Complex Function Algebras and Removable Spaces*

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Abstract

The compact Hausdorff space X has the *Complex Stone-Weierstrass Property* (CSWP) iff it satisfies the complex version of the Stone-Weierstrass Theorem. W. Rudin showed that all scattered spaces have the CSWP. We describe some techniques for proving that certain non-scattered spaces have the CSWP. In particular, if X is the product of a compact ordered space and a compact scattered space, then X has the CSWP if and only if X does not contain a copy of the Cantor set.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff.

Definition 1.1 *If X is compact, then $C(X) = C(X, \mathbb{C})$ is the algebra of continuous complex-valued functions on X, with the usual supremum norm. $\mathcal{A} \sqsubseteq C(X)$ means that $\mathcal{A}$ is a subalgebra of $C(X)$ which separates points and contains the constant functions. $\mathcal{A} \sqsubseteq_c C(X)$ means that $\mathcal{A} \sqsubseteq C(X)$ and $\mathcal{A}$ is closed in $C(X)$. X has the* Complex Stone-Weierstrass Property (CSWP) *iff every $\mathcal{A} \sqsubseteq C(X)$ is dense in $C(X)$.*

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Equivalently, X has the CSWP iff every $\mathcal{A} \mathrel{\underline{\subset}} C(X)$ equals $C(X)$. Note that if we replaced "$\mathbb{C}$" by "$\mathbb{R}$" in Definition 1.1, the property would be true of all compact X by the Stone-Weierstrass Theorem.

By 1960, it was known that the CSWP is true of all compact scattered spaces (Rudin [9]). It was not known whether having the CSWP was equivalent to being scattered, although two important examples were known of non-scattered spaces which failed the CSWP, namely, the Cantor set (Rudin [8]) and $\beta\mathbb{N}$ (Hoffman and Singer [4]). It was also well-known (and easy to see) that if X fails the CSWP, then so does every compact space containing X. In particular, every compact space containing a Cantor subset fails the CSWP. It follows, as Rudin pointed out, that for compact metric spaces, having the CSWP is equivalent to being scattered.

These results are discussed in more detail in [6], which shows that indeed there are non-scattered spaces with the CSWP. Of course, none of these can contain a Cantor subset.

Theorem 1.2 ([6]) *If X is a compact LOTS which does not contain a Cantor subset, then X has the CSWP.*

As usual, a LOTS is a linearly ordered set with its order topology. As a specific example, the double arrow space of Alexandroff and Urysohn, which is not scattered, has the CSWP. In this paper, we extend these results to a much wider class of spaces. Our results have, as a special case:

Theorem 1.3 *If L is a compact LOTS which does not contain a Cantor subset, and S is a compact scattered space, then $L \times S$ has the CSWP.*

Theorem 1.2 is the special case of Theorem 1.3 in which S is a singleton.

We shall in fact proceed by a generalization of the Cantor-Bendixson sequence. The standard Cantor-Bendixson sequence is obtained by removing isolated points: Let X' denote the set of non-isolated points of X. Starting with a space S, we iterate this construction to obtain $S^{(0)} = S$, $S^{(\alpha+1)} = (S^{(\alpha)})'$, and $S^{(\gamma)} = \bigcap_{\alpha<\gamma} S^{(\alpha)}$ for limit ordinals γ. Then S is *scattered* iff some $S^{(\alpha)}$ is empty.

We shall define a class of pseudo-removable (PR) spaces. There is a related class of PR-scattered spaces where one obtains the empty set after repeatedly removing open sets whose closure is PR (see Definitions 2.17 and 2.18). We shall show (Theorem 2.20) that every PR-scattered space has the CSWP. The one-point space is PR, so that every scattered space is PR-scattered. But also, we shall show (Lemma 2.19) that every compact LOTS which does not contain a Cantor subset is PR, from which Theorem 1.3 will be obvious.

Section 2 describes our basic techniques and the outline of the main proofs, leaving some more technical details to be verified in Sections 3 and 4. There are still many open questions about the CSWP; these are described in Section 5.

When we deal with scattered spaces in this paper, it will be more useful to use the equivalent definition that every non-empty subset contains an isolated point. Likewise, in the various generalizations of scattered, we shall use a definition of form (1) in the following simple observation:

Proposition 1.4 *Let $\mathfrak{K}$ be any class of compact spaces such that every space homeomorphic to a closed subspace of a member of $\mathfrak{K}$ is in $\mathfrak{K}$. Then for any compact X, the following are equivalent:*

1. *For all non-empty closed $F \subseteq X$, there is a non-empty $U \subseteq F$ which is (relatively) open in F such that $\overline{U} \in \mathfrak{K}$.*
2. *If one defines Z^* to be $Z \setminus \bigcup\{U \subseteq Z : U$ is open in Z and $\overline{U} \in \mathfrak{K}\}$, and then defines $X^{[0]} = X$, $X^{[\alpha+1]} = (X^{[\alpha]})^*$, and $X^{[\gamma]} = \bigcap_{\alpha<\gamma} X^{[\alpha]}$ for limit ordinals γ, then some $X^{[\alpha]}$ is empty.*

Proof. For (2) $\to$ (1), consider the largest α such that $F \subseteq X^{[\alpha]}$. ☕

Definition 1.5 *If $\mathfrak{K}$ is as in Proposition 1.4 and X is compact, then X is* scattered for $\mathfrak{K}$ *iff X satisfies* (1) (*or* (2)).

So, scattered spaces in the usual sense are scattered for the class of 1-point spaces.

2 Basic Techniques

These techniques involve *idempotents*, *measures*, and *removable spaces*.

As usual, $f \in C(X)$ is called an *idempotent* iff $f^2 = f$; equivalently, $f = \chi_H$, where H is a clopen subset of X (i.e., H is both closed and open). Thus, X is connected iff the only idempotents in $C(X)$ are the two *trivial* ones (the constant 0 and the constant 1 functions).

Definition 2.1 *The compact X has the* NTIP *iff every $\mathcal{A} \sqsubseteq^{c} C(X)$ contains a non-trivial idempotent.*

A connected space cannot have the NTIP, but if X is not connected, the CSWP implies the NTIP. Rudin [8] proved that the Cantor set fails the NTIP (and hence fails the CSWP). By [6], Lemma 3.5 (or see Proposition 5.7):

Lemma 2.2 *If X is compact and every perfect subset of X has the NTIP, then X has the CSWP.*

This simplifies the task of proving that a space has the CSWP, since one need only produce a non-trivial idempotent, although one must then deal with arbitrary perfect subsets of X. One can produce idempotents in some $\mathcal{A} \sqsubseteq C(X)$ by using Runge's Theorem on polynomial approximations (see [10]§13). We quote here only the special case we need:

Theorem 2.3 (Runge) *If $K_1, \ldots, K_n$ are disjoint compact convex subsets of $\mathbb{C}$, $\varepsilon > 0$, and $w_1, \ldots, w_n \in \mathbb{C}$, then is a complex polynomial $P(z)$ such that $|P(z) - w_\ell| < \varepsilon$ for each $\ell = 1, \ldots, n$ and each $z \in K_\ell$.*

Then, as in [8] or [4], composing functions with polynomials yields:

Lemma 2.4 *If X is compact, $\mathcal{A} \sqsubseteq C(X)$, and $\mathrm{Re}(h(X))$ is not connected for some $h \in \mathcal{A}$, then $\mathcal{A}$ contains a non-trivial idempotent.*

As usual, $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ denotes projection onto the real axis. Lemmas 2.2 and 2.4 were used in [6] to prove Theorem 1.2 in the case that X is separable.

Note that any X satisfying the hypotheses to Lemma 2.2 is totally disconnected. However, using *measures*, we can extend our results to apply to many connected spaces. A simple use of measures is contained in [6]; Lemma 2.6 below is Lemma 5.2 of [6].

Definition 2.5 *If μ is a regular complex Borel measure on the compact space X, then $|\mu|$ denotes its total variation, and $\mathrm{supt}(\mu) = \mathrm{supt}(|\mu|)$ denotes its (closed) support; that is, $\mathrm{supt}(\mu) = X \setminus \bigcup\{U \subseteq X : U \text{ is open} \quad \& \quad |\mu|(U) = 0\}$.*

Lemma 2.6 *Assume that X is compact and that $\mathrm{supt}(\mu)$ has the CSWP for all regular Borel measure μ. Then X has the CSWP.*

In fact, one can derive Lemmas 2.6 and 2.2 together by using the method of de Branges [2]; see Proposition 5.7. In [6], Lemma 2.6 was used to derive Theorem 1.2 from the separable case of it, by applying:

Lemma 2.7 *Assume that X is a compact LOTS and that μ is a regular Borel measure. Then $\mathrm{supt}(\mu)$ is separable.*

In Section 4, we shall continue to use the notion of "support" to reduce the problem of the CSWP for a "big" space X to that of a "small" subspace. Usually, the "small" subspace will be totally disconnected, and arguments involving idempotents will apply to it, whereas X itself may be connected. For example, it is easy to construct a connected compact LOTS which does not contain a Cantor subset (see Example 5.5), but every separable subspace of such a LOTS must be totally disconnected.

Theorem 1.3 will be proved using our notion of *removable spaces*. The definition of "removable" is given in terms of the Šilov boundary:

Definition 2.8 *Assume that $\mathcal{A} \sqsubseteq^{c} C(X)$. Let H be a closed subset of X. Then:*

- ☞ $\|f\|_H = \sup\{|f(x)| : x \in H\}$.
- ☞ *H is a* boundary *for $\mathcal{A}$ iff $\|f\|_H = \|f\|$ for all $f \in \mathcal{A}$.*
- ☞ *$Ш(\mathcal{A})$ denotes the* Šilov boundary*; this is the smallest closed set which is a boundary for $\mathcal{A}$.*

The existence of a smallest boundary, $Ш(\mathcal{A})$, is due to Šilov; see [1, 3, 5, 11]. Note that $Ш(\mathcal{A})$ is always non-empty, and cannot be finite unless X is finite.

Definition 2.9 *Given $\mathcal{A} \sqsubseteq C(X)$ and a closed subset $H \subseteq X$, let $\mathcal{A}{\restriction}H = \{f{\restriction}H : f \in \mathcal{A}\} \sqsubseteq C(H)$. $\overline{\mathcal{A}{\restriction}H} \sqsubseteq^{c} C(H)$ denotes the closure in the uniform topology.*

Observe that $\mathcal{A} \sqsubseteq^{c} C(X)$ does not in general imply that $\mathcal{A}{\restriction}H \sqsubseteq^{c} C(H)$. For example, if $X \subset \mathbb{C}$ is the closed unit disc and $\mathcal{A}$ is the usual disc algebra (see [3, 11]), then $\mathcal{A}{\restriction}[0,1]$ is dense in and not closed in $C([0,1])$. The following easy lemma is proved in [6]:

Lemma 2.10 *Suppose that $\mathcal{A} \sqsubseteq^{c} C(X)$ and $Ш(\mathcal{A}) \subseteq H$, where H is closed. Then $\mathcal{A}{\restriction}H \sqsubseteq^{c} C(H)$. Also, $\mathcal{A}{\restriction}H = C(H)$ iff $\mathcal{A} = C(X)$ (in which case $Ш(\mathcal{A}) = X$). If all idempotents of $\mathcal{A}$ are trivial, then the same is true of $\mathcal{A}{\restriction}H$.*

Definition 2.11 *A compact space H is* removable *iff for all $X, U, \mathcal{A}$, if:*

- ☞ *X is compact,*
- ☞ *$U \subsetneqq X$ and U is open,*
- ☞ *$\overline{U}$ is homeomorphic to a subspace of H, and*
- ☞ *$\mathcal{A} \sqsubseteq^{c} C(X)$ and all idempotents of $\mathcal{A}$ are trivial,*

then $Ш(\mathcal{A}) \subseteq X \backslash U$.

We remark that we disallow $U = X$ only because if X is a singleton, $U = X$ and $\mathcal{A} = C(X)$, then we would have a contradiction. If $U = X$ and X is not a singleton, then the conclusion, $Ш(\mathcal{A}) = \emptyset$, is still contradictory, but so are the hypotheses, since X would have the NTIP by Lemma 2.12 below. As stated, the hypotheses are non-vacuous, in the sense that if H is any compact space, then there are always $X, U, \mathcal{A}$ satisfying the hypotheses of Definition 2.11 with $U = \overline{U}$ homeomorphic to H; see Proposition 5.4.

Lemma 2.12 *Let H be removable.*

1. *If $K \subseteq H$ is closed, then K is removable.*
2. *If $|H| > 1$, then H has the NTIP.*
3. *H has the CSWP.*
4. *H is totally disconnected.*

Proof. (1) is immediate from the definition. For (2), suppose that we had $\mathcal{A} \overset{c}{\subseteq} C(H)$ with all idempotents trivial. Let $H = U \cup V$, where U, V are proper open subsets. Applying (1), $\overline{U}$ and $\overline{V}$ are removable, so $Ш(\mathcal{A}) \subseteq X \backslash U$ and $Ш(\mathcal{A}) \subseteq X \backslash V$, so $Ш(\mathcal{A}) = \emptyset$, a contradiction.

Now (3) is immediate from (2), (1), and Lemma 2.2. (4) is also immediate from (2) and (1), since no perfect subset of H is connected. ☕

Of course, for this notion to be useful, we need to show that there are some removable spaces. We begin with:

Lemma 2.13 *The one-point space is removable.*

Proof. Let $X, U, \mathcal{A}$ be as in Definition 2.11, with $\overline{U}$ a singleton, $\{p\}$, so that p is isolated in X. We need to show that $X \backslash \{p\}$ is a boundary. If not, then there is an $h \in \mathcal{A}$ with $\|h\|_{X \backslash \{p\}} = 1$ but $|h(p)| > 1$. We may assume that $h(p) \in \mathbb{R}$; but then $\mathrm{Re}(h(X))$ is not connected, a contradiction by Lemma 2.4. ☕

We can produce more removable spaces via a generalized Cantor-Bendixson analysis:

Definition 2.14 *The compact H is* R-scattered *iff H is scattered for the class of removable spaces* (*see Definition 1.5*).

Note that every closed subspace of an R-scattered space is R-scattered. By Lemma 2.13, every scattered space is R-scattered, and hence removable by:

Lemma 2.15 *H is R-scattered iff H is removable.*

Proof. For the non-trivial direction, let $X, U, \mathcal{A}$ be as in Definition 2.11. Then $\overline{U}$ is R-scattered. Let $K = \text{Ш}(\mathcal{A})$. We need to show that $U \cap K = \emptyset$, so assume that $U \cap K \neq \emptyset$. Note that X and K must be infinite.

By Lemma 2.10, $\mathcal{A}{\restriction}K \sqsubseteq C(K)$ and $\mathcal{A}{\restriction}K$ has no non-trivial idempotents. Also note that $\text{Ш}(\mathcal{A}{\restriction}K) = K$.

Let $W = U \cap K$. Then W is relatively open in K, and W is non-empty. $\overline{W} \subseteq \overline{U}$ and $\overline{U}$ is R-scattered, so choose $V \subseteq \overline{W}$ with V non-empty, V open in $\overline{W}$, and $\overline{V}$ removable. We may assume also that $V \subseteq W$ (otherwise, replace V by $V \cap W$), so that V is open in K. Now $V \neq K$ (since otherwise K would have the NTIP by Lemma 2.12), so $\text{Ш}(\mathcal{A}{\restriction}K) \subseteq K \backslash V$, a contradiction. ☕

It follows that every scattered space is removable, and thus has the CSWP, which was already known from Rudin [9]; in fact, all we have done is to redo the argument in [9], using somewhat more complicated terminology. For these methods to produce anything new, we need to produce some non-scattered removable spaces, which we do in Section 3. In particular, we shall prove there:

Lemma 2.16 *Every compact separable LOTS which does not contain a Cantor subset is removable.*

Proof of Theorem 1.3. Fix $X = L \times S$ as in Theorem 1.3.

Applying Lemma 2.6, it is sufficient to fix a regular Borel measure μ on X and prove that $Y := \text{supt}(\mu)$ has the CSWP. For this, it is sufficient to show that Y is R-scattered (and hence removable by Lemma 2.15).

Let $\pi_L : L \times S \to L$ be projection, and let $\mu\,\pi_L^{-1}$ be the induced measure on L. Let $L' = \text{supt}(\mu\,\pi_L^{-1})$. Then L' is separable (by Lemma 2.7), and $Y \subseteq L' \times S$.

To prove that Y is R-scattered, fix a non-empty closed $F \subseteq Y$. Since S is scattered, F contains a non-empty clopen subset of the form $L'' \times \{p\}$, where p is isolated in $\pi_S(F)$ and $L'' \subseteq L'$. This L'' is removable by Lemma 2.16 (a subspace of a separable LOTS is separable; see [7]). ☕

We can generalize Theorem 1.3 to Theorem 2.20 below, using the following:

Definition 2.17 *The compact H is* pseudo-removable (PR) *iff* $\text{supt}(\mu)$ *is removable whenever μ is a regular Borel measure on H.*

Definition 2.18 *The compact X is* PR-scattered *iff H is scattered for the class of PR spaces (see Definition 1.5).*

For example, applying Lemmas 2.7 and 2.16, we get:

Lemma 2.19 *Every compact LOTS which does not contain a Cantor subset is PR.*

We shall prove:

Theorem 2.20 *Every compact PR-scattered space has the CSWP.*

Given this, the proof of Theorem 1.3 is trivial, since it is immediate from Lemma 2.19 that $L \times S$ is PR-scattered. Now, our proof above of Theorem 1.3 actually showed that $L \times S$ is PR, but there are PR-scattered spaces which are not PR (see Example 5.5). For these spaces, the proof we gave of Theorem 1.3 will not work, but we need to use Theorem 2.20 instead.

Of course, we still need to prove Lemma 2.16 (in Section 3) and Theorem 2.20 (in Section 4). Theorem 2.20 makes use of the following easy remark:

Lemma 2.21 *There is no PR-scattered compact space F with more than one point, with a regular Borel measure ν and an $\mathcal{A} \sqsubseteq C(F)$ such that:*

1. $F = \mathrm{supt}(\nu)$.
2. *All idempotents of $\mathcal{A}$ are trivial.*
3. $F = Ш(\mathcal{A})$.

Proof. Let $U \subseteq F$ be non-empty and open, such that $\overline{U}$ is PR. $\overline{U}$ is actually removable, since it is the support of the measure ν restricted to $\overline{U}$. Then $\overline{U}$ cannot equal F (otherwise, F would have the NTIP by Lemma 2.12). It follows that $Ш(\mathcal{A}) \subseteq F \backslash U$, contradicting (3). ☕

The proof of Theorem 2.20 takes a PR-scattered space X and an $\mathcal{A} \sqsubseteq C(X)$ such that $\mathcal{A} \neq C(X)$, and produces an $F \subseteq X$ and a ν for which (1)(2)(3) hold for $\overline{\mathcal{A}\restriction F} \sqsubseteq C(F)$, which is contradictory. Now the methods discussed above can easily get $F, \nu, \overline{\mathcal{A}\restriction F}$ to satisfy (2)(3), and it is also easy to get (1)(2) (see Proposition 5.7), but obtaining (1)(2)(3) seems to require a different idea; we shall obtain ν as a measure which represents an element of the maximal ideal space of $\mathcal{A}$. The details are described in Section 4.

Sections 3 and 4 can be read independently of each other.

3 Some Removable Spaces

We describe a general technique for proving that certain spaces are removable. We begin with the observation, following Tychonov, that any subset $\mathcal{E} \subseteq C(X)$ defines a map from X into $\mathbb{C}^{\mathcal{E}}$:

Definition 3.1 *Let X be compact and $\mathcal{E} \subseteq C(X)$. Define $x \sim_{\mathcal{E}} y$ (or, $x \sim y$) iff $f(x) = f(y)$ for all $f \in \mathcal{E}$. Let $[x] = [x]_{\mathcal{E}} = \{y : x \sim y\}$, let $X/{\sim} = \{[x] : x \in X\}$, and let $\pi = \pi_{\mathcal{E}}$ be the natural map from X onto $X/{\sim}$. A subset $U \subseteq X$ is $\mathcal{E}$–*open *iff U is open and $U = \pi^{-1}\pi(U)$, and $H \subseteq X$ is $\mathcal{E}$–*closed *iff $X \backslash H$ is $\mathcal{E}$–open. For $U \subseteq X$, let $\mathrm{cl}_{\mathcal{E}}(U) = \pi^{-1}(\mathrm{cl}(\pi(U)))$.*

With the usual quotient topology, $X/{\sim}$ is a compact space and can be identified with the image of X under the evaluation map from X into $\mathbb{C}^{\mathcal{E}}$. Each $[x]$ is $\mathcal{E}$–closed. If $\mathcal{E}$ is countable, then $X/{\sim}$ will be second countable (equivalently, metrizable). $\mathrm{cl}_{\mathcal{E}}(U)$ is the smallest $\mathcal{E}$–closed set containing U. Note that even for $\mathcal{E}$–open U, $\mathrm{cl}_{\mathcal{E}}(U)$ might properly contain $\overline{U}$.

Definition 3.2 *Let X be compact, fix $\mathcal{E} \subseteq C(X)$ and $f \in C(X)$, and fix a real $c > 0$. Then $\mathrm{BIG}_{\mathcal{E}}(X, f, c) = \bigcup\{[x] \in X/{\sim} : \mathrm{diam}(f([x])) \geq c\}$.*

Here, "diam" refers to the usual notion of the diameter of a subset of $\mathbb{C}$. Note that $\mathrm{BIG}_{\mathcal{E}}(X, f, c) = \emptyset$ whenever $f \in \mathcal{E}$.

Lemma 3.3 *Each $\mathrm{BIG}_{\mathcal{E}}(X, f, c)$ is closed in X.*

Proof. Let $B = \{(x, y, z) \in X^3 : x \sim y \sim z \ \&\ |f(y) - f(z)| \geq c\} \subseteq X^3$. Then B is closed and $\mathrm{BIG}_{\mathcal{E}}(X, f, c)$ is the projection of B onto the first coordinate. ☕

Using these notions, we can define a class of spaces which are "close" to being metrizable:

Definition 3.4 *Let X be compact and $\mathcal{E} \subseteq C(X)$. $\mathcal{E}$ is* nice *iff*

1. *$[x]$ is scattered for all $[x] \in X$.*
2. *For all $f \in C(X)$, $f([x])$ is a singleton for all but at most countably many equivalence classes $[x] \in X/{\sim}$.*

X is nice *iff there is* countable *nice $\mathcal{E} \subseteq C(X)$.*

Lemma 3.5 *If $\mathcal{E} \subseteq C(X)$ is nice, $c > 0$, and $f \in C(X)$, then $\mathrm{BIG}_{\mathcal{E}}(X, f, c)$ is scattered.*

Proof. It is compact by Lemma 3.3, and a countable union of scattered subspaces. ☕

We shall show (Lemma 3.20) that if X is nice and does not contain a Cantor subset, then X is removable. Some examples of nice X: If X is scattered then X is nice, taking $\mathcal{E}$ to contain only constant functions (so there is only one equivalence class). If X is second countable, then X is nice, since there is a countable $\mathcal{E} \subset C(X)$ which separates points, so that each $[x]$ is a singleton. Of course, any second countable X is also scattered if it does not contain a Cantor subset. A more useful example is:

Lemma 3.6 *Every separable compact LOTS is nice.*

Proof. Let $D \subseteq X$ be dense in X and countable. Assume that X and D are infinite. List D as $\{d_n : n \in \mathbb{N}\}$. We may assume that d_0 is the first element of X and d_1 is the last element of X. Define $h : D \to [0,1]$ so that $h(d_0) = 0$, $h(d_1) = 1$, and $h(d_n) = (h(d_i) + h(d_j))/2$ when $n \geq 2$, where d_i is the largest element in $\{d_\ell : \ell < n \ \& \ d_\ell < d_n\}$, and d_j is the smallest element in $\{d_\ell : \ell < n \ \& \ d_\ell > d_n\}$.

This h extends to a continuous map from X into $[0,1]$, defined so that for $x \in X \backslash D$, $h(x) = \sup\{h(d_\ell) : d_\ell < x\} = \inf\{h(d_\ell) : d_\ell > x\}$. Let $\mathcal{E} = \{h\}$. (1) of Definition 3.4 holds because each $[x]$ has cardinality one or two. To prove (2), note that for any $f \in C(X)$ and each $c > 0$, $\mathrm{BIG}_{\mathcal{E}}(X, f, c)$ must be a *finite* union of two-element classes, since if it were infinite, it would have a limit point, which would contradict continuity of f. ☕

A class of nice spaces which is not related to ordered spaces or to scattered spaces is described in Example 5.6.

The definitions of "nice" and "BIG" refer only to $\sim_{\mathcal{E}}$, not $\mathcal{E}$, so it is reasonable to introduce:

Definition 3.7 *If $\mathcal{E}, \mathcal{F} \subseteq C(X)$, say that $\mathcal{E} \preccurlyeq \mathcal{F}$ iff $\sim_{\mathcal{F}}$ is finer than $\sim_{\mathcal{E}}$ (that is, every $\sim_{\mathcal{E}}$ class is a union of $\sim_{\mathcal{F}}$ classes). Say $\mathcal{E} \approx \mathcal{F}$ iff $\sim_{\mathcal{F}}$ and $\sim_{\mathcal{E}}$ are the same.*

Note that $\mathcal{E} \subseteq \mathcal{F} \to \mathcal{E} \preccurlyeq \mathcal{F}$, and $\mathcal{E} \approx \mathcal{F} \leftrightarrow \mathcal{E} \preccurlyeq \mathcal{F} \preccurlyeq \mathcal{E}$.

Lemma 3.8 *If $\mathcal{E} \preccurlyeq \mathcal{F}$, $\mathcal{E}$ is nice, and $\mathcal{F}$ is countable, then $\mathcal{F}$ is nice.*

Proof. In Definition 3.4, clause (1) for $\mathcal{F}$ is obvious. To verify clause (2), note that every $\mathcal{E}$ class, $[x]_{\mathcal{E}}$ is a countable union of $\mathcal{F}$ classes because $([x]_{\mathcal{E}})/{\sim_{\mathcal{F}}}$ is scattered and second countable, and hence countable. ☕

Next, we mention a few closure properties of the class of nice space:

Lemma 3.9 *If X is nice, then every closed subspace of X is nice.*

Lemma 3.10 *If X is compact and $X = \bigcup_{n\in\omega} H_n$, where each H_n is nice and is a closed G_δ in X, then X is nice.*

Proof. For each n, choose a countable nice $\mathcal{E}_n \subseteq C(H_n)$. By the Tietze Extension Theorem, we may assume that $\mathcal{E}_n = \mathcal{F}_n{\restriction}H_n$, where $\mathcal{F}_n \subseteq C(X)$. Since H_n is a G_δ, we may also assume that each H_n is $\mathcal{F}_n$–closed (adding another function to $\mathcal{F}_n$ if necessary). Then $\bigcup_n \mathcal{F}_n \subseteq C(X)$ is countable and nice. ☕

Corollary 3.11 *A finite disjoint sum of nice spaces is nice.*

Corollary 3.12 *If X is nice and Y is countable and compact, then $X \times Y$ is nice.*

The product of two nice spaces is not in general nice; for example, the square of the double arrow space is not nice by Proposition 5.1.

Next, we relate "nice" to function algebras.

Definition 3.13 *If X is compact and $\mathcal{E} \subseteq C(X)$, then $\langle\mathcal{E}\rangle$ is the intersection of all closed subalgebras of $C(X)$ which contain $\mathcal{E}$ and the constant functions.*

So, $\langle\mathcal{E}\rangle$ is, by definition, a closed subalgebra which contains the constant functions, but it will not separate points unless $\mathcal{E}$ does:

Lemma 3.14 *Let X be compact, $\mathcal{E} \subseteq C(X)$, and $\mathcal{E}^*$ the set of all complex conjugates of functions in $\mathcal{E}$. Let $\mathcal{A} = \langle\mathcal{E}\rangle$ and $\mathcal{A}' = \langle\mathcal{E} \cup \mathcal{E}^*\rangle$. Then $\mathcal{E} \approx \mathcal{A} \approx \mathcal{A}'$.*

Lemma 3.15 *Say $\mathcal{E} \subseteq C(X)$ is countable and $\mathcal{A} \sqsubseteq C(X)$. Then there is a countable $\mathcal{F} \subseteq \mathcal{A}$ such $\mathcal{E} \preccurlyeq \mathcal{F}$.*

Proof. By the Stone-Weierstrass Theorem, we may choose a countable $\mathcal{F} \subseteq \mathcal{A}$ such that $\mathcal{E} \subseteq \langle\mathcal{F} \cup \mathcal{F}^*\rangle$. Now, apply Lemma 3.14. ☕

Applying Lemma 3.15 and 3.8, we get

Lemma 3.16 *If X is nice and $\mathcal{A} \sqsubseteq C(X)$, then there is a countable nice $\mathcal{F} \subseteq \mathcal{A}$.*

In order to relate "nice" to "removable", we need to know that the nice $\mathcal{E}$ may be taken to have the following additional property:

Definition 3.17 *Suppose that $\mathcal{E} \subseteq \mathcal{A} \sqsubseteq C(X)$. Then $\mathcal{E}$ is* adequate *for $\mathcal{A}$ iff whenever we are given a finite m, an $f \in \mathcal{A}$, and sets H_i, W_i for $i < m$ satisfying:*

1. *The H_i are $\mathcal{E}$–closed subsets of X,*
2. *The W_i are open subsets of $\mathbb{C}$, and*
3. *$f(H_i) \subseteq W_i$ for each $i < m$,*

then there is a $g \in \mathcal{E}$ such that $g(H_i) \subseteq W_i$ for each $i < m$.

Lemma 3.18 *If $\mathcal{E} \subseteq \mathcal{A} \sqsubseteq C(X)$ and $\mathcal{E}$ is countable, then there is a countable $\mathcal{F}$ such that $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{F}$ is adequate for $\mathcal{A}$.*

Proof. We first reduce "adequate" to a countable number of instances. Let $\mathcal{W}$ be a countable base for the topology of $\mathbb{C}$. Assume that $\mathcal{W}$ is closed under finite unions. Let $\mathcal{U}_\mathcal{E}$ be a countable base for the space $X/{\sim}$, and also assume that $\mathcal{U}_\mathcal{E}$ is closed under finite unions. Let $\pi = \pi_\mathcal{E}$ (see Definition 3.1), and let $\mathcal{H}_\mathcal{E} = \{X \setminus \pi^{-1}(U) : U \in \mathcal{U}_\mathcal{E}\}$. Then all sets in $\mathcal{H}_\mathcal{E}$ are $\mathcal{E}$–closed.

Note that in order for $\mathcal{E}$ to be adequate for $\mathcal{A}$, it is sufficient that Definition 3.17 holds with the $H_i \in \mathcal{H}_\mathcal{E}$ and the $W_i \in \mathcal{W}$: To see this, suppose we started with arbitrary H_i and W_i. Each H_i is the intersection of the sets in $\mathcal{H}_\mathcal{E}$ which contain it, so by compactness, we can find $H_i' \supseteq H_i$ in $\mathcal{H}_\mathcal{E}$ with $f(H_i') \subseteq W_i$. Again by compactness, we can find $W_i' \subseteq W_i$ in $\mathcal{W}$ with $f(H_i') \subseteq W_i'$. Then if we get $g \in \mathcal{E}$ with each $g(H_i') \subseteq W_i'$, we will also have $g(H_i) \subseteq W_i$.

Now, starting from $\mathcal{E}$, we get $\mathcal{E} = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$, where each $\mathcal{E}_n$ is countable. Given $\mathcal{E}_n$, we obtain $\mathcal{E}_{n+1}$ so that whenever m, f and the H_i, W_i satisfy (1)(2)(3) of Definition 3.17 with the $H_i \in \mathcal{H}_{\mathcal{E}_n}$ and the $W_i \in \mathcal{W}$, there is a $g \in \mathcal{E}_{n+1}$ such that $g(H_i) \subseteq W_i$ for each $i < m$. Let $\mathcal{F} = \bigcup_n \mathcal{E}_n$. ☕

Finally, we need an easy consequence of Runge's Theorem 2.3:

Lemma 3.19 *Suppose that $\mathcal{A} \sqsubseteq C(X)$, $V \subseteq X$, $p, q \in X$, $p \neq q$, and $\mathcal{A}$ contains a function k with $|k(p)|, |k(q)| > \sup\{|k(x)| : x \in V\}$. Fix $a, b, c \in \mathbb{C}$ and $\varepsilon > 0$. Then $\mathcal{A}$ contains a function f with $f(p) = a$, $f(q) = b$, and $f(V) \subseteq B(c; \varepsilon)$.*

Proof. Since $\mathcal{A}$ separates points, we may assume, by adding a small function to k, that $k(p) \neq k(q)$. Then, whenever $\delta > 0$, we may apply Theorem 2.3 to find a polynomial P such that, setting $h = P \circ k$, we have $h(p) \in B(a; \delta)$, $h(q) \in B(b; \delta)$, and $h(V) \subseteq B(c; \delta)$. The result now follows by choosing δ small enough and composing h with a linear polynomial. ☕

Lemma 3.20 *Assume that H is nice and does not contain a Cantor subset. Then H is removable.*

Proof. Fix $X, U, \mathcal{A}$ satisfying the conditions in Definition 2.11. Then $\overline{U}$ is nice and $X \backslash U \neq \emptyset$, and we need to prove that $X \backslash U$ is a boundary. Assume that it is not a boundary, and we shall derive a contradiction by producing a non-trivial idempotent in $\mathcal{A}$. Fix a $k \in \mathcal{A}$ such that $\|k\| > 1$ but $\|k\|_{X \backslash U} \leq 1$. By replacing k with a power of k, we assume also that $\|k\| > 3$. Multiplying k by some $e^{i\theta}$, we can assume that some $k(x)$ is a real number in $(3, \infty)$.

Let $T = \{x \in X : |k(x)| \geq 2\}$ and $V = X \backslash T = \{x \in X : |k(x)| < 2\}$. Then $T \subseteq U$, so T is nice. Applying Lemma 3.16 to $\mathcal{A} \restriction T$, let $\mathcal{E}_0 \subset \mathcal{A}$ be countable, with $\mathcal{E}_0 \restriction T$ nice as a subfamily of $C(T)$. Then by Lemma 3.18, let $\mathcal{E}$ be countable with $\mathcal{E}_0 \cup \{k\} \subseteq \mathcal{E} \subset \mathcal{A}$ and $\mathcal{E}$ adequate for $\mathcal{A}$. Note that $\mathcal{E} \restriction T$ is nice by Lemma 3.8. In the following, $[x]$ and $\sim$ always mean $[x]_{\mathcal{E}}$ and $\sim_{\mathcal{E}}$. Since $k \in \mathcal{E}$, each $[x]$ is a subset of either V or T.

$\{x \in X : |k(x)| \geq 3\}$ is not scattered (since $\mathrm{Re}(k(X))$ is connected by Lemma 2.4), so it cannot be second countable (since it does not contain a Cantor subset), so fix distinct $p, q \in \{x \in X : |k(x)| \geq 3\}$ with $p \sim q$. Then, by Lemma 3.19, fix $f \in \mathcal{A}$ with $f(V) \subseteq B(0; 1/2)$ and $f(p) = 2$ and $f(q) = -2$.

Let $H = \mathrm{BIG}_{\mathcal{E} \restriction T}(T, f \restriction T, 1/2) \subseteq T$. H is scattered by Lemma 3.5. It follows that $f(H) - f(H) = \{f(x) - f(y) : x, y \in H\}$ is a continuous image of the scattered space $H \times H$, and is thus scattered in $\mathbb{C}$, so fix a $b \in (1, 2)$ with $b \notin \mathrm{Re}(f(H) - f(H))$.

Then, $\pm b \notin \mathrm{Re}(f([x]) - f([x]))$ for each $x \in X$: If $x \in H$, this follows by the choice of b. If $x \in T \backslash H$, then $\mathrm{diam}(f([x])) < 1/2$, so that $f([x]) - f([x]) \subseteq B(0; 1)$. If $x \in V$, then $f([x]) \subseteq f(V) \subseteq B(0; 1/2)$.

Next, for each $x \in X$, there are $\mathcal{E}$–open $O = O_x$ containing $[x]$ and open $W = W_x \subseteq \mathbb{C}$ such that $f(\mathrm{cl}_{\mathcal{E}}(O)) \subseteq W$ and $\pm b \notin \mathrm{Re}(W - W)$. To do this, choose $\mathcal{E}$-open O_n so that $O_0 \supseteq \mathrm{cl}_{\mathcal{E}}(O_1) \supseteq O_1 \supseteq \mathrm{cl}_{\mathcal{E}}(O_2) \cdots$ and $\bigcap_n O_n = [x]$. Since $\pm b \notin \mathrm{Re}(f([x]) - f([x]))$, we can apply compactness to choose n so that $\pm b \notin \mathrm{Re}(f(\mathrm{cl}_{\mathcal{E}}(O_n)) - f(\mathrm{cl}_{\mathcal{E}}(O_n))$. Then let O_x be that O_n, and let W be any open superset of $f(\mathrm{cl}_{\mathcal{E}}(O_n))$ such that $\pm b \notin \mathrm{Re}(W - W)$.

Then, by compactness, we have a finite m and $O^i = O_{x_i}$ for $i < m$ such that the O^i cover X. We may assume that $x_0 \in V$ and $O^0 = V$, and $W_{x_0} = B(0; 1/2)$. Let $W^i = W_{x_i}$. $f(\mathrm{cl}_{\mathcal{E}}(O^i)) \subseteq W^i$ and $\pm b \notin \mathrm{Re}(W^i - W^i)$. Since $\mathcal{E}$ is adequate, choose $g \in \mathcal{E}$ such that $g(\mathrm{cl}_{\mathcal{E}}(O^i)) \subseteq W^i$ for all $i < m$. Let $h = f - g$. Then $h(X) \cap B(0; 1) \neq \emptyset$ (because of x_0), so $\mathrm{Re}(h(X)) \cap (-1, 1) \neq \emptyset$. Let $z = g(p) = g(q)$. Then $h(p) = 2 - z$ and $h(q) = -2 - z$, so $\mathrm{Re}(h(X))$ meets either $[2, \infty)$ or $(-\infty, -2]$. But also, for each x, there is an i such that $x \in O^i$; so that

$f(x), g(x) \in W^i$ and $h(x) \in W^i - W^i$; hence, $\pm b \notin \mathrm{Re}(h(X))$. Thus, $\mathrm{Re}(h(X))$ is disconnected, so $\mathcal{A}$ has a non-trivial idempotent by Lemma 2.4. ☕

Proof of Lemma 2.16. Immediate by Lemmas 3.20 and 3.6. ☕

4 Supports of Measures

We use the notion of "support" in two different ways to reduce the problem of the CSWP for a "big" space X to that of a "small" subspace. First, we can apply it to measures which annihilate a subspace of $C(X)$:

Definition 4.1 *If μ is a complex Borel measure on X and $\mathcal{E} \subseteq C(X)$, then $\mu \perp \mathcal{E}$ means that $\int f\, d\mu = 0$ for all $f \in \mathcal{E}$.*

The proof in [6] of Lemma 2.6 starts with $\mathcal{A} \mathrel{\underline{\subset}} C(X)$ and $\mathcal{A} \neq C(X)$, and uses a measure $\mu \perp \mathcal{A}$ to conclude that $\mathrm{supt}(\mu)$ fails to have the CSWP.

Our second application uses measures associated with elements of the *maximal ideal space*, $\mathcal{M}(\mathcal{A})$. Elements of $\mathcal{M}(\mathcal{A})$ may be viewed either as maximal ideals of $\mathcal{A}$, or as homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. See [3, 5, 10, 11].

Recall (see [3], Theorem 4.1, or [10] §5.22) that if $\mathcal{A} \mathrel{\underline{\subset}} C(X)$ and $\varphi \in \mathcal{M}(\mathcal{A})$, then there is always a regular Borel probability measure ν on X such that $\varphi(f) = \int f\, d\nu$ for all $f \in \mathcal{A}$. However, ν is not uniquely determined from φ. We can define a notion of "support" which depends directly on φ, not on an arbitrary choice of measure:

Definition 4.2 *If $\mathcal{A} \mathrel{\underline{\subset}} C(X)$ and $\varphi \in \mathcal{M}(\mathcal{A})$, then a closed $H \subseteq X$ is a* pre-support *of φ iff $|\varphi(f)| \leq \|f\|_H$ for all $f \in \mathcal{A}$. A closed $H \subseteq X$ is a* support *of φ if H is a pre-support but no proper subset of H is a pre-support.*

Applying Zorn's Lemma,

Lemma 4.3 *If H is a pre-support of φ, then there is a closed $K \subseteq H$ which is a support of φ.*

For a pre-support H, $\mathcal{A}{\restriction}H$ need not be closed in $C(H)$, but using $|\varphi(f)| \leq \|f\|_H$, we see that φ defines an element of $\mathcal{M}(\overline{\mathcal{A}{\restriction}H})$, and is thus represented by a probability measure on H. Thus we have

Lemma 4.4 *If $\mathcal{A} \underline{\subseteq}^{c} C(X)$, $\varphi \in \mathcal{M}(\mathcal{A})$, and H is a support of φ, then there is a regular Borel probability measure ν with $H = \mathrm{supt}(\nu)$ and $\varphi(f) = \int f\, d\nu$ for all $f \in \mathcal{A}$.*

For example, if $X \subset \mathbb{C}$ is the closed unit disc and $\mathcal{A}$ is the disc algebra, then $\mathcal{M}(\mathcal{A}) = X$, so every $\varphi \in \mathcal{M}(\mathcal{A})$ has a singleton as one of its supports. Say φ is evaluation at 0. Then $\{0\}$ is a support of φ, but so is every simple closed curve in X which winds around 0.

Next we note that $\overline{\mathcal{A}\restriction H}$ has no non-trivial idempotents – in fact, no non-trivial real-valued functions (i.e., is "anti-symmetric" in the terminology of [4]):

Lemma 4.5 *Assume that $\mathcal{A} \underline{\subseteq}^{c} C(X)$, $\varphi \in \mathcal{M}(\mathcal{A})$, and X is a support of φ. Let $f \in \mathcal{A}$ be real-valued. Then f is constant.*

Proof. If not, then by re-scaling, we can assume that $f : X \to [0, 1]$, and $0, 1 \in f(X)$. Let ν be as in Lemma 4.4. Then $X = \mathrm{supt}(\nu)$, so that $0 < \varphi(f) < 1$. Say $\varphi(f) = 1 - 2\varepsilon$, where $0 < \varepsilon < 1/2$. Let $g = (1 + 2\varepsilon)f$. Then $\varphi(g) < 1$, so $\varphi(g^n) \to 0$ as $n \to \infty$. But also, $g(x) > (1+\varepsilon)$ on a set of positive measure (since $X = \mathrm{supt}(\nu)$), so $\varphi(g^n) = \int g^n\, d\nu \to \infty$ as $n \to \infty$. ☕

Next, we note that a support of φ has no isolated points unless it is a singleton:

Lemma 4.6 *Assume that $\mathcal{A} \underline{\subseteq}^{c} C(X)$, $\varphi \in \mathcal{M}(\mathcal{A})$, and X is a support of φ. Then:*

1. *$X = Ш(\mathcal{A})$.*
2. *X has no isolated points unless X is a singleton.*

Proof. (1) follows from the definition of "support" as a minimal pre-support, since $Ш(\mathcal{A})$ is always a pre-support. For (2), assume that $p \in X$ is isolated. Since $\{p\}$ is removable (Lemma 2.13), $X = Ш(\mathcal{A})$ implies that $\mathcal{A}$ must have a non-trivial idempotent, contradicting Lemma 4.5. ☕

In view of these lemmas and our methods in Section 3 for producing non-trivial idempotents, it is important to show that in many cases, there is some $\varphi \in \mathcal{M}(\mathcal{A})$ such that at least one of its supports is not a singleton. Of course, this cannot be true if $\mathcal{A} = C(X)$. We do not know if $\mathcal{A} \neq C(X)$ is sufficient for obtaining such a φ, but Lemma 4.8 below is a partial result in this direction which is strong enough for our purposes. For this, we need the following well-known theorem of Šilov [12] (or, see [1, 5]):

Theorem 4.7 (Šilov) *Suppose that $\mathcal{A} \sqsubseteq C(X)$ and every $\varphi \in \mathcal{M}(\mathcal{A})$ is a point evaluation. Then $\mathcal{A}$ contains the characteristic function of every clopen subset of X.*

Lemma 4.8 *Suppose that $\mathcal{A} \sqsubseteq C(X)$ and μ is a non-zero regular complex Borel measure on X with $\mu \perp \mathcal{A}$ and $X = \operatorname{supt}(\mu)$. Suppose that some clopen $K \subseteq X$ has the CSWP, where $\emptyset \subsetneqq K \subsetneqq X$. Then some $\varphi \in \mathcal{M}(\mathcal{A})$ has a support which is not a singleton.*

Proof. If some $\varphi \in \mathcal{M}(\mathcal{A})$ fails to be a point evaluation, then *every* support of φ is not a singleton. Thus, we may assume that $\mathcal{M}(\mathcal{A}) = X$. But then $\chi_K \in \mathcal{A}$ by Theorem 4.7. Since K has the CSWP, $\mathcal{A}$ must contain every continuous function which vanishes on $X \backslash K$, contradicting $\mu \perp \mathcal{A}$ (since $|\mu|(K) \neq 0$). ☕

Proof of Theorem 2.20. Suppose that X is PR-scattered, $\mathcal{A} \sqsubseteq C(X)$, and $\mathcal{A} \neq C(X)$. We shall derive a contradiction. Fix a non-zero regular Borel measure μ with $\mu \perp \mathcal{A}$, let $Y = \operatorname{supt}(\mu)$, and let $\mathcal{B} = \overline{\mathcal{A}\restriction Y} \sqsubseteq C(Y)$. Note that $(\mu\restriction Y) \perp \mathcal{B}$.

Now, there is a non-empty open $U \subseteq Y$ such that $\overline{U}$ is pseudo-removable, and hence removable, since $\overline{U}$ is the support of a measure. It follows that $\overline{U}$ is zero-dimensional, so there is a non-empty clopen $K \subseteq Y$ with $K \subseteq \overline{U}$. Then K has the CSWP, so we can apply Lemma 4.8 to get a $\varphi \in \mathcal{M}(\mathcal{B})$ with some support $F \subseteq Y$ which is not a singleton. Applying Lemma 4.4, we get ν with $F = \operatorname{supt}(\nu)$ and $\varphi(f) = \int f\, d\nu$ for all $f \in \mathcal{B}$. Let $\mathcal{B}'$ be the closure of $\mathcal{B}\restriction F$. Applying Lemmas 4.5 and 4.6, all idempotents of $\mathcal{B}'$ are trivial and $F = Ш(\mathcal{B}')$, so we have a contradiction by Lemma 2.21. ☕

5 Remarks and Examples

We do not know whether the CSWP for X and Y implies the CSWP for $X \times Y$, or even for the disjoint sum of X and Y.

The notion of "nice" is closed under disjoint sums (by Corollary 3.11), but not under products. For example, let X be a compact separable LOTS which is not second countable (e.g., the double arrow space), and let Y be scattered and uncountable. Then Y is trivially nice and X is nice by Lemma 3.6, but $X \times Y$ and $X \times X$ are not nice by:

Proposition 5.1 *If X is not scattered and Y is not second countable then $X \times Y$ is not nice.*

Proof. For $f \in C(X \times Y)$, define $\widehat{f} : Y \to C(X)$ by $(\widehat{f}(y))(x) = f(x, y)$. Then $\widehat{f}$ is continuous, so $\widehat{f}(Y) \subseteq C(X)$ is a compact metric space, and hence second countable.

Now, suppose that $\mathcal{E} \subseteq C(X \times Y)$ is countable. Then $\{\widehat{f} : f \in \mathcal{E}\}$ is a countable family of maps into second countable spaces. Since Y is not second countable, there are $b, c \in Y$ with $b \neq c$ and $\widehat{f}(b) = \widehat{f}(c)$ for all $f \in \mathcal{E}$. Thus, if $\sim$ is $\sim_{\mathcal{E}}$, we have $(x, b) \sim (x, c)$ for all $x \in X$.

If $\mathcal{E}$ were nice, then since each $[(x, y)]$ is closed and scattered, while X is not scattered, $X \times \{b\}$ must meet uncountably many equivalence classes. But then any $f \in C(X \times Y)$ such that $f(x, b) \neq f(x, c)$ for all $x \in X$ would contradict clause (2) of Definition 3.4. ☕

It is easy to see from Lemma 2.15 that any finite union of removable spaces is removable, but we do not know about products. We also do not know if removable is equivalent to CSWP plus totally disconnected.

If X is any compact space, there are trivially compact H such that the disjoint sum $X \oplus H$ fails the CSWP; for example, H itself can fail the CSWP. Proposition 5.4 below shows that we can always find H so that $X \oplus H$ fails the NTIP.

Definition 5.2 *If $f \in C(X \times Y)$, let $f_x(y) = f(x, y)$ (for $x \in X$) and $f^y(x) = f(x, y)$ (for $y \in Y$).*

Lemma 5.3 *Let X be an arbitrary compact space. Assume that Y is compact and that $\mathcal{A} \underline{\subseteq}^{c} C(Y)$, satisfying*

1. $\mathrm{Re}(h(Y))$ *is connected for all $h \in \mathcal{A}$.*
2. *Some $\varphi \in \mathcal{M}(\mathcal{A})$ is not a point evaluation.*

Then there is a $\mathcal{B} \underline{\subseteq}^{c} C(X \times Y)$ such that

a. $\mathrm{Re}(f(X \times Y))$ *is connected for all $f \in \mathcal{B}$.*
b. *For each $y \in Y$, $\{f^y : f \in \mathcal{B}\} = C(X)$.*

In particular $X \times Y$ fails the NTIP for all compact X.

Proof. Let

$$\mathcal{B} = \{f \in C(X \times Y) : \exists z \in \mathbb{C}\, \forall x \in X [f_x \in \mathcal{A} \ \&\ \varphi(f_x) = z]\} \quad .$$

Observe that $\mathcal{B}$ is a closed subalgebra of $C(X \times Y)$ and $\mathcal{B}$ contains all constant functions. To prove (b), fix $y_0 \in Y$. Since φ is not a point evaluation, fix $h_0 \in \mathcal{A}$ with $c := \varphi(h_0) \neq d := h_0(y_0)$. Let $h = (h_0 - c)/(d - c)$; then $h \in \mathcal{A}$ and $\varphi(h) = 0$

and $h(y_0) = 1$. Then whenever $k \in C(X)$, the function $f(x, y) = k(x)h(y)$ is in $\mathcal{B}$, and $f^{y_0} = k$.

To prove that $\mathcal{B}$ separates points (so that $\mathcal{B} \sqsubseteq C(X \times Y)$), fix $(x_1, y_1) \neq (x_2, y_2)$. If $y_1 \neq y_2$, fix $h \in \mathcal{A}$ such that $h(y_1) \neq h(y_2)$, and define $f(x, y) = h(y)$; then $f \in \mathcal{B}$ and separates $(x_1, y_1), (x_2, y_2)$. If $y_1 = y_2$, then $\mathcal{B}$ separates $(x_1, y_1), (x_2, y_2)$ by (b).

To prove (a), let $f \in \mathcal{B}$ be an idempotent; it is enough to show that f is trivial. By (1), each f_x is either identically 0 or identically 1. But $\varphi(f_x)$ is independent of x, so f itself is either identically 0 or identically 1. ☕

We remark that if Y is connected then (1) is trivial, whereas if Y is not connected, then (1) implies (2) by Šilov's Theorem 4.7.

Proposition 5.4 *If X is compact, then there is a compact Z containing a clopen copy of X and a $\mathcal{B} \sqsubseteq C(Z)$ such that*

1. *All idempotents of $\mathcal{B}$ are trivial.*
2. *$\mathcal{B}\restriction X = C(X)$.*

Proof. This follows from Lemma 5.3, if we choose Y to have an isolated point. For example, Y can be the Cantor set plus one point (using [4, 8]), or Y can be $\mathbb{T} \cup \{0\}$, where $\mathcal{A}$ is the disc algebra. ☕

We next give an example of a PR-scattered space which is not PR:

Example 5.5 *There is a compact X with a countable set I of isolated points such that if $L = X \backslash I$, then L is an infinite compact connected LOTS which does not contain a Cantor subset. This X is the support of a measure μ, and X is PR-scattered but not PR.*

Proof. Once X is constructed, it is the support of any μ which gives positive measure to the points in I. X is PR-scattered because L is PR (by Lemma 2.19). X is not removable because it is not totally disconnected (see Lemma 2.12). Then, X is also not PR because $X = \mathrm{supt}(\mu)$.

Let I be any countably infinite set. First,we describe L: As usual, for $y, z \subseteq I$, define $y \subseteq^* z$ iff $y \backslash z$ is finite, and define $y =^* z$ iff $y \Delta z$ is finite. Let C be a family of subsets of I satisfying:

1. C contains $\emptyset$ and I.
2. No two distinct elements of C are $=^*$.
3. C is totally ordered by $\subseteq^*$.
4. C is maximal with respect to (1)(2)(3).

Observe that C, as ordered by $\subseteq^*$, is a dense total order with first element $\emptyset$ and last element I. Let L be the Dedekind completion of C. Then L is a compact connected LOTS and no element of L has cofinality ω from both sides, so that L does not contain a Cantor subset. Let $X = C \cup I$. As a subbase for the topology of X, take all sets of the form $\{i\}$ and $X \backslash \{i\}$ for $i \in I$, together with all sets of the form $x \cup [\emptyset, x)$ and $(I \backslash x) \cup (x, I]$ for $x \in C$, where $[\emptyset, x)$ and $(x, I]$ denote intervals in L. ☕

Next, we show that the notion of "nice" provides results which are not obtainable just by considering ordered spaces. Specifically, call the compact H *LPR* iff it does not contain a Cantor subset, and whenever μ is a regular Borel measure on H, $\mathrm{supt}(\mu)$ is homeomorphic to compact separable LOTS. So, LPR spaces are PR by Lemmas 3.6 and 3.20. Then, call the compact X *LPR-scattered* iff X is scattered for the class of LPR spaces. Note that the only concrete examples we have given for spaces with the CSWP are contained in Theorem 1.3 and Example 5.5, but these spaces are actually LPR-scattered. Example 5.6 provides a space Z' which has the CSWP but is not LPR-scattered. The CSWP will follow from Z' being nice and not having a Cantor subset. To make Z' not LPR-scattered, we make sure that every non-empty open subset of Z' fails to be a LOTS, and that Z' is separable, and hence the support of a measure (e.g., a countable sum of point masses).

Example 5.6 *There is a compact separable nice space Z' which does not contain a Cantor set such that every non-empty open subset of Z' fails to be homeomorphic to a LOTS.*

Proof. We describe a generalization of the double arrow space construction. Assume that X is compact and has no isolated points, and let D be any finite set of "directions". As a set, Z will be $X \times D$, and we display elements of Z as d_x instead of (x, d). For the double arrow space construction, $X = [0, 1]$ and $D = \{\mathsf{l}, \mathsf{r}\}$, so Z contains "left and right copies", $\mathsf{l}_x, \mathsf{r}_x$ for each $x \in [0, 1]$. To define the topology, choose, for each $x \in X$ and $\mathsf{d} \in D$, open sets $U_x^{\mathsf{d}} \subseteq X \backslash \{x\}$ so that for each $x \in X$, the U_x^{d} are pairwise disjoint and $\{x\} \cup \bigcup_{\mathsf{d} \in D} U_x^{\mathsf{d}}$ is also open. In the double arrow space, $U_x^{\mathsf{r}} = (x, 1]$ and $U_x^{\mathsf{l}} = [0, x)$. Let $\pi : Z \to X$ be the natural projection. Give Z the topology whose subbase is all sets of the form $\pi^{-1}(V)$ such that V is open in X together with all sets of the form $\{\mathsf{d}_x\} \cup \pi^{-1}(U_x^{\mathsf{d}})$. Then Z is Hausdorff, π is continuous, and Z is compact by the Alexander Subbase Lemma. Z may have isolated points, but we can discard them, forming Z'; then Z' has no isolated points (since X has none), and $\pi(Z') = X$. In the double arrow space, the isolated points are l_0 and r_1.

If we form a base by taking finite intersections from the subbase, then every basic open set in Z is of the form $\pi^{-1}(V) \cup F$, where V is open in X and F is finite. Thus, if S is dense in X, then $\pi^{-1}(S) \cap Z'$ is dense in Z'. Thus, Z' will be separable whenever X is separable. Z itself need not be separable, since it might have uncountably many isolated points.

If X is compact metric then Z is nice: To see that, obtain the countable $\mathcal{E} \subset C(Z)$ of Definition 3.4 by composing $\pi : Z \to X$ with any countable subfamily of $C(X)$ which separates the points of X. Then $[z] = \pi^{-1}(\pi(z))$, which is finite. To verify condition (2) of Definition 3.4, it is sufficient to fix $f \in C(Z)$ and a real $c > 0$, and show that $\mathrm{BIG}_{\mathcal{E}}(Z, f, c)$ is a finite union of equivalence classes. If not, then $\pi(\mathrm{BIG}_{\mathcal{E}}(Z, f, c)) = \{x \in X : \mathrm{diam}(f(\pi^{-1}\{x\})) \geq c\}$ is infinite. Since X is compact, we can choose distinct $x_n \in \pi(\mathrm{BIG}_{\mathcal{E}}(Z, f, c))$ which converge to some point $y \in X$. Passing to a subsequence, we may assume that there is a $\mathsf{d} \in D$ such that each $x_n \in U_y^{\mathsf{d}}$. But then, by continuity, $\mathrm{diam}(f(\pi^{-1}\{x_n\})) < c$ for all but finitely many n, a contradiction.

Call our choice of the U_x^{d} *antisymmetric* iff one never has both $x \in U_y^{\mathsf{d}}$ and $y \in U_x^{\mathsf{d}}$ for any $\mathsf{d} \in D$ and $x, y \in X$. Observe that this is true for the double arrow space construction. If the choice is antisymmetric, then Z does not contain a Cantor set; in fact, no uncountable $E \subseteq Z$ can be second countable in its relative topology. To prove this, assume that E is uncountable and second countable. We may assume (shrinking E if necessary) that there is a fixed $\mathsf{d} \in D$ such that $E = \{\mathsf{d}_x : x \in G\}$, where G is an uncountable subset of X. Since E is second countable, it is metrizable, so let $\rho : E \times E \to \mathbb{R}$ be a metric on E which induces the topology on E. Shrinking E if necessary, we may assume that there is a fixed $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq (\{\mathsf{d}_x\} \cup \pi^{-1}(U_x^{\mathsf{d}})) \cap E$ for all $\mathsf{d}_x \in E$. But then, by antisymmetry, $\rho(\mathsf{d}_x, \mathsf{d}_y) \geq \varepsilon$ for all distinct $\mathsf{d}_x, \mathsf{d}_y \in E$, which is impossible.

Finally, we need to obtain an example where every non-empty open $W \subseteq Z'$ fails to be a LOTS. Since Z' is separable (assuming X is), W will be separable also, so it is sufficient to make sure that W fails to be hereditarily separable (HS) (since every separable LOTS is HS by [7]). To do this, we modify the well-known proof that the square of the double arrow space fails to be HS (although this square is also not nice by Proposition 5.1). Let C be a Cantor set in the real line with the property that any finite subset of C is linearly independent over $\mathbb{Q}$. Let $X = C \times C$. Observe that whenever $(x_1, x_2) \in X$, a, b are distinct positive rationals, and t is a non-zero real, then $(x_1 \pm at, x_2 \pm bt) \notin X$. Fix $a = 1.0$ and $b = 1.1$, say; then, for $x = (x_1, x_2) \in X$, the lines through x with slopes ± 1.1 partition $X \backslash \{x\}$ into four quadrants, north, east, south, and west of x. Let

$D = \{\mathsf{n}, \mathsf{e}, \mathsf{s}, \mathsf{w}\}$. If $x = (x_1, x_2) \in X$, define:

$$\begin{array}{l} U_x^{\mathsf{n}} = \{(w_1, w_2) \in X : w_2 > x_2 \ \&\ a(w_2 - x_2) > b|w_1 - x_1|\} \\ U_x^{\mathsf{s}} = \{(w_1, w_2) \in X : w_2 < x_2 \ \&\ a(x_2 - w_2) > b|w_1 - x_1|\} \\ U_x^{\mathsf{e}} = \{(w_1, w_2) \in X : w_1 > x_1 \ \&\ b(w_1 - x_1) > a|w_2 - x_2|\} \\ U_x^{\mathsf{w}} = \{(w_1, w_2) \in X : w_1 < x_1 \ \&\ b(x_1 - w_1) > a|w_2 - x_2|\} \end{array} .$$

Every non-empty open $W \subseteq Z'$ contains a subset of the form $\pi^{-1}V \cap Z'$ for some open $V \subseteq X$. Inside of this open set, the points of the form n_x for x on a horizontal line (i.e., with fixed x_2) form an uncountable discrete set; thus, W is not HS. ☕

L. de Branges [2] (or see [1]) used the Krein-Milman Theorem to give a quick proof of the Stone-Weierstrass Theorem. This proof can be modified to obtain directly (1) and (2) of Lemma 2.21. It also provides an alternate proof of Lemma 2.6 and a strengthened version of Lemma 2.2.

Proposition 5.7 *Suppose that $\mathcal{A} \sqsubseteq C(X)$ and $\mathcal{A} \neq C(X)$. Then there is a non-zero complex Borel measure μ on X with $\mu \perp \mathcal{A}$, such that if $F = supt(\mu)$ and $\mathcal{B} = \overline{\mathcal{A}\restriction F}$, then all real-valued functions in $\mathcal{B}$ are constant.*

Proof. We identify $(C(X))^*$ with the space of measures on X; note that $\|\mu\| = |\mu|(X)$. Let $K = \{\sigma \in (C(X))^* : \sigma \perp \mathcal{A} \ \&\ \|\sigma\| \leq 1\}$. Then K is convex and K is compact in the weak* topology. Let $\mu \in K$ be any non-zero extreme point. ☕

We do not see how to achieve all of (1)(2)(3) of Lemma 2.21 directly, avoiding the argument of Section 4. Note that the proof of Proposition 5.7 might result in $F = X \neq Ш(\mathcal{A})$. For example, let $X = \mathbb{T} \cup \{0\}$ and let $\mathcal{A}$ be the disc algebra restricted to X; then $Ш(\mathcal{A}) = \mathbb{T}$. Let $\mu = \frac{1}{2}\lambda - \frac{1}{2}\delta_0$, where δ_0 is the unit point mass at 0 and λ is the usual (Haar) measure on $\mathbb{T}$. Then $\text{supt}(\mu) = X$ and μ is an extreme point of K.

Thus, the F obtained in Proposition 5.7 might have isolated points. To obtain Lemma 2.2 from this, restrict the $\mathcal{B}$ guaranteed by Proposition 5.7 to its Šilov boundary $Ш(\mathcal{B})$, which in this case is perfect, and note that $\mathcal{B}\restriction Ш(\mathcal{B}) \sqsubseteq C(Ш(\mathcal{B}))$ still has all real-valued functions constant.

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6 Appendix

Here we collect a few remarks on Section 5 which don't seem worth putting in the published version of the paper, since Section 5 is itself essentially an appendix.

Note that in Proposition 5.4, one cannot also have $\mathcal{B} \restriction (Z \backslash X) = C(Z \backslash X)$, by the following:

Proposition 6.1 *If X is compact, $\mathcal{A} \sqsubseteq C(X)$, and $X = H \cup K$, where H, K are both closed in X and:*

1. *$\mathcal{A} \restriction H = C(H)$.*
2. *$\mathcal{A} \restriction K$ is dense in $C(K)$.*

Then $\mathcal{A} = C(X)$.

Proof. If $g \in C(K)$ and $g \restriction (H \cap K) \equiv 0$, define $e(g) \in C(X)$ so that $e(g) \restriction K = g$ and $e(g) \restriction H \equiv 0$. Let

$$\mathcal{I} = \{g \in C(K) : g \restriction (H \cap K) \equiv 0 \ \& \ e(g) \in \mathcal{A}\} \quad .$$

$\mathcal{I}$ is clearly a closed subalgebra of $C(K)$. It is also an ideal. To verify this, fix $g \in \mathcal{I}$. It is sufficient to show that $\{k \in C(K) : k \cdot g \in \mathcal{I}\}$ is dense in $C(K)$, so, by (2), it is sufficient to show that $(f \restriction K) \cdot g \in \mathcal{I}$ whenever $f \in \mathcal{A}$. But then $e((f \restriction K) \cdot g) = f \cdot e(g) \in \mathcal{A}$.

Since $\mathcal{I}$ is a closed ideal in $C(K)$, there is a closed $L \subseteq K$ such that $\mathcal{I} = \{f \in C(K) : f \restriction L \equiv 0\}$. Then $L \supseteq H \cap K$.

If $L = H \cap K$, then $\mathcal{A} = C(X)$. To verify this, fix $f \in C(X)$. By (1), fix $f' \in \mathcal{A}$ such that $f' \restriction H = f \restriction H$. Let $g = (f - f') \restriction K$. Then $g \restriction (H \cap K) \equiv 0$, so $g \in \mathcal{I}$, so $f - f' = e(g) \in \mathcal{A}$. Thus $f \in \mathcal{A}$.

Now, suppose $L \supsetneqq H \cap K$, and fix $p \in L \setminus (H \cap K)$. For any $f_1, f_2 \in \mathcal{A}$, if $f_1 \restriction H = f_2 \restriction H$, then $(f_1 - f_2) \restriction K \in \mathcal{I}$, so $f_1(p) = f_2(p)$. One can then define $\varphi : C(H) \to \mathbb{C}$ so that $\varphi(h) = f(p)$ for some (any) $f \in \mathcal{A}$ such that $f \restriction H = h$. Then $\varphi \in \mathcal{M}(C(H))$, so fix $q \in H$ such that $\varphi(h) = f(q)$ for all $h \in C(H)$. But then $p \neq q$ and $f(p) = f(q)$ for all $f \in \mathcal{A}$, a contradiction. ☕

In this proposition, one cannot weaken (1) to just "$\mathcal{A} \restriction H$ is dense in $C(H)$". For example, let X be the Cantor set and let $\mathcal{A} \sqsubseteq C(X)$ be the algebra defined by Rudin [8]. Then by Hoffman and Singer [4]§4, this $\mathcal{A}$ is *pervasive*; that is $\mathcal{A} \restriction H$ is dense in $C(H)$ for *every* proper closed $H \subset X$.

We remark that for the particular X obtained in Example 5.5, one can verify the CSWP without using the notion of PR; L has the CSWP by Theorem 1.2;

then, since L is the perfect kernel of X, X also has the CSWP by [6], Cor. 3.7. However, we can modify X by replacing each point of I by a copy of the double arrow space; that is, we build $Y = (X \times A)/\approx$, where A is the double arrow space and $\approx$ is the equivalence relation on $X \times A$ which identifies $\{x\} \times A$ to a point whenever $x \in L$. Then Y is also the support of a measure (since A is) and Y is PR-scattered but not PR, and Y is perfect.

To verify that the μ obtained in the proof of 5.7 really satisfies the proposition, let ν be the restriction of μ to $F = \mathrm{supt}(\mu)$. Note that ν is an extreme point of $N := \{\sigma \in (C(F))^* : \sigma \perp \mathcal{B} \ \& \ \|\sigma\| \leq 1\}$. Also, $\|\nu\| = 1$ (since ν is an extreme point).

Whenever $g \in C(F)$, let $g\nu$ denote the measure defined by $\int h\, d(g\nu) = \int hg\, d\nu$. Note that $\|g\nu\| = \int |g|\, d|\nu|$. Now, assume that $\mathcal{B}$ contains a non-constant real-valued function g. By rescaling, we may assume $g : F \to [0,1]$. Let $r = \|g\nu\| = \int g\, d|\nu|$; then $0 < r < 1$ (since $F = \mathrm{supt}(\nu)$), and if $s = 1 - r$ and $k(x) = 1 - g(x)$, then $s = \|k\nu\|$. But then $\nu = r \cdot (g/r)\nu + s \cdot (k/s)\nu$, and $(g/r)\nu$ and $(k/s)\nu$ are in N, which is impossible, since ν is an extreme point.

To verify that the μ described in the paragraph following Proposition 5.7 is really an extreme point, suppose that $\mu = (\mu^0 + \mu^1)/2$, where $\mu^0, \mu^1 \in K$. Let $\mu^j = \frac{1}{2}\nu^j - \frac{1}{2}c^j\delta_0$, where ν^j is a measure on $\mathbb{T}$ $(j = 0, 1)$. Then $\lambda = (\nu^0 + \nu^1)/2$ and $1 = (c^0 + c^1)/2$. Note that $\|\mu^j\| = 1$ and $\|\mu^j\| = \frac{1}{2}\|\nu^j\| + \frac{1}{2}|c^j|$. But also, $\mu^j \perp \mathcal{A}$ implies that $\int 1\, d\mu^j = 0$, so that $\nu^j(\mathbb{T}) = c^j$; so, $\|\nu^j\| \geq |c^j|$, and hence $1 = \|\mu^j\| \geq |c^j|$. This plus $1 = (c^0 + c^1)/2$ yields $c^0 = c^1 = 1$. Now we have $\|\nu^j\| = 1$ and $\nu^j(\mathbb{T}) = 1$, so that ν^j is a positive measure. But also $f(0) = \int_{\mathbb{T}} f d\nu^j$ for all $f \in \mathcal{A}$. Since ν^j is positive, we have $\nu^j = \lambda$ (see [10] §5.24), so $\mu^0 = \mu^1 = \mu$.