Wars of Attrition with Endogenously Determined Budget Constraints*

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Abstract

This paper models a war of attrition where participants first choose contest investment levels that act as a constraint for how long they can compete. To include a measure of the resource's transferability to other uses (e.g. other contests), expenditures are a convex combination of investment decisions and their 'bid' in the contest. It is shown in the symmetric equilibrium that participants use a mixed strategy for their resource investments and plan to exhaust those resources in the contest. Implications of an investment constraint on equilibrium strategies in a structured, tournament-style sequence of contests are also explored, where it is shown that increasing the number of contests in the tournament does not necessarily increase participants' investments in expectation. These modifications to the standard model allow for important insights into a variety of pre-calculated and budgeted all-pay contests.

Keywords: War of attrition, All-pay auction, Contest design, Budget constraints

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One of the most versatile models of dynamic competition is the war of attrition, in which $n + k$ participants vie for k indivisible prizes. As a type of all-pay auction, it is assumed that participants must make unrecoverable expenditures (i.e. place bids) at a constant rate until n participants choose to exit, at which point the prizes are awarded to the k remaining particiapnts. The purpose of this paper is to modify the standard model by considering wars of attrition that are non-spontaneous, and thus can be better understood by including a strategic decision over the allocation of resources toward the contest. This is accomplished through a multi-stage model in which the participants first privately choose their resource investments, then compete with their investments acting as a self-imposed budget constraint. As a way of adding realism to this modification, it is also assumed that all resources left over at the end of the contest can be transferred to other uses, such as future contests, at some fraction of their original value.

Since most economic conflicts are non-spontaneous, there are several topic areas in the war of attrition literature that can be represented by the variation proposed here. For instance, price competition, advertisement campaigns, patent races, political lobbying, and military tactics are almost always pre-calculated endeavors through time that include considerations over the resources needed to win. Moreover, in each of these examples, any residual resources left over at the end of the contest may transfer to other contests with varying degrees of success. As examples, a negative advertisement against the iPhone by Samsung does not help the firm win market share from non-Apple competitors in the future, but a pro-Samsung ad might; negative political campaigns against an opponent in a primary election do not transfer well to the general election, but positive ads about one's own platform might; and resources for desert-based military conflicts do not transfer well to naval conflicts, but they might transfer better to a tropic-based conflict. As a result, understanding the ability for a participant to transfer unused resources to a subsequent contest is an important aspect of this model.

With these modifications, this paper establishes the unique symmetric perfect-Bayesian equilibrium for a resource investment stage followed by a single contest. It is shown that participants choose resource investments using a mixed strategy and then intend to bid their entire investment in the contest. When unused resources are fully sunk (i.e. have no transferability) this contest becomes the familiar first-price all-pay auction, and when unused resources have full transferability this contest becomes the standard war of attrition. Furthermore, this paper explores equilibrium strategies when a single investment is used in a tournament-style series of contests where winning one contest allows the participant to compete in a subsequent contest with their remaining resources. In the special case provided, it is shown that a sequence of contests does not necessarily increase the rate at which participants invest in resources, relative to a single contest. This surprising result has practical significance to repetitious time-based competition when resources must be pre-committed. One such application is the Bertrand-Edgeworth model of firm competition in which firms choose capacity constraints and then compete over price.

The paper proceeds as follows. First, the relevant literature to wars of attritions and other all-pay contests is discussed. Second, the main model for a single contest is presented, followed by the theoretical results. Third, this model is then extended to explore investment and bidding strategies over a tournament-style sequence of contests, and a symmetric equilibrium is presented for a special case of this extension. Fourth, concluding remarks provide further research direction and application opportunities.

Literature Review

The variety of applications for the war of attrition is impressive. While suitable for modeling fighting among fiddler crabs (Morrell et al. 2005), it is equally suited for modeling macroeconomic stabilization (Alesina and Drazen 1991). Economists have gleamed theoretical insights from this model in funding public goods (Andreoni 2006), markets facing decreasing net present value (Fudenberg and Tirole 1986), labor strikes (Kennan and Wilson 1989), patent races (Leininger 1991), political lobbying (Dekel et al. 2008, 2009) and economic diplomacy (Ponsatí 2004).

Originally introduced by biologists to describe animal conflicts over such things as mating rights (Smith 1974), the war of attrition first relied on an apparent asymmetry between the the animals' abilities to compete, allowing for costless determination of the victor (the relatively strong participant). This was defined as the *handicap principle* in Zahavi (1975, 1977), which states that costly signals can help minimize animal conflicts. On the other hand, when there is no salient asymmetry between the participants the conflict can result in costly investments to determine the winner.

Economists began employing the war of attrition in the mid-1980s by applying it in several applications. To better understand firm exits and industry structure, Ghermawat and Nalebuff (1985), Whinston (1988), and Ghermawat and Nalebuff (1990) showed that declining future value markets put pressure on the largest firms to shrink to the size of the (formerly) smaller rivals. In an effort to explain patent races, Harris and Vickers (1985) modeled two participants competing for a prize that is awarded to the one who reached the 'finish line' first (i.e. to complete some task first). And Bliss and Nalebuff (1984) suggested that finding a contributor for a public good is very similar to a war of attrition because until the public good is funded the participants are not benefiting from it, which is costly.

Many of the notable theoretical advances in the standard model of the war of attrition have been within the context of auction theory. When there is a single prize awarded in the contest, the war of attrition with open-loop strategies (i.e. ones that are chosen at the beginning of the contest and cannot be updated through time) is equivalent to a second-price all-pay auction and thus allows for direct comparisons to other all-pay auctions.[1] In Bulow and Klemperer (1999), which considers closed-loop strategies, a generalized model captures two prominent versions of the contest: the 'natural oligopoly' version in which participants incur no further costs upon exit, and the 'standards' version in which participants continue to incur costs until the prizes are awarded. In the former version, the ability to avoid continuation costs causes 'instant sorting' where all but one of the participants needed to exit do so immediately at the beginning of the contest. Despite valuations being private information, it is demonstrated this result is efficient. In the latter version, while the equilibrium is also efficient, there is much less incentive to exit, and thus the instant sorting does not take place.

Additionally, Bulow and Klemperer (1999) uses the Revenue Equivalence Theorem to understand total expenditures when bidders are risk-neutral and have i.i.d. valuations, which must equal those of the winner-pay equivalent, the English auction, in expectation. However, Krishna and Morgan (1997) demonstrates that when relaxing the assumption of independent valuations there are conditions (relatively weak signal associations) where all-pay auctions outperform winner-pay auctions in terms of expected revenue. It is also shown in their paper that the war of attrition can outperform the first-price all-pay auction.

It is often difficult to study the war of attrition using naturally-occurring data, and there exist relatively few experimental studies of behavior in controlled environments. Moreover, the experimental literature to date lacks corroborative evidence over point predictions. One experimental paper, Oprea et al. (2013), directly tests the exit model of an unsustainable duopoly presented in Fudenberg and Tirole (1986). Their experimental results show firm exits are often efficient (76% of the time higher cost firms exit first) and very closely resemble the point predictions of the model. Another such experimental paper, Hörisch and Kirchkamp (2010),

[1] In the more general case with k prizes, the war of attrition with open-loop strategies becomes a $(k+1)$-price all-pay auction.

compares the well-studied first-price all-pay auction to the war of attrition. Their study finds subjects typically over-bid in the first-price all-pay auction (a common result), but under-bid in the war of attrition. This result, however, contradicts the more recent research conducted on alternative mechanism design in charity auction settings, in which the war of attrition tends to outperform its theoretical predictions, particularly in the presence of sunk-cost sensitive bidders (Carpenter et. al 2014; Foster 2017).[2]

As in this paper, others have also given consideration to relaxing some of the common assumptions associated with the war of attrition. Che and Gale (1996, 1998) consider the impact of exogenously imposed budget constraints on bidding behavior in a variety of mechanisms. Across the pair of papers, the first-price all-pay auction is found to outperform the first-price winner-pay auction in expected revenue, which in turn outperforms the second-price winner-pay auction. In Anderson et al. (1998), a model of a two-player war of attrition with a common and exogenously fixed budget constraint over bids yields a symmetric mixed-strategy equilibrium when there is a common value for the prize.

Another paper that seeks to endogenize likely choice variables in the war of attrition is Hörner and Sahuguet (2011), which suggests a single and continuous cost paid per time unit is an unreasonable assumption to make for many situations. In an alternative proposition, the authors evaluate how strategies would change when participants are allowed to vary the amount they choose to spend in a given time period. In their setup, these bids reveal information about valuations, and participants must match their opponents' total expenditures or else exit. Games of complete information are given the primary focus, along with the mixed strategies thereof - though there are asymmetric equilibria where a participant will give up with a high probability at the beginning. The authors find that expected delay is shorter and rent dissipation is smaller with these assumptions. This is partly due to the ability for participants to jump bid.

Like this paper, Burkett (2015) also considers the effect of endogenous budget constraints. However, it does so by assuming they are imposed by the auctioneer. By studying first-price and second-price winner-pay auctions, it is shown these mechanisms can often agree with the theoretical results without constraints. Another paper with similar modifications to this one is Amann and Leininger (1996), which makes a participant's total expenditure a convex combination of the contest's duration and their own investment strategy.[3] However, a few fundamental

[2]Baye et al. (2012) establishes the symmetric equilibrium for a broad class of two-player contests with rank-order spillovers, including the war of attrition. Their paper is particularly relevant to situations with behavioral considerations, such as these charity auctions and other contests involving inter-dependent payoffs among participants.

[3]Morath and Münster (2008) is another paper that makes costs a linear combination of both bids to show expected revenues are lower when bidders' valuations are private rather than common knowledge.

differences exist between the previous literature and this paper. First, resource investment decisions are endogenously determined by each participant at the beginning of the contest. Second, the payments made by *all participants* are a convex combination of the loser's bid and their own resource investment decision according to an exogenously determined repurpose value.

Setup

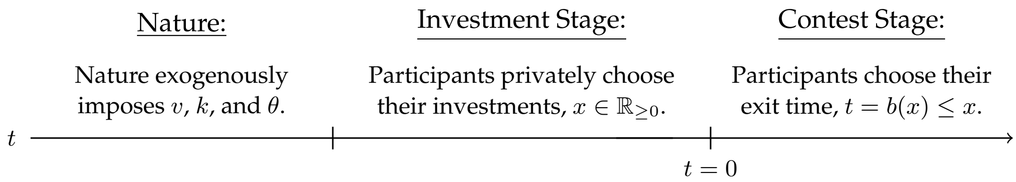
In this war of attrition there are $k + 1$ risk-neutral participants looking to claim a single prize with a pure common value, v. Participants individually choose a wait time t, which has a normalized cost of 1 per time unit, and the prize is awarded to the participant with the longest wait time. Those participants who do not claim the prize pay a cost according to their wait time, and the winning participant pays a cost according to the second longest wait time, which is the outcome that determines when the contest ends.

A key feature of this model is that, before the contest begins, participants have one opportunity to privately choose a resource investment $x \in \mathbb{R}_{\geq 0}$, which acts functionally as a self-imposed constraint on how long the participant may compete. Thus, participant i's choice on when to exit the contest must be less than or equal to their resource investment ($t_i \leq x_i$), and x_i is unobservable to all participants $j \neq i$. Therefore, let a participant's choice of exit time t (i.e. their 'bid') be defined as a strictly increasing function of their investment, $b|_x : x \to t$. While there exist asymmetric equilibria in this model, which will be discussed briefly in this paper, the analysis focuses primarily on symmetric equilibrium strategies.

The addition of an investment decision to this model captures the value loss trade-offs that are often an important component of many pre-calculated wars of attrition. In many applications for the war of attrition, the resources allocated to one contest may not transfer perfectly to other uses (e.g. another contest). As such, let each time unit of x that a participant holds at the conclusion of the contest have a repurpose value of $\theta \in [0, 1]$. Practically, a participant may hold a surplus of resource investments at the end of a contest for two reasons: one is that the participant may have chosen to exit without using their entire investment; another is that the participant may win the contest before their investment is fully exhausted. Both cases, though particularly the latter, represent important considerations for situations where unused resources may be applied to a future contest. Figure 1 summarizes the timing of decision stages of this contest.

By rank-ordering bids (i.e. wait times) in descending order $\{b(x_1), ..., b(x_k), b(x_{k+1})\}$, the

Figure 1: Timing of Nature and Decision Stages



utility function for participant i can be represented as

$$u(x_i) = \begin{cases} v - \theta b(x_2) - (1-\theta)x_i & \text{if} \quad b(x_i) = b(x_1) > b(x_2), \\ -\,\theta b(x_i) - (1-\theta)x_i & \text{if} \quad b(x_i) < b(x_1), \\ \frac{v}{m} - \theta b(x_i) - (1-\theta)x_i & \text{if} \quad b(x_i) = b(x_1) = \cdots = b(x_m) > b(x_{m+1}). \end{cases} \tag{1}$$

The utility to participant i is a function of their resource investment x_i from the *investment stage*, which is expressed in three cases. In the first case, participant i submitted the largest of $k+1$ bids in the *contest stage*. Under the assumption that bids are strictly increasing in x, this is equivalent to making the largest of $k+1$ resource investments such that $x_i = x_1$. Note that this participant does not pay their bid in this case, as they only compete as long the participant with the second largest bid ($b(x_2)$) competes. In the second case, participant i submits one of the k smallest bids and thus does not claim the prize and must pay their bid ($b(x_i)$). In the third case, participant i ties with $m-1$ other participants for the largest bid and splits the prize equally among them. In all three cases, the utility function reveals all participants immediately sacrifice $(1-\theta)x$ of their investment, which is the result of unused resources having a repurpose value of θ. As a result, this also means the resources that are bid during the contest are discounted to θ.

Lemma 1: *The symmetric equilibrium over contest resource investments, x, must be in mixed strategies.*

Proof: Assume there is a symmetric pure strategy equilibrium over contest resource investments, $\tilde{x}$. Then each participant receives a utility of $\tilde{u} = \frac{v}{k+1} - \theta b(\tilde{x}) - (1-\theta)\tilde{x}$. If $\tilde{u} \geq 0$, then it stands that a participant could unilaterally benefit from increasing their investment to $\tilde{x} + d\tilde{x}$, thus claiming v alone and receiving $v - \theta b(\tilde{x}) - (1-\theta)(\tilde{x} + d\tilde{x}) \geq \tilde{u}$, which holds for $d\tilde{x} \leq \frac{vk}{(1-\theta)(k+1)}$. If $\tilde{u} < 0$ for some $\tilde{x} > 0$, then a participant would be better off choosing $x = 0$ and receiving $u = 0$. Therefore, there is no $x = \tilde{x}$ that can be a pure strategy equilibrium. $\square$

Lemma 2: *The symmetric equilibrium over contest resource investments, x, contains no mass points, making ties impossible.*

Proof: Assume there is a symmetric equilibrium where all participants put positive mass on an investment of $\tilde{x}$. Then there is some probability the contest ends in a tie. However, from Lemma 1 we know participants have an incentive to unilaterally break ties, a contradiction. □

Given the symmetric equilibrium over investment decisions will involve mixed strategies, let x be drawn independently from a continuous and differentiable distribution F with density f. Analysis proceeds by establishing equilibrium behavior via backward induction, beginning with the *contest stage*.

The Contest Stage

According to Lemmas 1 and 2, participants will have privately chosen their resource investments according to a mixed strategy $F(x)$ before the *contest stage* begins. Due to the private information from the *investment stage*, analysis of the *contest stage* is restricted to symmetric perfect-Bayesian equilibria by modifying Bulow and Klemperer (1999).[4] The expected payout to a participant with an investment of x who bids as though they have an investment of r is

$$u(r \mid \theta, k, v) = \int_0^r [v - \theta b(z) - (1-\theta)x] k F(z)^{k-1} f(z) dz - (\theta b(r) + (1-\theta)x) \left[1 - F(r)^k\right] \quad s.t.\ x - b(r) \geq 0. \tag{2}$$

The first term in (2) is the expected payout from winning the contest when the k other participant choose to bid an amount less than $b(r)$, while the second term is the expected payout from losing by being outbid by at least one other participant. In addition, we see the repurpose value of unused resource investments, θ, causes participants' costs to be a convex combination of their investment and the bid function. Also, note that when $\theta = 1$, this war of attrition defoliates to the standard model, and when $\theta = 0$, this war of attrition defoliates to a first-price all-pay auction.

The constraint participants' face in the *contest stage* reflects the impossibility of bidding more

[4]For discussions of asymmetric equilibria, see Nalebuff and Riley (1985) regarding wars of attrition; see Baye et al. (1996) regarding first price all-pay auctions.

than one's investment, $b(r) \leq x$. The necessary conditions for the optimal bidding function, based off the Lagrangian function associated with (2), are

$$\begin{aligned} \mathcal{L}_r = vkf(r)F(r)^{k-1} - b'(r)\left(\theta\left[1 - F(r)^k\right] + \lambda\right) = 0 \\ \lambda\left[x - b(r)\right] = 0 \\ x - b(r) \geq 0 \\ \lambda \geq 0 \end{aligned} \tag{3}$$

where λ is the Kuhn-Tucker multiplier on the constraint $x - b(r) \geq 0$. Imposing $r = x$ according to the revelation principle (Myerson 1981) and examining the feasibility of the complementarity condition cases yields Lemma 3.

Lemma 3: *The symmetric equilibrium bidding function binds against a participant's investment x when the symmetric distribution of investments $F(x)$ is sufficiently large. Specifically,*

$$b(x \mid \theta, k, v) = \begin{cases} x & \text{if } F(x) \geq \left[1 - \exp(-\frac{\theta}{v}x)\right]^{\frac{1}{k}} \\ \frac{vk}{\theta}\int_0^x \frac{f(z)F(z)^{k-1}}{1-F(z)^k}dz & \text{otherwise.} \end{cases} \tag{4}$$

Proof: Under the assumption the bidding constraint does not bind against the participant's investment, then $x - b(x) > 0$ and $\lambda = 0$. Rearranging $\mathcal{L}_r$ in (3) for $b'(r)|_{r=x}$ yields $b'(x) = \frac{vk}{\theta}\frac{f(x)f(x)^{k-1}}{1-F(x)^k}$. Integrating this equation to recover the bid function, and imposing that $b(0) = 0$ yields $b(x) = b(0) + \frac{vk}{\theta}\int_0^x \frac{f(z)F(z)^{k-1}}{1-F(z)^k}dz = \frac{vk}{\theta}\int_0^x \frac{f(z)F(z)^{k-1}}{1-F(z)^k}dz$. By substituting this bid function into the bidding constraint gives the inequality $F(x) < \left[1 - \exp\left(-\frac{\theta}{v}x\right)\right]^{\frac{1}{k}}$ when $F(0) = 0$, which is true from Lemma 2. This result demonstrates the bidding function is not binding if the mixed strategy over resource investments is sufficiently small. Conversely, under the assumption the bidding constraint is binding, then $x - b(r)|_{r=x} = 0$ and $\lambda \geq 0$. Solving this set of inequalities begins by assuming the bidding constraint is binding, allowing $\lambda \geq 0$ and $b(r)|_{r=x} = x$. This implies $b'(x) = 1$ and $\mathcal{L}_r$ in (3) can be rearranged for $\lambda = vkF(x)^{k-1}f(x) - \theta(1 - F(x)^k)$. Substituting this expression for λ into the inequality $\lambda \geq 0$ from (3) demonstrates this result holds for $F(x) \geq \left[1 - \exp(-\frac{\theta}{v}x)\right]^{\frac{1}{k}}$, the complement condition to the first case.

To demonstrate the solution is a maximum, a bordered Hessian is constructed for both cases.

In the case where the bidding constraint is binding, we find

$$\begin{bmatrix} 0 & b'(x) \\ b'(x) & \mathcal{L}_{rr}|_{r=x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & kF(x)^{k-2}f(x)\left[(k-1)v + \theta F(x)\right] \end{bmatrix}$$

whose determinant is -1, indicating a maximum. A similar Hessian can be constructed for the case in which the bidding constraint is not binding to show its determinant is also unambiguously negative. However, its analytical form is too cumbersome to present, and thus is omitted. □

Lemma 3 demonstrates that bidding constraints are binding when the probability other participants are choosing smaller investments is sufficiently large. If the condition in (4) is not met, then participants will bid less than their investment according to the hazard function presented in (4), which is adjusted by $\frac{vk}{\theta}$.

While not the focus of this paper's analytical interest, it is also important to characterize equilibria with asymmetric strategies. Due to the asymmetry among participants' resource investments, the asymmetric equilibrium strategy in the contest stage would be for those participants with the k lowest investments to exit immediately and the participant with the largest resource investment to bid an amount greater than the second largest resource investment. Note this outcome is efficient in that there are no resources utilized in the contest stage, a result also found in the standard war of attrition.

In the next section we backward induct optimal investment decisions given this bidding function. We begin by first assuming that the bidding constraint is binding such that $b(x) = x$.

The Investment Stage

Given the optimal bidding behavior from the *contest stage*, it is possible to backward induct optimal investment behavior. Substituting the bid function $b(x) = x$ into (2) yields

$$u(x \mid \theta, k, v) = \int_0^x [v - \theta z - (1-\theta)x]kF(z)^{k-1}f(z)dz - x[1 - F(x)^k].$$

Deriving the first order condition yields the differential equation

$$0 = vkF(x)^{k-1}f(x) + \theta F(x)^k - 1. \tag{5}$$

Solving the differential equation in (5) and imposing $F(0) = 0$ gives Proposition 1.

Proposition 1: *There is a unique symmetric perfect-Bayesian equilibrium in which*

$$F(x \mid \theta, k, v) = \left(\frac{1}{\theta}\left[1 - \exp\left(-\frac{\theta}{v}x\right)\right]\right)^{\frac{1}{k}}, \text{ and} \qquad (6)$$
$$b(x) = x.$$

Proof: The proof begins by assuming there is a symmetric equilibrium in which the bid constraint is not binding. Substituting the optimal bidding strategy for this case of $\frac{vk}{\theta}\int_0^x \frac{f(z)F(z)^{k-1}}{1-F(z)^k}dz$ into (2) reduces the expression for expected utility to $-(1-\theta)$, thus there is no solution that can satisfy Lemma 1.

To demonstrate the mixed strategy equilibrium over investments is unique, begin by assuming there are two distinct mixed strategy equilibria over investments with cumulative densities of $F(x)$ and $G(x)$. Without loss of generality allow $G(x) < F(x)$ over some interval, and label x^* as the critical value in that interval for where the difference $F(x) - G(x)$ is maximized. At x^* we must find $f(x^*) = g(x^*)$ and $g'(x^*) \geq f'(x^*)$. By rearranging (5) for $f(x)$ and differentiating with respect to x we find $f'(x) = -\frac{\left(k-1+\theta F(x)^k\right)f(x)}{vkF(x)^k}$. Substituting this expression into the condition $g'(x^*) \geq f'(x^*)$, imposing $f(x^*) = g(x^*)$, and simplifying yields $\frac{k-1}{G(x^*)} \leq \frac{k-1}{F(x^*)}$, which cannot be true for $k > 1$. For $k = 1$, again assume two solutions $F(x)$ and $G(x)$ exist. Then, given (5) we know $f(x) - g(x) = \frac{\theta}{v}\left[G(x) - F(x)\right] \Rightarrow G(x) - F(x) = Ce^{\frac{\theta}{v}x}$. But $G(0) - F(0) = 0 - 0 = 0 \Rightarrow C = 0$. Thus $G(x) - F(x) = 0 \Rightarrow F(x) = G(x)$. □

The equations in (6) show the expected investment from the *investment stage* is increasing in θ. In the limit as θ approaches zero, and all resource investments are fully sunk, the mixing distribution becomes the familiar first-price all-pay equilibrium over bids for $k+1$ participants, $F(x) = \left(\frac{x}{v}\right)^{\frac{1}{k}}$, as shown in Baye et al. (1996). When $\theta = 1$ and unused investments may be fully repurposed, $F(x)$ becomes the standard war of attrition bidding equilibrium for $k+1$ participants, $F(x) = \left(1 - \exp(-\frac{x}{v})\right)^{\frac{1}{k}}$.[5]

[5] One may wish to consider a variant where all past play is common knowledge, including investment levels, in which case the equilibrium outcomes over investments and bidding create a relatively simple contest. Namely, any heterogeneity in investments would eliminate the symmetric equilibrium in the bidding stage. Participants would use pure strategies in which those with all but the largest investment remaining exit immediately, while the participant with the largest endowment bids enough to outlast any other participant (yet spend nothing in the contest). However, due to the losses in resource value equal to $1 - \theta$ per unit after the contest, participants still face a cost to competing. Using this to backward induct equilibrium investments strategies, it can be shown that a mixed strategy emerges in

Finally, it is straight-forward to show that the expected total expenditure by all participants, E, is equal to v, a common feature of contests for prizes with pure common values. This can be expressed as

$$
\begin{aligned}
E &= \underbrace{(k+1)\int_0^{\bar{x}} x f(x) dx}_{\text{Total Expected Investment}} \; - \; \underbrace{\theta(k+1)\int_0^{\bar{x}} x\left(f_{(1)}(x) - f_{(2)}(x)\right) dx}_{\text{Expected Investment Returned to Winner}} \\
&= (k+1)\int_0^{\bar{x}} x\left[1 + \theta F(x)^{k-1}\left(k - (k+1)F(x)\right)\right] f(x) dx \\
&= v.
\end{aligned}
\tag{7}
$$

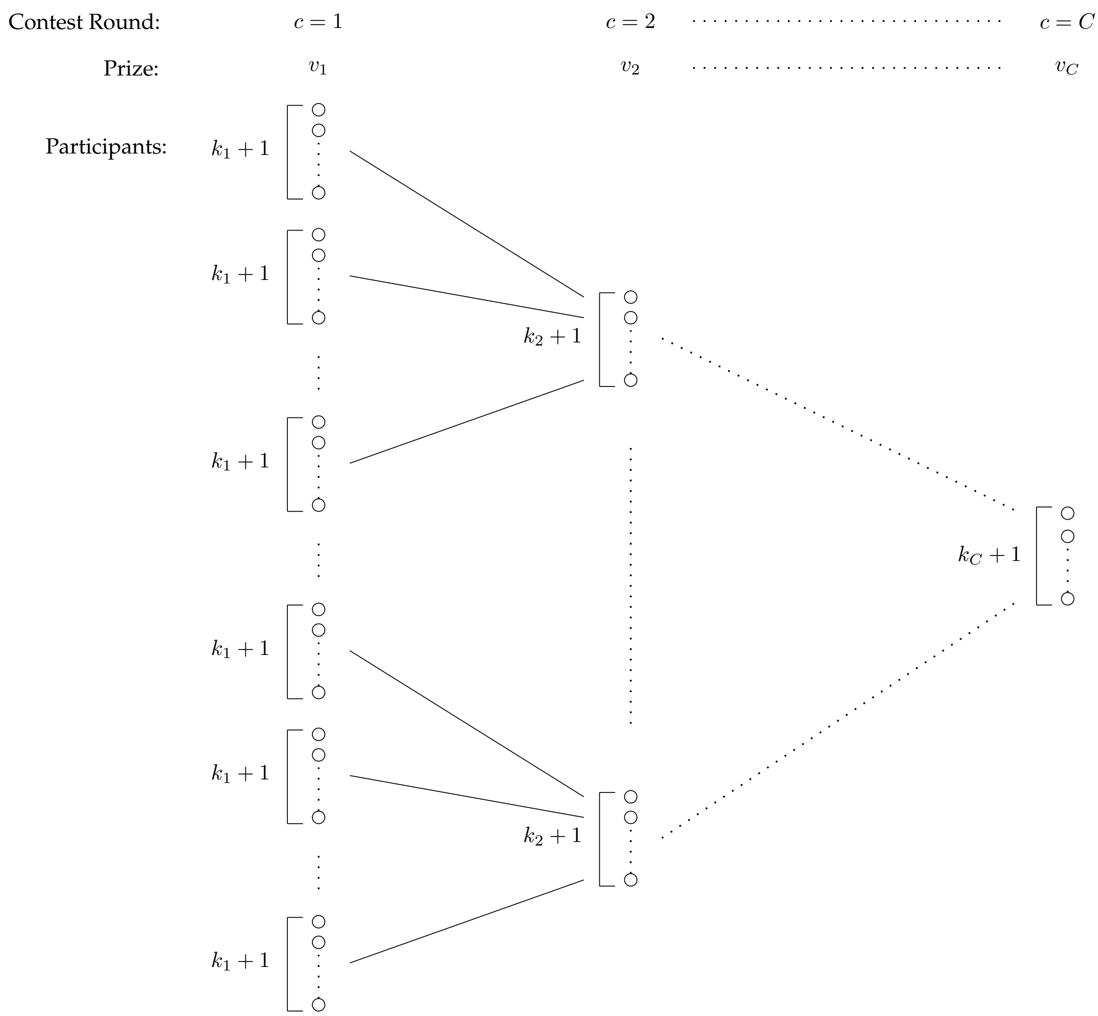
Here, $f_{(1)}(x)$ and $f_{(2)}(x)$ define the probability density function of the largest and second largest of $k+1$ independent draws from F, and where $\bar{x}$ gives us $F(\bar{x}) = 1 \Rightarrow \bar{x} = -\frac{v}{\theta}\ln[1-\theta]$. We find that $\lim_{\theta\to 0+} \bar{x} = v$ and when $\lim_{\theta\to 1-} \bar{x}$ tends to infinity, which corresponds to the results of the first-price all-pay auction and standard war of attrition, respectively. Note, θ is the rate at which unused resources may be transferred to other purposes. Thus, the winner's total expenditure for the contest must be adjusted by θ accordingly. The first term in the first line of (7) represents the expected total investment from all participants, while the second term represents the expected investment that will be returned to the winner, who is the only participant with unused resources.

One Investment, A Tournament of Sequential Contests

A natural extension of this model is to consider equilibrium resource investment strategies when a participant could potentially enter a sequence of contests which share a single budget constraint. To explore this, we assume there is a structured sequence of C contests that can be organized in a tournament, as described in Figure 2. In a given contest $c \in \{1, 2, ..., C\}$ there is a prize of v_c awarded to the participant who, among all $k_c + 1$ participants, had the largest bid. For contests $c < C$, this participant is also awarded with the ability to compete in contest $c+1$. Since a participant must win the $c^{\text{th}} < C$ contest to participate in the $(c+1)^{\text{th}}$ contest, all bids in subsequent contests are utilizing the residual resources from previous contests, based on their original investment of x.

equilibrium similar to that of a first-price all-pay auction whose distribution adjusts with θ. For instance, when $\theta = 0$, the theoretical results for the symmetric equilibrium are identical in this variant and the one proposed in the paper. As θ increases, participants increase their amount invested since the cost of investments has dropped. When $\theta = 1$, there is no longer a unique pure strategy equilibrium over bids, as all participants find it optimal to invest in an infinite amount of resources and must then coordinate over who will never exit.

Figure 2: Sequential Contest Tournament Design



If participants begin the first contest with an investment drawn from F, then for each of the subsequent contests a new distribution over winners' residual investment resources must be established. To do this, we calculate the cumulative density function for the winning participant in contest c retaining x_{c+1} investment resources or less, where $x_{c+1} = \theta^c x - \sum_{i=1}^{c} \theta^i b(x_{k_i,i})$ and $b(x_{k_i,i})$ is the bid from participant with the k_i^{th} highest investment in the i^{th} contest. Thus, there is a distribution over the expected residual investments in contest $c > 1$ defined by $F_c(x)$. Next, we describe the determination of $F_c(x)$.

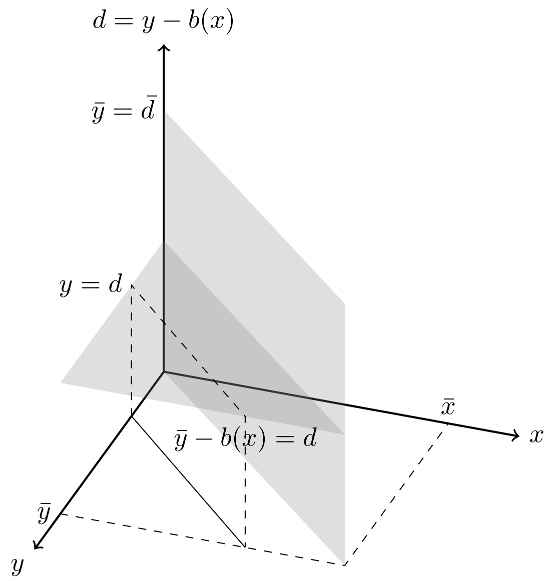
Determination of the c^{th} Contest's Resource Distribution

Using the joint distribution of the first and second order statistics for contest $c+1$ we can calculate the distribution of differences, d, between the largest investment and the bid from the second largest investment in the c^{th} contest for any underlying distribution of investments, $F_c(x)$. For any two largest investments $x < y$ from $k_c + 1$ draws of F_c, this difference is $d = y - b(x)$. The joint density over these investments is given by

$$f_{c+1}(x, y) = k_c(k_c + 1)f_c(x)f_c(y)F_c(x)^{k_c - 1}.$$

Figure 3 symbolically describes how this distribution is determined by exploring level curves of $d = y - b(x)$.

Figure 3: Level Curves of Differences in the Joint Distribution of Order Statistics



Because $b(x) < y$, analysis in the figure is restricted to the area left of the vertical partition. The range of possible values for d starts at zero, where investments x and y are equal, and increases to $\bar{d}$, where the largest investment is at its maximum value ($\bar{y}$) and the second largest investment is at its minimum value ($x = 0$, thus $b(0) = 0$). For some difference in the largest investment and the bid from the second largest investment $d = y - b(x)$, the joint probability distribution can be re-expressed in terms of y as

$$f_{c+1}(d,y) = k_c(k_c+1)f_c(b^{-1}(y-d))f_c(y)F_c(b^{-1}(y-d))^{k_c-1}$$

where b^{-1} is the inverse of the bid function. Finally, to determine the probability with which some difference d or less occurs we integrate over all combinations of $b(x)$ and y where $d = y - b(x)$. This is equivalent to integrating over the solid line in Figure 3 from $y = d$ to $\bar{y}$. Thus, the new distribution is

$$F_{c+1}(d) = k_c(k_c+1)\int\int_d^{\bar{y}} f_c(b^{-1}(y-d))f_c(y)F_c(b^{-1}(y-d))^{k_c-1}dydd \tag{8}$$

which includes a constant that ensures $F(0) = 0$ since ties are impossible.

Establishing the distribution of residual investments allows for the consideration of bidding behavior in sequential contests according to (2). In the next subsection, we apply this analysis to a special case to show how it affects investment decisions in equilibrium.

Special Case: $C = 2$, $k_c = 1$, $v_c = v$, $\theta = 1$

In this special case there are $C = 2$ contests, both with $k_c+1 = 2$ participants and a pure common value prize of v. At the end of each contest, unused investments have a full redemption value of $\theta = 1$. For tractability, analysis focuses on open-loop strategies in which participants choose their exit times for all of the contests they enter before the tournament begins.

Relying on backward induction, analysis begins with the last contest. In this contest, (2) once again derives Proposition 1 to determine the equilibrium bidding behavior in the second contest. Given this bidding strategy, we now consider bidding strategies in the first contest. To do so, we modify (2) to include a continuation payoff of $u_2(r - z \mid 1, 1, v)$ - the expected return of the second contest according to $F_2(x)$. This gives us

$$\begin{gathered} u_1(r \mid 1,1,v) = \int_0^r [v - b(z) + u_2(r - b(z) \mid 1,1,v)]f(z)dz - b(r)[1-F(r)] \\ s.t.\ x - b(r) \geq 0. \end{gathered} \tag{9}$$

By assuming $F(x)$, $F_2(x) \geq 1 - \exp(-\frac{1}{v}x)$ for all x, it is ensured that the bidding constraints binding and giving $b(x) = x$ in both contests. Using (8) to define $F_2(x)$ where $b^{-1}(x) = y -$

d under our temporary assumption, it can be shown there is a symmetric perfect-Bayesian equilibrium in which

$$F(x \mid 1,1) = 1 - \exp\left(-\frac{1}{v}x\right), \text{ and}$$
$$F_2(x \mid 1,1) = 1 - \exp\left(-\frac{1}{v}x\right), \text{ and}$$
$$b(x) = x \text{ for both contests.}$$

Interestingly, the results of this special case indicate that the initial investment distribution $F(x)$ leads to an identical residual investment distribution for the second contest. Moreover, this distribution is identical to the distribution we would have anticipated for the case where $C = 1$ - just a single contest - which we know from previous analysis incentivizes binding bidding constraints. Finally, since the original investment distribution is self-replicating in the residual investment distributions, it is clear from this special case that this result will hold for any number of contests C with this structure. As a result, one might expect empirical tests of this model to show no difference in the investments between participants in one contest or a sequence of contests of any size.

Conclusion

In pre-calculated wars of attrition an important strategic decision for participants is to decide how much they are willing to invest in the contest. Importantly, this investment decision can also constrain a participant's ability to compete. In this paper, symmetric equilibrium strategies give insight into how resource investments are chosen when they enter a single contest or, for a special case, a series of contests. The results have important implications for understanding many areas of dynamic competition in economics, including but limited to industrial organization, military campaigns, and public goods.

This paper demonstrates for the modified war of attrition presented here there is a unique symmetric perfect-Bayesian equilibrium where participants rely on a mixed strategy for their resource investments and use the pure strategy of bidding all of that investment in the contest. This result suggests that the preceding stages to a pre-calculated war of attrition may be the most important ones, as in equilibrium a majority of the expenditure often comes directly from investments, and the contest itself is, in some ways, a formality of the strategies employed in its preparation.

Finally, we see a natural extension of this work to be an examination of how participants might choose a repurpose value (θ) when there exist salient trade-offs in resource effectiveness across present and future contests. One such method may be to pair this model with the endogenous effort levels model in Hörner and Sahuguet (2011).

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