Aronszajn Compacta *

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Abstract

We consider a class of compacta X such that the maps from X onto metric compacta define an Aronszajn tree of closed subsets of X.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. We begin by defining an *Aronszajn compactum*, along with a natural tree structure, by considering a space embedded into a cube. An equivalent definition, in terms of elementary submodels, is considered in Section 2.

Notation 1.1 *Given a product* $\prod_{\xi<\lambda} K_\xi$*: If* $\alpha \le \beta \le \lambda$*, then* π^β_α *denotes the natural projection from* $\prod_{\xi<\beta} K_\xi$ *onto* $\prod_{\xi<\alpha} K_\xi$*. If we are studying a space* $X \subseteq \prod_{\xi<\lambda} K_\xi$ *then* X_α *denotes* $\pi^\lambda_\alpha(X)$*, and* σ^β_α *denotes the restricted map* $\pi^\beta_\alpha \restriction X_\beta$*; so* $\sigma^\beta_\alpha : X_\beta \twoheadrightarrow X_\alpha$*.*

Definition 1.2 *An* embedded Aronszajn compactum *is a closed subspace* $X \subseteq [0,1]^{\omega_1}$ *with* $w(X) = \aleph_1$ *and* $\chi(X) = \aleph_0$ *such that for some club* $C \subseteq \omega_1$*: for each* $\alpha \in C$ $\mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma^{\omega_1}_\alpha)^{-1}\{x\}| > 1\}$ *is countable. For each such* X*, define* $T = T(X) := \bigcup\{\mathcal{L}_\alpha : \alpha \in C\}$*, and let* $\lhd$ *denote the following order: if* $\alpha, \beta \in C$*,* $\alpha < \beta$*,* $x \in \mathcal{L}_\alpha$ *and* $y \in \mathcal{L}_\beta$*, then* $x \lhd y$ *iff* $x = \pi^\beta_\alpha(y)$*.*

The $\sigma^{\omega_1}_\alpha$ for which $|\mathcal{L}_\alpha| \le \aleph_0$ are called *countable rank* maps in [2, 9]. Observe that $\langle T(X), \lhd\rangle$ is an Aronszajn tree: Each level $\mathcal{L}_\alpha$ is countable by definition, each $\mathcal{L}_\alpha$ is non-empty because $w(X) = \aleph_1$, and every chain in $T(X)$ is countable because

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$\chi(X) = \aleph_0$ and an uncountable path through $T(X)$ would yield a point of uncountable character in X. Of course, a compactum X of weight $\aleph_1$ may be embedded into $[0,1]^{\omega_1}$ in many ways, but if one copy of X is an Aronszajn compactum, then so are the others.

Lemma 1.3 *If $X, Y \subseteq [0,1]^{\omega_1}$, X is an embedded Aronszajn compactum, and Y is homeomorphic to X, then Y is an embedded Aronszajn compactum.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then use the fact that there is a club $D \subseteq \omega_1$ on which f commutes with projection; that is, for $\gamma \in D$, there is a homeomorphism $f_\gamma : X_\gamma \twoheadrightarrow Y_\gamma$ such that $\pi^{\omega_1}_\gamma \circ f = f_\gamma \circ \pi^{\omega_1}_\gamma$.

To get D: Let $\mathcal{B}$ be the base for $[0,1]^{\omega_1}$ consisting of all finite unions of rational open boxes. For $U \in \mathcal{B}$, let $\mathrm{supt}(U)$ denote the least α such that $U = (\pi^{\omega_1}_\alpha)^{-1}(\pi^{\omega_1}_\alpha(U))$. Then let $\mathcal{B}_\gamma = \{U \in \mathcal{B} : \mathrm{supt}(U) < \gamma\}$. Let $D \subseteq \omega_1$ be the set of all γ such that $\mathcal{B}_\gamma$ is closed under homeomorphic images, by f and f^{-1}, of disjoint pairs; more precisely, $\gamma \in D$ iff for each pair $U_0, U_1 \in \mathcal{B}_\gamma$, if $\mathrm{cl}(f(U_0 \cap X)), \mathrm{cl}(f(U_1 \cap X))$ are disjoint, then there are disjoint $V_0, V_1 \in \mathcal{B}_\gamma$ separating them, *and* if $\mathrm{cl}(f^{-1}(U_0 \cap Y)), \mathrm{cl}(f^{-1}(U_1 \cap Y))$ are disjoint, then there are disjoint $V_0, V_1 \in \mathcal{B}_\gamma$ separating them. ☕

The proof of this lemma shows that the Aronszajn trees derived from X and from Y are isomorphic on a club.

Definition 1.4 *An* Aronszajn compactum *is a compact X such that $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ and for some (equivalently, for all) $Z \subseteq [0,1]^{\omega_1}$ homeomorphic to X, Z is an embedded Aronszajn compactum.*

The next lemma is immediate from the definition. Further closure properties of the class of Aronszajn compacta are considered in Section 4.

Lemma 1.5 *A closed subset of an Aronszajn compactum is either second countable or an Aronszajn compactum.*

The Dedekind completion of an Aronszajn line is an Aronszajn compactum (see Section 2), and the associated tree is essentially the same as the standard tree of closed intervals. A special case of this is a compact Suslin line, which is a well-known compact L-space; that is, it is HL (hereditarily Lindelöf) and not HS (hereditarily separable). The line derived from a special Aronszajn tree is much different topologically, since it is not even ccc.

In Section 5 we shall prove:

Theorem 1.6 *Assuming $\diamondsuit$, there is an Aronszajn compactum which is both HS and HL.*

Our construction is flexible enough to build in additional properties for the space and its associated tree, which may be either Suslin or special; see Theorem 5.8. The form of the tree is (up to club-isomorphism) a topological invariant of X, but seems to be unrelated to more conventional topological properties of X; for example, X may be totally disconnected, or it may be connected and locally connected, with $\dim(X)$ finite or infinite.

Question 1.7 *Is there, in ZFC, an HL Aronszajn compactum?*

We would expect a ZFC example to be both HS and HL. Note that an Aronszajn compactum is dissipated in the sense of [10], so it cannot be an L-space if there are no Suslin lines by Corollary 5.3 of [10].

To refute the existence of an HL Aronszajn compactum, one needs more than just an Aronszajn tree of closed sets, since this much exists in the Cantor set:

Proposition 1.8 *There is an Aronszajn tree T whose nodes are closed subsets of the Cantor set 2^ω. The tree ordering is $\supset$, with root 2^ω. Each level of T consists of a pairwise disjoint family of sets.*

One can construct T inductively, as in the proof of Theorem 4 of Galvin and Miller [5], which is attributed there to Todorčević. The following proof, suggested by the referee, is simpler and gets a stronger result.

Proof. Let S be any special Aronszajn tree. Let $\varphi : S \to \mathbb{Q}$ be an order-preserving map that is 1-1 on each level $\mathcal{L}_\beta(S)$; φ is easy to get by adapting the natural embedding (described in Theorem 5, page 15, of [1]). Let $\varphi^*(s) = \{\varphi(t) : t \le s\}$. Then define $\Phi : S \to \mathcal{P}(\mathbb{Q}) \cong 2^{\mathbb{Q}}$ by $\Phi(s) = \{\varphi^*(s) \cup B : B \subseteq (\varphi(s), \infty) \cap \mathbb{Q}\}$; these are the subsets of $\mathbb{Q}$ which have $\varphi^*(s)$ as an initial segment. Note that $\Phi(s_1) \cap \Phi(s_2) = \emptyset$ whenever s_1, s_2 are incomparable in S; this follows from the fact that φ is 1-1 on each level. ☕

2 Elementary Submodels

We consider Aronszajn compacta from the point of view of elementary submodels. Assume that X is compact, with X (and its topology) in some suitably large $H(\theta)$. If X is first countable, so that $|X| \le \mathfrak{c}$ and its topology is a set of size $\le 2^{\mathfrak{c}}$, then θ can be any regular cardinal larger than $2^{\mathfrak{c}}$, assuming that the set X is chosen so that its transitive closure has size $\le \mathfrak{c}$.

If $X \in M \prec H(\theta)$, then there is a natural quotient map $\pi = \pi_M : X \twoheadrightarrow X/M$ obtained by identifying two points of X iff they are not separated by any function in $C(X, \mathbb{R}) \cap M$. Furthermore, X/M is second countable whenever M is countable.

Lemma 2.1 *Assume that X is compact, $w(X) = \aleph_1$, and $\chi(X) = \aleph_0$. Then the following are equivalent:*

1. *X is an Aronszajn compactum.*
2. *Whenever M is countable and $X \in M \prec H(\theta)$, there are only countably many $q \in X/M$ such that $\pi^{-1}\{q\}$ is not a singleton.*
3. *(2) holds for all M in some club of countable elementary submodels of $H(\theta)$.*

Proof. For (1) $\to$ (2), note that $X \in M \prec H(\theta)$ implies that M contains some club satisfying Definition 1.2.

This elementary submodel characterization will help us determine when a compact LOTS (linearly ordered topological space) is an Aronszajn compactum. First, note the following more explicit description of the quotient:

Lemma 2.2 *Let X be a compact LOTS, $X \in M \prec H(\theta)$, and $x, y \in X$ with $x < y$. Then $\pi(x) = \pi(y)$ iff $|[x, y] \cap M| \leq 1$; hence, each equivalence class is convex. Furthermore, if X is first countable, then $|[x, y] \cap M|$ can never equal 1.*

Proof. If $|[x, y] \cap M| \geq 2$, fix $a, b \in M \cap X$ with $x \leq a < b \leq y$. There is then an $h \in M \cap C(X, \mathbb{R})$ which maps $(-\infty, a]$ to 0 and $[b, +\infty)$ to 1, so $\pi(x) \neq \pi(y)$. Conversely, if $\pi(x) \neq \pi(y)$, fix $h \in M \cap C(X, \mathbb{R})$ with $|h(x) - h(y)| \geq 1$. To prove that $|[x, y] \cap M| \geq 2$, find, in M, a cover of X by convex open sets $U_1, \ldots U_n$ such that each $\mathrm{diam}(h(U_i)) \leq 1/3$.

For the "furthermore", fix $b \in [x, y] \cap M$; WLOG $x < b \leq y$. If b has an immediate predecessor, that predecessor must be in M; otherwise, b is a limit from the left of an ω–sequence of elements of $M \cap X$. In either case, $|[x, y] \cap M| \geq 2$.

The LOTS version of an Aronszajn compactum is a compact*ed* Aronszajn line. The term "*compact* Aronszajn line" is not common in the literature. An *Aronszajn line* is usually defined to be a LOTS of size $\aleph_1$ with no increasing or decreasing ω_1–sequences and no uncountable subsets of *real type* (that is, order-isomorphic to a subset of $\mathbb{R}$). Such a LOTS cannot be compact, but its Dedekind completion is a compact*ed* Aronszajn line.

Definition 2.3 *A* compacted Aronszajn line *is a compact LOTS X such that $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$, and the closure of every countable set is second countable.*

Note that by $\chi(X) = \aleph_0$ plus compactness, X has no increasing or decreasing ω_1–sequences. But our definition allows for the possibility that X contains uncountably many disjoint intervals isomorphic to $[0, 1]$.

Lemma 2.4 *A LOTS X is an Aronszajn compactum iff X is a compacted Aronszajn line.*

Proof. For $\leftarrow$: suppose that $X \in M \prec H(\theta)$ and M is countable. Then X/M is a compact metric LOTS, and is hence order-embeddable into $[0,1]$. Suppose there were an uncountable $E \subseteq X/M$ such that $|\pi^{-1}\{y\}| \geq 2$ for all $y \in E$. Say $\pi^{-1}\{y\} = [a_y, b_y] \subset X$ for $y \in E$, where $a_y < b_y$. If D is a countable dense subset of E then $\mathrm{cl}(\{a_y : y \in D\}) \subseteq X$ would not be second countable, a contradiction.

We use the standard definition of a *Suslin line* as any LOTS which is ccc and not separable; this is always an L-space. Then a *compact Suslin line* is just a Suslin line which happens to be compact. A compacted Aronszajn line may be a Suslin line, but a compact Suslin line need not be a compacted Aronszajn line. For example, we may form X from a connected compact Suslin line Y by doubling uncountably many points lying in some Cantor subset of Y. More generally, for any compact Suslin line, its subset D of points having a right nearest neighbor determines whether the line is an Aronszajn compactum.

Lemma 2.5 *Let X be a compact Suslin line. Then X is a compacted Aronszajn line iff $D := \{x \in X : \exists y > x\,([x,y] = \{x,y\})\}$ does not contain an uncountable subset of real type.*

Proof. If D contains an uncountable set E real type, let $B \subseteq E$ be countable and dense in E. Then whenever M is countable and $X, B \in M \prec H(\theta)$, there are uncountably many $y \in X/M$ such that $|\pi^{-1}\{y\}| \geq 2$, so that X is not an Aronszajn compactum.

Conversely, if X is not an Aronszajn compactum, consider any countable M with $X \in M \prec H(\theta)$ and $A := \{y \in X/M : |\pi^{-1}\{y\}| \geq 2\}$ uncountable. Let $A' := \{y \in X/M : |\pi^{-1}\{y\}| > 2\}$. Since each $\pi^{-1}\{y\}$ is convex, A' is countable by the ccc, and the left points of the $\pi^{-1}\{y\}$ for $y \in A \backslash A'$ yield an uncountable subset of D of real type.

A zero dimensional compact Suslin line formed in the usual way from a binary Suslin tree will also be a compacted Aronszajn line.

3 Normalizing Aronszajn Compacta

The club C and tree T derived from an Aronszajn compactum X in Definition 1.2 can depend on the embedding of X into $[0,1]^{\omega_1}$. To standardize the tree, we choose a nice embedding. For $X \subseteq [0,1]^{\omega_1}$, C cannot in general be ω_1, since $C = \omega_1$ implies that $\dim(X) \leq 1$. Replacing $[0,1]$ by the Hilbert cube, however, we can assume $C = \omega_1$, which simplifies our tree notation. In particular, the levels will be indexed by ω_1, so that $\mathcal{L}_\alpha$ will be level α of the tree in the usual sense.

Definition 3.1 *Q denotes the Hilbert cube, $[0,1]^\omega$. If $X \subseteq Q^{\omega_1}$ is closed and $\alpha < \omega_1$, then $\mathcal{L}_\alpha = \mathcal{L}_\alpha(X) = \{x \in X_\alpha : |(\sigma^{\omega_1}_\alpha)^{-1}\{x\}| > 1\}$. $\mathcal{W}(X) = \{\alpha < \omega_1 : |\mathcal{L}_\alpha| \leq \aleph_0\}$.*

So, X is an Aronszajn compactum iff $\mathcal{W}(X)$ contains a club; $\mathcal{W}(X)$ itself need not be closed, and $\mathcal{W}(X)$ depends on how X is embedded into Q^{ω_1}. Now, using the facts that $Q \cong Q^\omega$ and that an Aronszajn tree can have only countably many finite levels:

Lemma 3.2 *Every Aronszajn compactum is homeomorphic to some $X \subseteq Q^{\omega_1}$ such that $\mathcal{W}(X) = \omega_1$ and $|\mathcal{L}_\alpha| = \aleph_0$ for all $\alpha > 0$.*

Of course, $\mathcal{L}_0 = X_0 = \{\emptyset\} = Q^0$, and $\emptyset$ is the root node of the tree.

Definition 3.3 *If $X \subseteq Q^{\omega_1}$ is an Aronszajn compactum and $\mathcal{W}(X) = \omega_1$, let $\widehat{\mathcal{L}}_\alpha = \{x \in \mathcal{L}_\alpha : w((\sigma^{\omega_1}_\alpha)^{-1}\{x\}) = \aleph_1\}$, and let $\widehat{T} = \bigcup_\alpha \widehat{\mathcal{L}}_\alpha$.*

Since X is not second countable, each $\widehat{\mathcal{L}}_\alpha \neq \emptyset$ and $\widehat{T}$ is an Aronszajn subtree of T. Repeating the above argument, we get

Lemma 3.4 *Every Aronszajn compactum is homeomorphic to some $X \subseteq Q^{\omega_1}$ such that $\mathcal{W}(X) = \omega_1$, and $|\widehat{\mathcal{L}}_\alpha| = \aleph_0$ for all $\alpha > 0$, and each $x \in \mathcal{L}_\alpha \backslash \widehat{\mathcal{L}}_\alpha$ is a leaf, and each $x \in \widehat{\mathcal{L}}_\alpha$ has $\aleph_0$ immediate successors in $\widehat{\mathcal{L}}_{\alpha+1}$.*

This normalization can also be obtained with elementary submodels. Start with a continuous chain of elementary submodels, $M_\alpha \prec H(\theta)$, for $\alpha < \omega_1$, with $X \in M_0$ and each $M_\alpha \in M_{\alpha+1}$. Let $X_\alpha = X/M_\alpha$, let $\pi_\alpha : X \twoheadrightarrow X_\alpha$ be the natural map, and let $\mathcal{L}_\alpha = \{y \in X_\alpha : |\pi_\alpha^{-1}\{y\}| > 1\}$. We may view each X_α as embedded topologically into Q^α, in which case $\mathcal{L}_\alpha$ has the same meaning as before. If $\pi_\alpha^{-1}\{y\}$ is second countable, then (since $M_\alpha \in M_{\alpha+1}$), all the points in $\pi_\alpha^{-1}\{y\}$ are separated by functions in $C(X) \cap M_{\alpha+1}$, so $y \in \mathcal{L}_\alpha \backslash \widehat{\mathcal{L}}_\alpha$ is a leaf.

If X is a compacted Aronszajn line, then $X_{\alpha+1}$ is formed by replacing each $y \in \mathcal{L}_\alpha$ by a compact interval I_y of size at least 2. If $y \in \mathcal{L}_\alpha \backslash \widehat{\mathcal{L}}_\alpha$, then $\pi_\alpha^{-1}\{y\}$ is second countable and is isomorphic to I_y. Note that the tree may have uncountably many leaves; we do not obtain the conventional normalization of an Aronszajn tree, where the tree is uncountable above every node.

Next, we consider the ideal of second countable subsets of X:

Definition 3.5 *For any space X, $\mathcal{I}_X$ denotes the family of all $S \subseteq X$ such that S, with the subspace topology, is second countable.*

$\mathcal{I}_X$ need not be an ideal. It is obviously closed under subsets, but need not be closed under unions (consider $\omega \cup \{p\} \subset \beta\omega$).

Lemma 3.6 *Assume that $X \subseteq Q^{\omega_1}$ is an HL Aronszajn compactum, as in Lemma 3.2. Then $\mathcal{I}_X$ is a σ–ideal, and, for all $S \subseteq X$, the following are equivalent:*

1. *$S \in \mathcal{I}_X$.*
2. *For some $\alpha < \omega_1$, $\sigma^{\omega_1}_\alpha(S) \cap \mathcal{L}_\alpha = \emptyset$.*
3. *There is a $G \supseteq S$ such that $G \in \mathcal{I}_X$ and G is a G_δ subset of X.*
4. *There is an $f \in C(S, Q)$ such that f is 1-1.*
5. *There is an $f \in C(S, Q)$ such that $f^{-1}\{y\}$ is second countable for all $y \in Q$.*

Proof. It is easy to verify (2) → (3) → (1) → (4) → (5). In particular, for (2) → (3): Fix α and let $G = (\sigma^{\omega_1}_\alpha)^{-1}(X_\alpha \backslash \mathcal{L}_\alpha)$. Then G is a G_δ set and $\sigma^{\omega_1}_\alpha : G \twoheadrightarrow X_\alpha \backslash \mathcal{L}_\alpha$ is a 1-1 closed map, and hence a homeomorphism.

For (1) → (2): Fix an open base for S of the form $\{V_n \cap S : n \in \omega\}$, where each V_n is open in X. X is HL, so V_n is an F_σ. We can thus fix $\xi < \omega_1$ such that each $V_n = (\sigma^{\omega_1}_\xi)^{-1}(\sigma^{\omega_1}_\xi(V_n))$. It follows that $\sigma^{\omega_1}_\xi$ is 1-1 on S. We may then choose α with $\xi < \alpha < \omega_1$ such that $\sigma^{\omega_1}_\alpha(S) \cap \mathcal{L}_\alpha = \emptyset$.

Now, $\mathcal{I}_X$ is a σ–ideal by (1) ↔ (2).

To prove (5) → (2): Fix f as in (5). Let $\{U_n : n \in \omega\}$ be an open base for Q; then $f^{-1}(U_n) = S \cap V_n$, where V_n is open in X and hence an F_σ set. We can thus fix $\alpha < \omega_1$ such that $V_n = (\sigma^{\omega_1}_\alpha)^{-1}(\sigma^{\omega_1}_\alpha(V_n))$. It follows that f is constant on $S \cap (\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ for all $z \in X_\alpha$. Thus, $S \cap (\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ is second countable for all $z \in X_\alpha$. But then S is contained in the union of $\bigcup\{S \cap (\sigma^{\omega_1}_\alpha)^{-1}\{y\} : y \in \mathcal{L}_\alpha\}$ and $(\sigma^{\omega_1}_\alpha)^{-1}(X_\alpha \backslash \mathcal{L}_\alpha) \cong (X_\alpha \backslash \mathcal{L}_\alpha)$, so $S \in \mathcal{I}_X$ because $\mathcal{I}_X$ is a σ–ideal.

This proof shows that every Aronszajn compactum is an ascending union of ω_1 Polish spaces: namely, the $(\sigma^{\omega_1}_\alpha)^{-1}(X_\alpha \backslash \mathcal{L}_\alpha)$.

We needed X to be Aronszajn in Lemma 3.6; HS and HL are not enough to prove the equivalence of (1)(3)(4)(5). If S is the Sorgenfrey line contained in the double arrow space X, then (4)(5) are true but (1)(3) are false. Similar remarks hold for similar spaces which are both HS and HL. For example, assuming CH, Filippov [3] constructed a locally connected continuum which is HS and HL but not second countable. The space was obtained by replacing a Luzin set of points in $[0, 1]^2$ by circles. If S contains one point from each of the circles, then S satisfies (4)(5) but fails (1)(3). In both examples, the space X itself satisfies (5) but not (1)(3)(4).

More generally, any space X that has an $f \in C(X, Q)$ as in (5) cannot be an Aronszajn compactum. Thus, a ZFC example of an HL Aronszajn compactum would settle in the negative the following well-known question of Fremlin ([4] 44Qc): is it consistent that for every HL compactum X, there is an $f \in C(X, Q)$ such that $|f^{-1}\{y\}| < \aleph_0$ for all $y \in Q$? In [6], Gruenhage gives some of the history related to this question, and points out some related results suggesting that the answer might be “yes” under PFA.

4 Closure Properties of Aronszajn Compacta

Closure under subspaces was already mentioned in Lemma 1.5. For products, Lemma 2.1 implies:

Lemma 4.1 *Assume that X is an Aronszajn compactum and Y is an arbitrary space. Then $X \times Y$ is an Aronszajn compactum iff Y is compact and countable.*

Regarding quotients, we first prove:

Lemma 4.2 *Assume that X, Y are compact, $\varphi : X \twoheadrightarrow Y$, and $X, Y, \varphi \in M \prec H(\theta)$. Let $\sim$ denote the M equivalence relation on X and on Y. Then*

1. *If $x_0, x_1 \in X$ and $x_0 \sim x_1$, then $\varphi(x_0) \sim \varphi(x_1)$; so, the inverse image of an equivalence class of Y is a union of equivalence classes of X.*
2. *If $y_0, y_1 \in Y$ and $x_0 \not\sim x_1$ for all $x_0 \in \varphi^{-1}\{y_0\}$ and all $x_1 \in \varphi^{-1}\{y_1\}$, then $y_0 \not\sim y_1$.*

Proof. For (1): If $f \in C(Y) \cap M$ separates $\varphi(x_0)$ from $\varphi(x_1)$ then $f \circ \varphi \in C(X) \cap M$ separates x_0 from x_1.

For (2): For each $x_0 \in \varphi^{-1}\{y_0\}$ and $x_1 \in \varphi^{-1}\{y_1\}$, there is an $f \in C(X, [0, 1]) \cap M$ such that $f(x_0) \neq f(x_1)$. By compactness of $\varphi^{-1}\{y_0\} \times \varphi^{-1}\{y_1\}$, there are $f_0, \ldots, f_{n-1} \in C(X, [0, 1]) \cap M$ for some $n \in \omega$ such that: for all $x_0 \in \varphi^{-1}\{y_0\}$ and $x_1 \in \varphi^{-1}\{y_1\}$, there is some $i < n$ such that $f_i(x_0) \neq f_i(x_1)$. These yield an $\vec{f} \in C(X, [0, 1]^n) \cap M$ such that $\vec{f}(\varphi^{-1}\{y_0\}) \cap \vec{f}(\varphi^{-1}\{y_1\}) = \emptyset$. Since M contains a base for $[0, 1]^n$, there are open $U_0, U_1 \subseteq [0, 1]^n$ with each $U_i \in M$ such that $\overline{U_0} \cap \overline{U_1} = \emptyset$ and each $\vec{f}(\varphi^{-1}\{y_i\}) \subseteq U_i$, so that $\varphi^{-1}\{y_i\} \subseteq (\vec{f})^{-1}(U_i)$. Let $V_i = \{y \in Y : \varphi^{-1}\{y\} \subseteq (\vec{f})^{-1}(U_i)\}$. Then the V_i are open in Y, each $V_i \in M$, each $y_i \in V_i$, and $\overline{V_0} \cap \overline{V_1} = \emptyset$. There is thus a $g \in C(Y) \cap M$ such that $g(\overline{V_0}) \cap g(\overline{V_1}) = \emptyset$, so that $g(y_0) \neq g(y_1)$. Thus, $y_0 \not\sim y_1$. ☕

Theorem 4.3 *Assume that X is an Aronszajn compactum, $\varphi : X \twoheadrightarrow Y$, $w(Y) = \aleph_1$, and $\chi(Y) = \aleph_0$. Then Y is an Aronszajn compactum.*

Proof. It is sufficient to check that for a club of elementary submodels M, all but countably many M–classes of Y are singletons. Fix M as in Lemma 4.2; so all but countably many M–classes of X are singletons. Then for all but countably many classes $K = [y]$ of Y: all M–classes of X inside of $\varphi^{-1}(K)$ are singletons, so that, by the lemma, K is a singleton. ☕

Note that we needed to assume that $\chi(Y) = \aleph_0$. Otherwise, when X is not HL, we would get a trivial counterexample of the form X/K, where K is a closed set which is not a G_δ.

Examining whether an Aronszajn compactum may be both HS and HL reduces to considering zero dimensional spaces and connected spaces, by the following lemma.

Lemma 4.4 *Assume that X is an HL Aronszajn compactum, $\varphi : X \twoheadrightarrow Y$. Then either Y is an Aronszajn compactum or some $\varphi^{-1}\{y\}$ is an Aronszajn compactum.*

Proof. Y will be an Aronszajn compactum unless it is second countable. But if it is second countable, then some $\varphi^{-1}\{y\}$ will be not second countable by Lemma 3.6, and then $\varphi^{-1}\{y\}$ will be an Aronszajn compactum. ☕

Corollary 4.5 *Suppose there is an Aronszajn compactum X which is HS and HL. Then there is an Aronszajn compactum Z which is HS and HL and which is either connected or zero dimensional.*

Proof. Get $\varphi : X \twoheadrightarrow Y$ by collapsing all connected components to points. Then Z is either Y or some component. ☕

Note that the cone over X is also connected, but is not an Aronszajn compactum by Lemma 4.1.

5 Constructing Aronszajn Compacta

We begin this section by constructing a space X which proves Theorem 1.6. We construct $X = X_{\omega_1}$ as an inverse limit as a closed subspace of Q^{ω_1}. To make X both HS and HL, we shall apply the following lemma:

Lemma 5.1 *Assume that X is compact and for all closed $F \subseteq X$, there is a compact metric Y and a map $g : X \twoheadrightarrow Y$ such that $g \restriction g^{-1}(g(F)) : g^{-1}(g(F)) \twoheadrightarrow g(F)$ is irreducible. Then X is both HS and HL.*

Proof. By irreducibility, $g^{-1}(g(F)) = F$, so that F is a G_δ and F is separable. Thus, X is a compact HL space in which all closed subsets are separable, so X is HS. ☕

In applying the lemma to $X = X_{\omega_1}$, g will be some $\pi^{\omega_1}_{\alpha} \restriction X$. We shall use $\diamondsuit$ to capture all closed $F \subseteq Q^{\omega_1}$ so that all closed $F \subseteq X$ will be considered. This method was also employed in [7], which constructed some compacta which were HS and HL but not Aronszajn.

As in standard inverse limit constructions, we inductively construct $X_\alpha \subseteq Q^\alpha$, for $\alpha \le \omega_1$. To ensure that X will be Aronszajn, at each stage $\alpha < \omega_1$, we carefully select a countable set $\mathcal{E}_\alpha \subseteq X_\alpha$ of "expandable points", and at each stage $\beta > \alpha$, we construct $X_\beta \subseteq Q^\beta$ so that $|(\sigma^\beta_\alpha)^{-1}\{x\}| = 1$ whenever $x \notin \mathcal{E}_\alpha$. Then the $\mathcal{L}_\alpha$ of Definition 3.1 will be subsets of $\mathcal{E}_\alpha$ and hence countable.

These preliminaries are included in the following conditions:

Conditions 5.2 *X_α, for $\alpha \le \omega_1$, and $\mathcal{P}_\alpha, F_\alpha, \mathcal{E}_\alpha, q_\alpha$, for $0 < \alpha < \omega_1$, satisfy:*

1. *Each X_α is a closed subset of Q^α.*
2. *$\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $\alpha \le \beta \le \omega_1$.*
3. *$\mathcal{P}_\alpha$ is a countable family of closed subsets of X_α, and $F_\alpha \in \mathcal{P}_\alpha$.*
4. *For all $P \in \mathcal{P}_\alpha$:*
   a. $\sigma^{\alpha+1}_\alpha \restriction ((\sigma^{\alpha+1}_\alpha)^{-1}(P)) : (\sigma^{\alpha+1}_\alpha)^{-1}(P) \twoheadrightarrow P$ is irreducible, and
   b. $(\sigma^\beta_\alpha)^{-1}(P) \in \mathcal{P}_\beta$ whenever $\alpha \le \beta < \omega_1$.
5. *For all closed $F \subseteq X$, there is an α with $0 < \alpha < \omega_1$ such that $\sigma^{\omega_1}_\alpha(F) = F_\alpha$.*
6. *$\mathcal{E}_\alpha$ is a countable dense subset of X_α, and $q_\alpha \in \mathcal{E}_\alpha$.*
7. *$\mathcal{E}_\beta \subseteq (\sigma^\beta_\alpha)^{-1}(\mathcal{E}_\alpha)$ whenever $0 < \alpha \le \beta < \omega_1$.*
8. *$|(\sigma^{\alpha+1}_\alpha)^{-1}\{x\}| = 1$ whenever $0 < \alpha < \omega_1$ and $x \in X_\alpha \backslash \{q_\alpha\}$.*
9. *$|(\sigma^{\alpha+1}_\alpha)^{-1}\{q_\alpha\}| > 1$.*

We discuss below how to satisfy these conditions. Conditions (1) and (2) simply determine our $X = X_{\omega_1} \subseteq Q^{\omega_1}$ with each $X_\alpha = \pi^{\omega_1}_\alpha(X)$. $\diamondsuit$ is used for (5). Constructing an X that satisfies Conditions (1 - 9) is enough to prove Theorem 1.6:

Lemma 5.3 *Conditions $(1-9)$ imply that $X = X_{\omega_1}$ is an Aronszajn compactum and is both HS and HL.*

Proof. By (4) and induction on β, $\sigma^\beta_\alpha \restriction ((\sigma^\beta_\alpha)^{-1}(P)) : (\sigma^\beta_\alpha)^{-1}(P) \twoheadrightarrow P$ is irreducible whenever $\alpha \le \beta \le \omega_1$ and $P \in \mathcal{P}_\alpha$. Then X is HS and HL by Lemma 5.1 and (5)(3).

By (6)(7)(8) and induction, $|(\sigma^\beta_\alpha)^{-1}\{x\}| = 1$ whenever $0 < \alpha \le \beta \le \omega_1$ and $x \in X_\alpha \backslash \mathcal{E}_\alpha$. So, $\mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma^{\omega_1}_\alpha)^{-1}\{x\}| > 1\} \subseteq \mathcal{E}_\alpha$, which is countable by (6).

Finally, $w(X) = \aleph_1$ by (9), and $\chi(X) = \aleph_0$ because X is HL. ☕

To obtain Conditions $(1-9)$, we must add some further conditions so that the natural construction avoids contradictions. For example, satisfying Conditions (6) and (7) at stage β requires $\bigcap_{\alpha<\beta}(\sigma^\beta_\alpha)^{-1}(\mathcal{E}_\alpha) \ne \emptyset$. So we add Conditions (10 - 12) below making the $\mathcal{E}_\alpha$ into the levels of a tree; the selection of the $\mathcal{E}_\alpha$ will resemble the standard inductive construction of an Aronszajn tree.

The sets F_α may be scattered or even singletons. This cannot be avoided, because we are using the F_α to ensure that *all* closed sets are G_δ sets, so that X is HL; making just the perfect sets G_δ could produce a Fedorchuk space (as in [8]), which is not even first countable. If $x \in P \in \mathcal{P}_\alpha$ and x is isolated in P, then the irreducibility condition in (4) requires that $|(\sigma^{\alpha+1}_\alpha)^{-1}\{x\}| = 1$, but that contradicts (9) if $x = q_\alpha$. Now, if every point of $\mathcal{E}_\alpha$ is isolated in some $P \in \mathcal{P}_\alpha$, then we cannot choose $q_\alpha \in \mathcal{E}_\alpha$, as required by (6). We shall avoid these problems by requiring that if $x \in \mathcal{E}_\alpha$ and $P \in \mathcal{P}_\alpha$, then either $x \notin P$ or x is in the perfect kernel of P. This can be ensured by choosing F_α first (as given by $\diamondsuit$), and then choosing $\mathcal{E}_\alpha$; for limit α, our Aronszajn tree construction will give us plenty of options for choosing the points of $\mathcal{E}_\alpha$, and we shall make F_α trivial for successor α. The additional conditions that handle this will employ the notation in the following:

Definition 5.4 *If F is compact and not scattered, let $\ker(F)$ denote the perfect kernel of F; otherwise, $\ker(F) = \emptyset$.*

To satisfy Condition (8), we construct $X_{\alpha+1}$ from X_α by choosing an appropriate $h_\alpha \in C(X_\alpha \backslash \{q_\alpha\}, Q)$, and letting $X_{\alpha+1} = \mathrm{cl}(h_\alpha)$. Identifying $Q^{\alpha+1}$ with $Q^\alpha \times Q$ and h_α with its graph, $h_\alpha(x)$ is the $y \in Q$ such that $x^\frown y \in X_{\alpha+1}$. Note that h_α is indeed continuous because its graph is closed.

Thus, to construct X so that Conditions (1 - 9) are met, we add the following:

Conditions 5.5 *h_α and r^n_α, for $0 < \alpha < \omega_1$ and $n < \omega$, satisfy:*

10. $(\sigma^\beta_\alpha)(\mathcal{E}_\beta) = \mathcal{E}_\alpha$ *whenever* $0 < \alpha \le \beta < \omega_1$.
11. $|\mathcal{E}_{\alpha+1} \cap (\sigma^{\alpha+1}_\alpha)^{-1}\{q_\alpha\}| > 1$.
12. *If* $x \in \mathcal{E}_\alpha$, *then* $(\sigma^{\alpha+n}_\alpha)(q_{\alpha+n}) = x$ *for some* $n \in \omega$.
13. X_α *has no isolated points whenever* $\alpha > 0$.
14. $F_\alpha = \emptyset$ *whenever* α *is a successor ordinal.*
15. $\mathcal{P}_\beta = \{F_\beta\} \cup \{(\sigma^\beta_\alpha)^{-1}(P) : 0 < \alpha < \beta \;\&\; P \in \mathcal{P}_\alpha\}$.
16. $\mathcal{E}_\alpha \cap (P \backslash \ker(P)) = \emptyset$ *whenever* $P \in \mathcal{P}_\alpha$.
17. $r^n_\alpha \in X_\alpha \backslash \{q_\alpha\}$ *and the sequence* $\langle r^n_\alpha : n \in \omega\rangle$ *converges to* q_α.
18. $h_\alpha \in C(X_\alpha \backslash \{q_\alpha\}, Q)$, *and* $X_{\alpha+1} = \mathrm{cl}(h_\alpha)$.
19. *If* $q_\alpha \in P \in \mathcal{P}_\alpha$, *then* $r^n_\alpha \in \ker(P)$ *for infinitely many* n, *and every* $y \in Q$ *with* $q_\alpha^\frown y \in X_{\alpha+1}$ *is a limit point of the sequence* $\langle h_\alpha(r^n_\alpha) : n \in \omega \;\&\; r^n_\alpha \in \ker(P)\rangle$.

Observe that (10)(11)(12) will give us the following:

Lemma 5.6 *$\mathcal{L}_\alpha = \mathcal{E}_\alpha$ whenever $0 < \alpha < \omega_1$.*

In the tree $T(X)$, although only the node $q_\alpha \in \mathcal{L}_\alpha$ has more than one successor in $\mathcal{L}_{\alpha+1}$, (12) ensures that at limit levels γ, there are $2^{\aleph_0}$ choices for the elements of $\mathcal{E}_\gamma$, so that we may avoid the points in $F_\gamma \backslash \ker(F_\gamma)$, as required by (16).

By (14)(15), $\emptyset \in \mathcal{P}_\alpha$ for all $\alpha > 0$, and non-empty sets are added into the $\mathcal{P}_\alpha$ only at limit α.

The following proof gives the bare-bones construction; refinements of it produce the spaces of Theorem 5.8.

Proof of Theorem 1.6. Before we start, use $\diamondsuit$ to choose a closed $\widetilde{F}_\alpha \subseteq Q^\alpha$ for each $\alpha < \omega_1$, so that $\{\alpha < \omega_1 : \pi^{\omega_1}_\alpha(F) = \widetilde{F}_\alpha\}$ is stationary for all closed $F \subseteq Q^{\omega_1}$.

To begin the induction: X_0 must be $\{\emptyset\} = Q^0$, and $\mathcal{P}_\alpha, F_\alpha, \ldots\ldots$ are only defined for $\alpha > 0$.

Now, fix β with $0 < \beta < \omega_1$, and assume that all conditions have been met below β. We define in order X_β, F_β, $\mathcal{P}_\beta$, $\mathcal{E}_\beta$, q_β, r^n_β, h_β.

If β is a limit, then X_β is determined by (1)(2) and the X_α for $\alpha < \beta$. X_1 can be any perfect subset of Q^1. If $\beta = \alpha + 1 \geq 2$, then $X_\beta = \mathrm{cl}(h_\alpha)$, as required by (18). Now let $F_\beta = \widetilde{F}_\beta$ if $\widetilde{F}_\beta \subseteq X_\beta$ and β is a limit; otherwise, let $F_\beta = \emptyset$. $\mathcal{P}_\beta$ is now determined by (15).

$\mathcal{E}_1$ can be any countable dense subset of X_1. If $\beta = \alpha + 1 \geq 2$, let $\mathcal{E}_\beta = (\sigma^\beta_\alpha)^{-1}(\mathcal{E}_\alpha \backslash \{q_\alpha\}) \cup \mathcal{D}_\beta$, where $\mathcal{D}_\beta$ is any subset of $(\sigma^\beta_\alpha)^{-1}\{q_\alpha\}$ such that $2 \leq |\mathcal{D}_\beta| \leq \aleph_0$. Observe that $\mathcal{E}_\beta$ is dense in X_β (without using $\mathcal{D}_\beta$), so (6) is preserved, and $\mathcal{D}_\beta$ guarantees that (11) is preserved. To verify (16) at β, note that by (15) at α, every non-empty set in $\mathcal{P}_\beta$ is of the form $\widehat{P} := (\sigma^\beta_\alpha)^{-1}(P)$ for some $P \in \mathcal{P}_\alpha$. So, if (16) fails at β, fix $P \in \mathcal{P}_\alpha$ and $x \in \mathcal{E}_\beta \cap (\widehat{P} \backslash \ker(\widehat{P}))$. Then $x \in (\sigma^\beta_\alpha)^{-1}\{q_\alpha\}$, so $q_\alpha \in P$, and hence $q_\alpha \in \ker(P)$; but then by (19), x is a limit of a sequence of elements of $\ker(\widehat{P})$, so that $x \in \ker(\widehat{P})$.

For limit β, let $\mathcal{E}_\beta = \{x^* : x \in \bigcup_{\alpha<\beta} \mathcal{E}_\alpha\}$, where, x^*, for $x \in \mathcal{E}_\alpha$, is some $y \in X_\beta$ such that $\pi^\beta_\alpha(y) = x$ and $\pi^\beta_\xi(y) \in \mathcal{E}_\xi$ for all $\xi < \beta$. Any such choice of the x^* will satisfy (10). But in fact, using (11)(12), for each such x there are $2^{\aleph_0}$ possible choices of x^*, so we can satisfy (16) by avoiding the countable sets $P \backslash \ker(P)$ for $P \in \mathcal{P}_\beta$.

To facilitate (12), list each $\mathcal{E}_\alpha$ as $\{e^j_\alpha : j \in \omega\}$; let $e^j_0 = \emptyset \in X_0$. Then, if β is a successor ordinal of the form $\gamma + 2^i 3^j$, where γ is a limit or 0, choose $q_\beta \in \mathcal{E}_\beta$ so that $\sigma^\beta_{\gamma+i}(q_\beta) = e^j_{\gamma+i}$. For other β, $q_\beta \in \mathcal{E}_\beta$ can be chosen arbitrarily.

Next, we may choose the r^n_β to satisfy (19) because if $q_\beta \in P \in \mathcal{P}_\beta$, then $q_\beta \in \ker(P)$ by (16), so that q_β is also a limit of points in $\ker(P)$.

Finally, we must choose $h_\beta \in C(X_\beta \backslash \{q_\beta\}, Q)$. Conditions (18)(19) only require that h_β have a discontinuity at q_β with the property that every limit point of the function at q_β is also a limit of each of the sequences $\langle h_\beta(r^n_\beta) : n \in \omega \ \& \ r^n_\beta \in \ker(P)\rangle$. Since X_β is a compact metric space with no isolated points, we may accomplish this by making every point of Q a limit point of each $\langle h_\beta(r^n_\beta) : n \in \omega \ \& \ r^n_\beta \in \ker(P)\rangle$. ☕

If we choose each h_β as above and also set $X_1 = Q$, then our X will be connected, and it is fairly easy to choose the h_β so that X fails to be locally connected. The next theorem shows how to make X connected and locally connected. We construct X so that each X_α is homeomorphic to the *Menger sponge*, MS, and all the maps σ^β_α are monotone. The Menger sponge [11] is a one dimensional locally connected metric continuum; the properties of MS used in inductive constructions such as these are summarized in [8], which contains further references to the literature. A map is *monotone* iff all point inverses are connected. Monotonicity of the σ^β_α will imply that X is locally connected.

At successor stages, to construct $X_{\alpha+1} \cong \mathsf{MS}$, we assume that $X_\alpha \cong \mathsf{MS}$ and apply the following special case of Lemmas 2.7 and 2.8 of [8]:

Lemma 5.7 *Assume that $q \in X \cong \mathsf{MS}$ and that for each $j \in \omega$, the sequence $\langle r_j^n : n \in \omega\rangle$ converges to q, with each $r_j^n \neq q$. Let $\pi : X \times [0,1] \twoheadrightarrow X$ be the natural projection. Then there is a $Y \subseteq X \times [0,1]$ such that:*

1. *$Y \cong \mathsf{MS}$ and $\pi(Y) = X$.*
2. *$|Y \cap \pi^{-1}\{x\}| = 1$ for all $x \neq q$.*
3. *$\pi^{-1}\{q\} = \{q\} \times [0,1]$.*
4. *Let $Y \cap \pi^{-1}\{r_j^n\} = \{(r_j^n, u_j^n)\}$. Then, for each j, every point in $[0,1]$ is a limit point of $\langle u_j^n : n \in \omega\rangle$.*

Constructing X as such an inverse limit of Menger sponges will make X one dimensional. The results quoted from [8] about MS were patterned on an earlier construction of van Mill [12], which involved an inverse limit of Hilbert cubes; replacing MS by Q here would yield an infinite dimensional version of this Aronszajn compactum. The following summarizes several possibilities for X and its associated tree:

Theorem 5.8 *Assume $\diamondsuit$. For each of the following $2 \cdot 3 = 6$ possibilities, there is an Aronszajn compactum X with associated Aronszajn tree T such that X is HS and HL. Possibilities for T:*

- *a. T is Suslin.*
- *b. T is special.*

Possibilities for X:

- *α. $\dim(X) = 0$.*
- *β. $\dim(X) = 1$ and X is connected and locally connected.*
- *γ. $\dim(X) = \infty$ and X is connected and locally connected.*

Proof. We refine the proof of Theorem 1.6, To obtain (a) or (b), the refinement is in the choice of the $\mathcal{E}_\beta$ for limit β. To obtain (α) or (β) or (γ), the refinement is in the choice of X_1 and the functions h_α. Since these refinements are independent of each other, the discussion of $(a)(b)$ is unrelated to the discussion of $(\alpha)(\beta)(\gamma)$.

For (a): We use $\diamondsuit$ to kill all potential uncountable maximal antichains $A \subset T$. Fix a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ such that each A_α is a countable subset of $Q^{<\alpha}$ and such that for all $A \subseteq Q^{<\omega_1}$: if each $A \cap Q^{<\alpha}$ is countable, then $\{\alpha < \omega_1 : A \cap Q^{<\alpha} = A_\alpha\}$ is stationary.

Let $T_\beta = \bigcup\{\mathcal{L}_\alpha : \alpha < \beta\} = \bigcup\{\mathcal{E}_\alpha : \alpha < \beta\}$ (see Lemma 5.6), and use $\lhd$ for the tree order. For each limit $\beta < \omega_1$, modify the construction of $\mathcal{E}_\beta$ in the proof of Theorem 1.6 as follows: We still have $\mathcal{E}_\beta = \{x^* : x \in T_\beta\}$, where, x^*, for $x \in T_\beta$, is chosen so that $x \lhd x^*$ and x^* defines a path through T_β. But now, *if* $A_\beta \subseteq T_\beta$ *and* A_β is a maximal antichain in T_β, *then* make sure that each x^* is above some element of A_β. To do this, use maximality of A_β first to choose $x^\dagger \in T_\beta$ so that $x \lhd x^\dagger$ and $x^\dagger$ is above some element of A_β, and then choose x^* so that $x \lhd x^\dagger \lhd x^*$. There are still $2^{\aleph_0}$ possible choices for x^*, so we can satisfy (16) by avoiding the countable sets $P \backslash \ker(P)$ as before. Now, the usual argument shows that T is Suslin.

For (b): Let Lim denote the set of countable limit ordinals, and let $T^{\mathrm{Lim}} = \bigcup\{\mathcal{L}_\alpha : \alpha \in \mathrm{Lim}\} = \bigcup\{\mathcal{E}_\alpha : \alpha \in \mathrm{Lim}\}$. To make T special, inductively define an order preserving map $\varphi : T^{\mathrm{Lim}} \to \mathbb{Q}$. To make the induction work, we also assume inductively:

$$\forall \gamma, \beta \in \mathrm{Lim}\, \forall x \in \mathcal{L}_\gamma\, \forall q \in \mathbb{Q}\, [\gamma < \beta \;\&\; q > \varphi(x) \;\to\; \exists y \in \mathcal{L}_\beta\, [x \lhd y \;\&\; \varphi(y) = q]] \quad (*)$$

To start the induction, $\varphi \restriction \mathcal{L}_\omega : \mathcal{L}_\omega \to \mathbb{Q}$ can be arbitrary.

For $\beta = \alpha + \omega$, where α is a limit ordinal: First, determine the x^* exactly as in the proof of Theorem 1.6. Then, note that for each $x \in \mathcal{L}_\alpha$, the set $S_x := \{y \in \mathcal{E}_\beta : x \lhd y\}$ has size $\aleph_0$, so we can let $\varphi \restriction S_x$ map S_x *onto* $\mathbb{Q} \cap (\varphi(x), \infty)$.

For $\beta < \omega_1$ which is a limit of limit ordinals: Let

$$\mathcal{E}_\beta = \{x_q^* : x \in T_\beta \;\&\; q \in \mathbb{Q} \cap (\varphi(x), \infty)\} \quad ,$$

where each x_q^* is chosen so that $x \lhd x_q^*$ and x_q^* defines a path through T_β and the x_q^* are all different as q varies. We let $\varphi(x_q^*) = q$, which will clearly preserve $(*)$, but we must make sure that φ remains order preserving. For this, choose x_q^* so that $q > \sup\{\varphi(z) : z \in T^{\mathrm{Lim}} \;\&\; z \lhd x_q^*\}$. Such a choice is possible using $(*)$ on T_β. As before, there are $2^{\aleph_0}$ possible choices of x_q^*, so we can still avoid the countable sets $P \backslash \ker(P)$.

For (α), just make sure that X_α is homeomorphic to the Cantor set 2^ω whenever $0 < \alpha < \omega_1$. In view of (13), this is equivalent to making X_α zero dimensional. For $\alpha = 1$, we simply choose X_1 so that $X_1 \cong 2^\omega$. Then, for larger α, just make sure that in (9), we always have $|(\sigma_\alpha^{\alpha+1})^{-1}\{q_\alpha\}| = 1$, which will hold if in (18), we choose $h_\alpha \in C(X_\alpha \backslash \{q_\alpha\}, 2)$ (identifying $2 = \{0, 1\}$ as a subset of Q). To make this choice, and satisfy (19): First, let A_j, for $j \in \omega$, be disjoint infinite subsets of ω such that for each $P \in \mathcal{P}_\alpha$, if $q_\alpha \in P$ then for some j, $r_\alpha^n \in \ker(P)$ for all $n \in A_j$. Next, let $X_\alpha = K_0 \supset K_1 \supset K_2 \supset \cdots$, where each K_i is clopen, $\bigcap_i K_i = \{q_\alpha\}$, and, for each j, there are infinitely many even i and infinitely many odd i such that $K_i \backslash K_{i+1} \cap \{r_\alpha^n : n \in A_j\} \neq \emptyset$. Now, let h_α be 0 on $K_i \backslash K_{i+1}$ when i is even and 1 on $K_i \backslash K_{i+1}$ when i is odd.

For (β), construct X so that each X_α is homeomorphic to the *Menger sponge*, MS, and all the maps σ_α^β are monotone. Then $\dim(X) = 1$ will follow from the fact that X is an inverse limit of one dimensional spaces.

For monotonicity of the σ_α^β, it suffices to ensure that each $\sigma_\alpha^{\alpha+1}$ is monotone. By Condition (8), that will follow if we make $(\sigma_\alpha^{\alpha+1})^{-1}\{q_\alpha\}$ connected; in fact we shall make $(\sigma_\alpha^{\alpha+1})^{-1}\{q_\alpha\}$ homeomorphic to $[0,1]$, as in the proof of Theorem 1.6. But we also need to verify inductively that $X_\alpha \cong \mathsf{MS}$. At limits, this follows from Lemma 2.5 of [8]. At successor stages, we assume that $X_\alpha \cong \mathsf{MS}$ and identify $[0,1]$ as a subspace of Q, so that $X_{\alpha+1}$ may be the Y of Lemma 5.7.

(γ) is proved analogously to (β). Construct $X_\alpha \cong Q$ rather than MS, applying the results about Q in [12]§3. As in [12]§2, all the σ_α^β are cell-like Z^*-maps. ☕

6 Chains of Clopen Sets

The double arrow space has an uncountable chain (under $\subset$) of clopen sets of real type. This cannot happen in an Aronszajn compactum:

Lemma 6.1 *If X is an Aronszajn compactum and $\mathcal{E}$ is an uncountable chain of clopen subsets of X, then $\mathcal{E}$ cannot be of real type.*

Proof. Suppose that $\mathcal{E}$ is such a chain. Deleting some elements of $\mathcal{E}$, we may assume that $(\mathcal{E}, \subset)$ is a dense total order. Let $\mathcal{D}$ be a countable dense subset of $\mathcal{E}$. Since X is an Aronszajn compactum, there is a map $\varphi : X \twoheadrightarrow Z$, where Z is a compact metric space, $A = \varphi^{-1}(\varphi(A))$ for all $A \in \mathcal{D}$, and $\{y \in Z : |\varphi^{-1}\{y\}| > 1\}$ is countable. Since $\mathcal{D}$ is dense in $\mathcal{E}$, the sets $\varphi(B)$ for $B \in \mathcal{E}$ are all different. Each $\varphi(B)$ is closed, and only countably many of the $\varphi(B)$ can be clopen. Whenever $\varphi(B)$ is not clopen, choose $y_B \in \varphi(B) \cap \varphi(X \backslash B)$. Since $\mathcal{D}$ is dense in $\mathcal{E}$, these y_B are all different points, so there are uncountably many such y_B. But $\varphi^{-1}\{y_B\}$ meets both B and $X \backslash B$, so each $|\varphi^{-1}\{y_B\}| \geq 2$, a contradiction. ☕

Note that if this argument is applied with a chain of clopen sets in the double arrow space, then the $|\varphi^{-1}\{y_B\}|$ will be exactly 2.

Lemma 6.2 *If X is any separable space, and $\mathcal{E}$ is an uncountable chain of clopen subsets of X, then $\mathcal{E}$ must be of real type.*

Proof. If $D \subseteq X$ is dense, then $(\mathcal{E}, \subset)$ is isomorphic to a chain in $(\mathcal{P}(D), \subset)$. ☕

Corollary 6.3 *If X is a separable Aronszajn compactum and $\mathcal{E}$ is a chain of clopen subsets of X, then $\mathcal{E}$ is countable.*

Note that if X is a zero dimensional compacted Aronszajn line which is also Suslin (see Lemma 2.5), then X has uncountable chain of clopen sets, but X is not separable.

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