)$. Then G is a dense G_δ, and f is refined by g, g for all $g \in G$, so G is disjoint from E. ☺

The situation for finite dimensional M seems more complicated. We do not actually have an example of an $E_0(X,\mathbb{R})$ which is second category but not a Banach space, although it is easy to produce a *consistent* example of this by forcing [10, 11]. In the ground model, V, let $X = L_\omega$ be the space constructed in the proof of Theorem 4.2(b), so $E_0(X,\mathbb{R}) = C(X,\mathbb{R})$. Let $V[G]$ add one Cohen real. Then, in $V[G]$, $E_0(X,\mathbb{R})$ is of second category, since it contains the ground model $C(X,\mathbb{R})$, which is of second category with this forcing. However, in $V[G]$, $E_0(X,\mathbb{R})$ is not all of $C(X,\mathbb{R})$, since $V[G]$ will contain a $g \in C(K_{()},\mathbb{R})$ which is 1-1 on $K_{()} \cap V$; if $f = g \circ \pi_0 \in C(X,\mathbb{R})$, then $\Omega_f = \emptyset$. To verify the details of this construction, one must compare X and $C(X,\mathbb{R})$ in both models, V and $V[G]$; this is described in §3 of [8].

The following lemma yields a class of examples where $E_0(X,\mathbb{R})$ is of first category.

5.4. Theorem. Let M be any Banach space, and let $X = \prod_{i\in\omega} X_i$, where each X_i is compact Hausdorff and has more than one point. Then $E_0(X,M)$ is of first category, and is dense in $C(X,M)$.

Proof. Let $P_n = \prod_{i=0}^{n} X_i$, and let σ_n be the projection from X onto P_n. Call a function f on X *n-supported* iff $f = g \circ \sigma_n$ for some function g on P_n.

To prove that $E_0(X,M)$ dense in $C(X,M)$, it is sufficient to show that $E_0(X,\mathbb{R})$ separates points. Fix two distinct points, $x, y \in X$. Since an infinite product has no isolated points, we may assume (by partitioning the index set into infinitely many infinite sets) that each X_i has no isolated points. We may also assume that $\sigma_0(x) \neq \sigma_0(y)$. We now produce an f in $E_0(X,\mathbb{R})$ which separates x, y.

Note that if $\sigma_n(\Omega_f) = P_n$ for all n, then Ω_f will be dense. To obtain this situation, we shall focus on the dyadic rationals. Let $D_n = \{j \cdot 2^{-n} : 0 \leq j \leq 2^n\}$; so, $D_0 = \{0,1\}$ and $D_1 = \{0, \frac{1}{2}, 1\}$. Inductively choose $f_n \in C(X,[0,1])$ so that:

1. $x \in int(f_0^{-1}\{0\})$ and $y \in int(f_0^{-1}\{1\})$.
2. f_n is n-supported.
3. $\|f_{n+1} - f_n\| \leq 2^{-n}$.

4. $f_n^{-1}\{q\} \subseteq f_{n+1}^{-1}\{q\}$ whenever $q \in D_n$.
5. $\bigcup\{\sigma_n(int(f_{n+1}^{-1}\{q\})) : q \in D_{n+1}\} = P_n$.

Let $f = lim_n f_n$. This limit exists by (3). $\sigma_n(\Omega_f) = P_n$ for all n by (4)(5). f separates x, y by (1). Condition (2) allows the inductive construction of f_{n+1}.

Now, we prove that $E_0(X, M)$ is of first category in $C(X, M)$. For each n, let U_n be the set of all $f \in C(X, M)$ such that for all $z \in P_n$, f is not constant on $\{x \in X : \sigma_n(x) = z\}$. Then U_n is dense and open in $C(X, M)$, and $\Omega_f = \emptyset$ whenever $f \in \bigcup_{n\in\omega} U_n$. ☺

We remark that the space $D(X, M, \mu)$ defined above is always dense in $C(X, M)$ (by modifying the proof of the Urysohn Separation Theorem), and is always of first category, except in the trivial case that μ is a countable sum of point masses, where $D(X, M, \mu) = C(X, M)$.

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