Day By Day Notes for MATH 206

Fall 2011

Day 1

Activity: Go over syllabus. Take roll. Functions activities. We will work in groups, and compare solutions.

Goals: Review course objectives: collect data, summarize information, and make inferences.

I have divided this course into three "units". Unit 1 (Days 1 through 8) is about basic functions. Unit 2 (Days 9 through 17) is about derivatives and their uses. Unit 3 (Days 18 through 28) is about integration and multi-variable functions.

I believe to be successful in this course, you must **READ** the text carefully, working many practice problems. Our activities in class will sometimes be unrelated to the homework you practice and/or turn in for the homework portion of your grade; instead they will be for **understanding** of the underlying principles. For example, on Day 9 you will draw sample graphs and derivatives, then try to reconstruct the original graph. This is something you would never do in practice, but which I think will demonstrate several lessons for us. In these notes, I will try to point out to you when we're doing something to gain **understanding**, and when we're doing something to gain **skills**.

Each semester, I encourage students to see me for help outside of class. I suspect some of you are embarrassed to seek help, or you may feel I will think less of you for not "getting it" on your own. Personally, I think that if you are struggling and cannot make sense of what we are doing, and **don't** seek help, you are cheating yourself out of your own education. I am here to help you learn mathematics. Please ask questions when you have them; I don't believe there are "stupid questions". Often other students have the same questions but are also too shy to ask them in class. If you are still too shy to ask questions in class, come to my office hours or make an appointment.

I believe you get out of something what you put into it. Very rarely will someone fail a class by attending every day, doing all the assignments, and working many practice problems; typically people fail by not applying themselves enough - either through missing classes, or by not allocating enough time for the material. Obviously I cannot tell you how much time to spend each week on this class; you must all find the right balance for you and your life's priorities. One last piece of advice: don't procrastinate. I believe mathematics is learned best by daily exposure. Cramming for exams may get you a passing grade, but you are only cheating yourself out of understanding and learning.

Today we will begin by discussing functions. Quite simply, a function is a rule. From an input value, a function gives the output value. The set of possible inputs is called the **domain**, and the set of output values is called the **range**. The input value is sometimes called the **independent** value, and the output value the **dependent** value. One of the chief goals of mathematics is to model real world phenomena with functions. Therefore it is important for us to be familiar with their uses and roles.

Throughout the course, we will try to look at functions from four different viewpoints. Data will be presented to us in tables, graphs, and equations, as well as orally. It will be up to us to determine the most appropriate method of describing the function. A common misconception that I hope to dispel is that equations are synonymous with functions. Equations are only one method of describing functions. Our text makes an honest effort to display functions for us in all four forms.

Today I would like to explore functions graphically, verbally, and algebraically. We will begin with a discussion of a hypothetical flight between two cities. Then I will have you work in groups on three projects.

In these notes, I will put the daily **task** in gray background.

I have three activities for us to become more familiar with functions.

Activity 1: Graphical and Verbal Description.

The value of a car goes down as the car gets older, so we can think of the value of a car, V, in thousands of dollars, as a function of the age of a car, a, in years. We have $V = f(a)$.

1) Draw a possible graph of V against a. You don't need scales on the axes, but label each axis as V or a.

2) What does the statement $f(5) = 6$ tell you about the value of the car? Be sure to use units for 5 and for 6. Label this as a point on your graph, and mark the 5 and the 6 on the appropriate axes.

3) Put a vertical intercept of 15 on your graph of the function. Explain the meaning of this vertical intercept in terms of the value of the car.

4) Put a horizontal intercept of 10 on your graph of the function. Explain the meaning of this horizontal intercept in terms of the value of the car.

Activity 2: Algebraic and Verbal Description.

From a 24-inch length of string, form two geometric shapes, a circle and a square. Your goal is to create the smallest total area enclosed by both shapes (add the area enclosed by the square to the area enclosed by the circle, and make that total area small. Note: you will need to know the formulas for areas of circles and squares.) To begin this activity, I suggest trying some specific values. For example, what if the string is not cut at all? How much area will just a circle contain (the square has zero area)? How much area will just a square contain (the circle has zero area)? What if the string is cut in half instead? Then how much will the square contain, and how much will the circle contain? Then try a different breakdown, like ¼ of the string for the square and ¾ for the circle. My belief is that if you can figure out the areas for specific numbers, you can figure it out for arbitrary values, like x and $24 - x$.

Activity 3: Graphical and Algebraic Description.

With our calculators, we have the tools available to explore limits. Specifically, we can hone our intuition about this important topic in calculus.

1) Calculate $\left(1+\frac{1}{n}\right)$ for n = 1, 10, 100, 1000, etc. What seems to be happening to the terms? Can you explain it intuitively?

2) Calculate $\left(1+\frac{1}{n}\right)^n$ for n = 1, 10, 100, 1000, etc. What seems to be happening to the terms? Can you explain it intuitively? This limit we see here is a very important limit in calculus and mathematics. We will encounter it and study it in more detail later.

3) Consider this series: 1, 1/2, 1/3, 1/4, etc. Add successive terms to get a new series of partial sums. That is, find 1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4, etc. What seems to be happening to these sums? Can you explain it intuitively?

4) Now try this series and repeat what you did in problem 3). 1/2, 1/4, 1/8, 1/16, 1/32, etc. That is, find 1/2, 1/2+1/3, 1/2+1/3+1/4, etc. (These are successively smaller powers of two.) What seems to be happening to these sums? Can you explain it intuitively?

5) Using $f(x)=\dfrac{x+1}{x^2-x-2}$, find the limits as you approach $x = 2$ from the right and left. (Approaching from the right means using values just above 2 and approaching from the left means using values just below 2.) Also find the value right at $x = 2$.

6) Repeat 5) using $g(x)=\dfrac{x^2-x-2}{x+1}$.

In these notes, I will put sections of computer commands in boxes, like this one. I'm actually hoping that you already are quite familiar with this machine, having already taken MBA I. In these notes, I refer to the calculator as the TI-83. The same commands apply to the TI-84.

Y = is found on the top row of buttons, on the left. You enter equations into whichever **Y**-variable you want to use. Be careful to enter what you want, that is, pay attention to parentheses, typos, etc! Each **Y**-variable whose **=** sign is highlighted will be graphed when the **GRAPH** button is pressed. In addition, if any plots at the top of the display are highlighted, those too will be plotted, whether you intended them to be or not!

GRAPH is found on the top row of buttons, on the right. This button toggles between the data / numerical entry screens and the graphing window. To leave the graphing window, press any key, or press **QUIT**, (found by pressing **2nd MODE**).

WINDOW is found on the top row of buttons, second from the left. This opens the windows setting screen, which tells you the dimensions and characteristics of the current graphing window. We will mostly change only 4 items: **Xmin**, **Xmax**, **Ymin**, and **Ymax**. If you like, you may tinker with the other settings.

TRACE is on the top row of buttons, second from the right. This key puts a cursor on the graphing window on one of your *y*-variables / functions. You may push right and left arrow to move sideways on the selected curve, or up and down arrow to select other curves (if you have entered more than one *y*-variable.) Be careful: **TRACE** is dependent on the current window settings. If you need precise values, after pressing **TRACE**, type the *x*-value you need evaluated. **TRACE** will calculate the functional value exactly.

ZoomFit (Zoom 0) Many times, you do not know which is the best viewing window. If you first specify the horizontal endpoints in the **WINDOW** settings screen (**Xmax** and **Xmin**), then you can press **ZoomFit** (under **ZOOM** menu, item **0**) to have the calculator find the appropriate **Ymin** and **Ymax** values. This function is quite handy; I use it a lot myself.

ZStandard (Zoom 6) If you are in love with the numbers between –10 and +10, you should use **ZStandard** in the **ZOOM** menu. Otherwise, you may find this key useless!

Goals: (In these notes, I will summarize each day's activity with a statement of goals for the day.)

Introduce the course, and the idea of a function. Appreciate the dynamics of collaboration. Understand the different problem solving strategies. Explore some basic limits.

Skills: (In these notes, each day I will identify **skills** I believe you should have after working the day's activity, reading the appropriate sections of the text, and practicing exercises in the text.

- **Use the "Guess and Check" method of problem solving.** This technique is the essence of the scientific method. There is nothing bad about guessing in order to learn. The better guessers, of course, tend to get quicker results, but if you have appropriate tools to evaluate your guesses, then even poor guesses can be refined adequately. By the way, your calculator in this class will essentially use this guess and check method to solve equations. It's just that your calculator works a bit faster than you can. Another related idea is using test numbers to start a process. That is, perhaps making up a sample situation will help you see what is going on. I encourage you to use this approach often; it is the most basic lesson my advisor taught me in graduate school. He used to say, "Start with a simple example." That often meant assuming some specific values for some variables, and working from there to understand the problem.

- **Physical modeling.** Many times being stuck on a real world problem can be alleviated by modeling the situation with physical items or by other simulations. Of course many situations are infeasible; you can't fly airplanes to simulate scheduling airline routes, but you can use appropriate diagrams or tokens representing airplanes. Sometimes actually physically representing something will get you over that mental block.

- **Collaboration.** One of the biggest problems I see semester after semester with math students is their reluctance to talk about their math frustration. **Talk** about things with each other! If you are too timid to talk to me, (or if you have other reasons for not wanting to chat with me) at least talk to your peers. Sometimes simply saying something out loud will open up doors you might not have otherwise opened, or an offhand remark may inspire someone else's imagination. Of course this doesn't mean that one person in a group of problem solvers should do all the work; but even if only one group member 'gets' a solution, the sharing is beneficial to all concerned. The sharer gets to **really** learn the concept as he/she is required to **explain** it; the others get to see a solution they missed. Ideally, everyone should be able to explain a group solution; until you can explain the solution, you haven't quite understood the method.

- **Evaluate limits numerically and graphically.** By using numbers closer and closer to the value in question, whether it is finite or infinite, your calculator or computer can help you to evaluate limits. There is a caution, however: you must still use your analytic skills to avoid being fooled. You may have observed this in Exercise 2 of Activity 3. Some limits are easy to evaluate (simply plug in and evaluate) while others are more complicated (the partial sums we saw in Exercises 3 and 4 of Activity 3 are often quite difficult to evaluate or to even decide if they converge.) One of the "big ideas" in calculus is differentiation, and we need to be comfortable with limits to understand derivatives. Another of the "big ideas" in calculus is integration, and we need to understand limits such as partial sums to understand integrals.

- **Recognize the harmonic series.** Even though terms in a series may be getting smaller and smaller, their partial sums may not converge to a finite number. The sequence in Exercise 3 of Activity 3 above is called the **harmonic series** and demonstrates this seeming paradox. Many partial sums will converge, though, as you saw in Exercise 4 of Activity 3, which is an example of a **geometric series**.

- **Understand the definition of the number *e*** $(e \approx 2.718)$**.** Exercise 2 of Activity 3 is the definition of the number *e*, which we will use again and again in calculus. Remember, though, *e* is just a number, nothing more. Don't be afraid of it!

Reading: Sections 1.1 to 1.3. Bring your calculator to class every day. It will be an invaluable tool.

Day 2

Activity: Using the Olympic data, fit a regression line to predict the 2012 race results. Interpreting Rates of Change.

Unit 1 is about building up a library of functions. To be an effective mathematical modeler, we must have a working knowledge of basic functions. These include linear functions, exponential functions, polynomials, and combinations of these. The simplest and most used (it is the basis for the derivative we will master in Chapter 2) is the linear function. You should already know a lot about linear functions. Just to make sure we all have the same background, today we will explore linear functions in detail.

To begin, I will list the useful forms for linear equations.

1) Slope/Intercept form: $y = mx + b$. In this form, m is the slope and b is the y-intercept.

2) Point/Slope form: $y - y_1 = m(x - x_1)$. In this form, m is the slope, and (x_1, y_1) is an ordered pair on the line.

3) Two Point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. In this form, ($x_1$, y_1) and (x_2, y_2) are two sets of ordered pairs on the line.

I will use do the Celsius/Fahrenheit conversion in class to demonstrate using these forms.

The chief technique for summarizing a linear relationship given data points on a scatter plot is **Least Squares Linear Regression**. This technique is also known as **Least Squares Regression**, **Best Fit Regression**, **Linear Regression**, etc. The important point is that we are going to describe the relationship with a straight line, so if the scatter plot shows some other shape, this technique will be inappropriate. Your tasks are to 1) come up with a line, either by hand or with technology, that "goes through" the data in some appropriate way, 2) to be able to use this model to describe the relationship verbally, and 3) to predict numerically y-values for particular x-values of interest.

While we won't cover regression in complete detail, the text does go over it on pages 80 to 82, so it might be worthwhile for you to look at those sections of the text in addition to the regular reading.

Activity 1: Graphical description: Using linear regression.

Begin by making a scatter plot of the race times. (Use STAT PLOT. See calculator commands below.) If you want a rough guess for the slope of the best fitting line through the data, you can connect two points spaced far apart (I will show you the details in class.)

Next, use the TI-83's regression features to calculate the **best fit**. The command is STAT CALC LinReg(ax+b), assuming the two lists are in L1 and L2. (L1 will be the horizontal variable, years in this case.) (For regression it is **vital** that you get the order of the variables correct; the idea here is that you are predicting the vertical variable from the known horizontal variable.)

Interpret what your two regression coefficients mean. Make sure you have units attached to your numbers to help with the meanings.

Have the calculator type this equation into your Y = menu (using VARS Statistics EQ RegEQ or use the commands below), and TRACE on the line to predict the future results. Specifically, see what your model says the 2008 time should have been. Compare to the actual time to see how predictive our model is.

Activity 2: Algebraic description: Using verbal description.

Taxicab rates. Given the following information on the side of a cab, develop an equation that will let you calculate the fare for any distance x. So, if someone tells you the distance they want to travel, your formula will tell them the fare.

Info on the side of a cab: $2.50 FOR THE FIRST 1/9 MILE, PLUS 25 CENTS FOR EACH ADDITIONAL 1/9 MILE OR FRACTION OF A MILE.

Activity 3: Tabular description: Using average rates of change.

A half-marathon runner records the following times during a race. Find the average speed of the runner from the start through mile 6. From the end of six miles through the end of the race. For the whole race. For just the last 3.1 miles. Report your answers on the board.

Mile	Time on Clock	Mile	Time on Clock
1	7:36	7	55:07
2	15:29	8	1:02:50
3	23:25	9	1:10:29
4	31:23	10	1:18:20
5	39:20	11	1:26:08
6	47:18	13.1	1:42:58

Men's and Women's 100-meter dash winning Olympic times:

1896	Thomas Burke, United States	12 sec		
1900	Francis W. Jarvis, United States	11.0 sec		
1904	Archie Hahn, United States	11.0 sec		
1908	Reginald Walker, South Africa	10.8 sec		
1912	Ralph Craig, United States	10.8 sec		
1920	Charles Paddock, United States	10.8 sec		
1924	Harold Abrahams, Great Britain	10.6 sec		
1928	Percy Williams, Canada	10.8 sec	Elizabeth Robinson, United States	12.2 sec
1932	Eddie Tolan, United States	10.3 sec	Stella Walsh, Poland (a)	11.9 sec
1936	Jesse Owens, United States	10.3 sec	Helen Stephens, United States	11.5 sec
1948	Harrison Dillard, United States	10.3 sec	Francina Blankers-Koen, Netherlands	11.9 sec
1952	Lindy Remigino, United States	10.4 sec	Marjorie, Jackson, Australia	11.5 sec
1956	Bobby Morrow, United States	10.5 sec	Betty Cuthbert, Australia	11.5 sec
1960	Armin Hary, Germany	10.2 sec	Wilma Rudolph, United States	11.0 sec
1964	Bob Hayes, United States	10.0 sec	Wyomia Tyus, United States	11.4 sec
1968	Jim Hines, United States	9.95 sec	Wyomia Tyus, United States	11.0 sec
1972	Valery Borzov, USSR	10.14 sec	Renate Stecher, E. Germany	11.07 sec
1976	Hasely Crawford, Trinidad	10.06 sec	Annegret Richter, W. Germany	11.08 sec
1980	Allen Wells, Britain	10.25 sec	Lyudmila Kondratyeva, USSR	11.6 sec
1984	Carl Lewis, United States	9.99 sec	Evelyn Ashford, United States	10.97 sec
1988	Carl Lewis, United States	9.92 sec	Florence Griffith-Joyner, United States	10.54 sec
1992	Linford Christie, Great Britain	9.96 sec	Gail Devers, United States	10.82 sec
1996	Donovan Bailey, Canada	9.84 sec	Gail Devers, United States	10.94 sec
2000	Maurice Greene, United States	9.87 sec	Marion Jones, United States	10.75 sec
2004	Justin Gatlin, United States	9.85 sec	Yuliya Nesterenko, Belarus	10.93 sec
2008	Usain Bolt, Jamaica	9.69 sec	Shelly-ann Fraser, Jamaica	10.78 sec
2012	?		?	

(a) A 1980 autopsy determined that Walsh was a man.

Men's and Women's 200-meter dash winning Olympic times:

1900	John Tewksbury, United States	22.2 sec		
1904	Archie Hahn, United States	21.6 sec		

1908	Robert Kerr, Canada	22.6 sec		
1912	Ralph Craig, United States	21.7 sec		
1920	Allan Woodring, United States	22 sec		
1924	Jackson Scholz, United States	21.6 sec		
1928	Percy Williams, Canada	21.8 sec		
1932	Eddie Tolan, United States	21.2 sec		
1936	Jesse Owens, United States	20.7 sec		
1948	Mel Patton, United States	21.1 sec	Fanny Blankers-Koen, Netherlands	24.4 sec
1952	Andrew Stanfield, United States	20.7 sec	Marjorie, Jackson, Australia	23.7 sec
1956	Bobby Morrow, United States	20.6 sec	Betty Cuthbert, Australia	23.4 sec
1960	Livio Berruti, Italy	20.5 sec	Wilma Rudolph, United States	24.0 sec
1964	Henry Carr, United States	20.3 sec	Edith McGuire, United States	23.0 sec
1968	Tommy Smith, United States	19.83 sec	Irena Szewinska, Poland	22.5 sec
1972	Valeri Borzov, USSR	20.00 sec	Renate Stecher, E. Germany	22.40 sec
1976	Don Quarrie, Jamaica	20.23 sec	Barbel Eckert, E. Germany	22.37 sec
1980	Pietro Mennea, Italy	20.19 sec	Barbara Wockel, E. Germany	22.03 sec
1984	Carl Lewis, United States	19.80 sec	Valerie Brisco-Hooks, United States	21.81 sec
1988	Joe DeLoach, United States	19.75 sec	Florence Griffith-Joyner, United States	21.34 sec
1992	Mike Marsh, United States	20.01 sec	Gwen Torrance, United States	21.81 sec
1996	Michael Johnson, United States	19.32 sec	Marie-Jose Perec, France	22.12 sec
2000	Konstantinos Kenteris, Greece	20.09 sec	Marion Jones, United States	21.84 sec
2004	Shawn Crawford, United States	19.79 sec	Veronica Campbell, Jamaica	22.05 sec
2008	Usain Bolt, Jamaica	19.30 sec	Veronica Campbell-Brown, Jamaica	21.74 sec
2012	?		?	

STAT EDIT To enter a list of numbers into your calculator, instead of an equation, use the **STAT** menu. **EDIT** is the display that allows you to enter lists of numbers. You may have up to 3 lists displayed in the **EDIT** window. It is convenient to use the built-in lists **L1** to **L6**, but actually any named lists may be used. You may want to refer to the calculator manual if you are interested in naming and saving your lists. (It might save you having to constantly re-enter data.)

STAT PLOT 1 On Use this screen to designate the plot settings. You can have up to three plots on the screen at once.

ZOOMStat (Zoom 9) To view a scatter plot of two lists, **ZoomStat** will create an appropriate viewing window. To use the TI-83 to effectively view scatter plots, I recommend turning off or de-selecting all **Y**-variables before pressing **ZoomStat**. There will be times however, when you will want to have both a scatter plot and an equation on the same viewing window, so it is not required to always de-select all functions.

STAT CALC ???Reg After two lists of numbers have been entered, we can fit lines or curves to the data with the **???Reg** commands. The TI-83 will fit 10 kinds of equations; the most common one is **LinReg**. Before you use any of the fitting routines, perform the following: Press **CATALOG** (found by pressing **2nd 0**), the letter **D**, down arrow eight times (to point to **DiagnosticOn**), and press **ENTER** twice.

If you want to store your fitted equation in the **Y=** list directly from the regression command, do this: press **STAT CALC ???Reg**, then indicate the lists (variables) you want to use, separated by commas, then press **VARS**, right arrow, **1**, and choose the desired **Y**-variable. Your fitted equation then appears in your list of **Y**-variables. An example command is: **LinReg(ax+b) L1, L2, Y1**. This will use **L1** as the *x*-values, **L2** as the *y*-values, and **Y1** as the equation to store the fitted equation in. Be aware though that this command will overwrite anything you already had stored in **Y1**. Make sure important stuff in **Y1** is saved elsewhere before you perform this command.

Goals: Practice using regression with the TI-83. We want the regression equation, the regression line superimposed on the plot, and we want to be able to use the line to predict new values. Understand the slope of the line is important to the Rate of Change.

Skills:

- **Fit a line to data.** This may be as simple as 'eyeballing' a straight line to a scatter plot. However, to be more precise, we will use least squares, **STAT CALC LinReg(ax+b)** on the TI-83, to calculate the coefficients, and **VARS Statistics EQ RegEQ** to type the equation in the **Y=** menu. You should also be able to sketch a line onto a scatter plot (by hand) by knowing the regression coefficients.

- **Interpret regression coefficients.** Usually, we want to only interpret slope, and slope is best understood by examining the units involved, such as inches per year or miles per gallon, etc. Because slope can be thought of as "rise" over "run", we are looking for the ratio of the units involved in our two variables. More precisely, the slope tells us the change in the response variable for a unit change in the explanatory variable. We don't typically bother interpreting the intercept, as zero is often outside of the range of experimentation.

- **Estimate/predict new observations using the regression line.** Once we have calculated a regression equation, we can use it to predict new responses. The easiest way to use the TI-83 for this is to **TRACE** on the regression line. You may need to use up and down arrows to toggle back and forth from the plot to the line. You may also just use the equation itself by multiplying the new *x*-value by the slope and adding the intercept. (This is exactly what **TRACE** is doing.) Note: when using **TRACE**, and the *x*-value you want is currently outside the window settings (lower than **Xmin** or above **Xmax**) you must reset the window to include your *x*-value first.

- **Convert a verbal description into an equation.** You should be able to recognize the ideas of slope / intercept or a description of several points on a line from a verbal description of a linear function. By recognizing which information is present, you then should be able to choose the proper form for the linear equation.

- **Be able to calculate average rates of change from tabular data.** Given a table of values, you should be able to calculate various rates of change. The important concept is that the average rate of change is simply the slope from a linear equation.

Reading: Section 1.4.

Day 3

Activity: Economics Examples. **Quiz 1 today.**

Several important business/economic applications use linear functions. Today we will look at profit, marginal costs, depreciation, and supply/demand curves. All of these topics can be modeled with non-linear functions, so we will encounter them later. For now, however, we will use only the linear functions.

Profit: In business settings, profit is calculated by subtracting costs from revenue.

Marginal Costs: The concept of marginal costs, revenues, etc. is a notion about the **next** item's cost, revenue, etc. Recognizing the difference between a marginal cost and an average cost is critical to using derivatives appropriately later (Chapter 2).

Depreciation: Items lose value over time, and we model this with different functions. With linear depreciation, we basically use a two-point form.

Supply/Demand Curves: Economists theorize that markets can be modeled with supply and demand curves, where the supply curve applies to producers of a commodity and the demand curve applies to the consumers. One interesting modification we can make to the supply and demand setting is adding various kinds of taxes. The basic question is how taxation affects market equilibrium. For the problems today, we will consider various points of view. For example, when we charge the producer the tax on an item, as opposed to charging the consumer, the producer behaves as if the product sells for less than the cost the consumer pays. Therefore we replace p with $p - t$, where t is the amount taxed per item. With the new equation, we now have a new equilibrium, and new total profits, which we can now compare to the values before the tax.

Today we will look at examples of each of the above topics. For each exercise, put your group's solution on the board. After the quiz, I will stay to answer any questions you might have, or to help you work through any problems you're having.

Revenue, Cost, Profit using linear functions. Marginal Cost/Revenue. Problem 9, page 30.

Linear Depreciation. Problem 16, page 30.

Supply/Demand using curves. Problem 20, page 31.

Modeling. Problems 27 and 28, page 32.

Supply/Demand using lines. Effect of taxes. Problems 35 to 37, page 32.

Goals: Recognize the application of linear functions to economic examples.

Skills:

- **Understand profit functions.** Profit is defined as the difference between Revenue and Cost. We often phrase these functions in terms of quantity produced, q. Revenue as a function of quantity is usually linear. Cost as a function of quantity is usually **not** linear, but today we will assume it is to make some calculations. Marginal cost (revenue, profit) is the cost (revenue, profit) of the **next** item produced. Marginal values are often different, based on current production levels. We will explore marginal values more in Chapter 2 on derivatives.

- **Understand linear depreciation.** In general, depreciation is the declining value of an item over time. The simplest form of depreciation is linear depreciation. The usual method of determining a linear equation for linear depreciation is to use the two-point form.

- **Understand supply and demand curves.** Economic theory suggests that prices and quantities produced or desired are related. The **demand** curve suggests that as price increases, fewer people will buy an item. The **supply** curve suggests that as price increases, more items will be produced. These two curves can be modeled with linear functions, and economic theory says they intersect at **equilibrium**. Later, we will explore non-linear supply and demand curves (Section 6.2, Day 21).

Reading: Sections 1.5 and 1.6.

Day 4

Activity: Exponential and Logarithmic Functions. **Homework 1 due today.**

In linear functions, as the x-value increases one unit, the y-value increases m units, where m is the slope of the line. This is **additive** growth. Another type of growth is **multiplicative**. In this kind of growth, when the x-value increases one unit, the y-value increases by a **factor** of b. That is, instead of **adding** a fixed value, we **multiply** by a fixed value. This kind of growth is called **exponential growth**.

Famous examples of exponential growth are populations. I will look at the US population. In Presentation 1, you will select an individual state and model its growth, perhaps efficiently with exponential curves. (Some populations do **not** grow exponentially; you will have to explore the growth rates to see.)

To use an exponential growth function, we start with a known x-value, such as a time. The exponential formula then gives us the height of the function, or the y-value. In many situations, however, we want to work in the other direction. That is, we know the height of the function (the y-value), but want the time when that happens (or the x-value). This **inverse** is called a **logarithmic function**. I have found that many students are rather confused by logarithms. I will try to alleviate this confusion by emphasizing the fact that exponentials and logarithms belong together, much like squares and square roots do, or multiplication and division do. There **are** rules we must learn to do algebra with exponential functions however; for example when we solve for time in an exponential growth model.

Today we will use the calculator to fit exponential curves to growth functions, like the US population over time. We will also explore *e*, and the log rules.

Activity 1: Modeling Population Growth.

The population for the US is on page 213. (I also have the numbers at the end of these notes.) Using ratios, find periods of time when the US population grew approximately exponentially. For your candidate eras, fit an exponential model using regression.

Activity 2: Discovering *e*.

As we saw on Day 2, the number *e* is a limit of the calculation $\left(1+\frac{1}{n}\right)^n$ as n gets large. However, you need to be careful not to let your calculator fool you. For example, try values of n from 10^{10} to 10^{14}. With such large values for n, your calculator's precision capabilities are exceeded. In your groups, try to come up with an explanation of what the calculator is having trouble with.

Activity 3: Rules.

Using test values, explore the rules for exponents and logs. Explore $x^{\left(a^b\right)}$, $x^{(a+b)}$, and x^{ab}. Now look at $\ln(a^b)$, $\ln(ab)$, and $\ln(a/b)$. I will "prove" each of the results using algebra. Practice the rules using 1-16 on page 43.

STAT CALC ExpReg This regression functions fits exponential curves. Again, the *x*-variable comes first, then the *y*-variable. The third parameter, if used, is the **Y**-variable where the equation will be stored. Example: **ExpReg(ax+b) L1, L2, Y1** uses data from lists **L1** and **L2** and stores the equation in **Y1**.

Goals: Explore exponential growth, and its inverse, the logarithm.

Skills:

- **Know the form of the exponential functions.** Exponential equations have two parameters, a *y*-intercept, and a base. The base is the multiplicative growth factor. The general equation is $y = ab^x$. You should be familiar with the shape of the exponential graphs, as well as the domain and range.

- **Know the multiplicative nature of exponential functions.** In contrast to linear functions growing at a steady rate over time, exponential functions grow at an increasing rate. The **ratio** of successive *y*-values for equally spaced *x*-values is a constant. This fact is especially useful for checking whether tabled values grow exponentially, but **only** if the table has equally spaced values of the independent variable.

- **Understand the relationship between exponential and logarithmic functions.** Logarithmic functions are inverses to exponential functions. This means that we reverse the *x* and *y* values and their associated facts. For example, the **range** of the exponential functions is only positive numbers; therefore the **domain** of the logarithmic functions is also only positive numbers.

- **Understand the definition of the number *e* (approximately 2.7182818).** Exercise 2 of Activity 3 from Day 1 is the definition of the number *e*, which we will use again and again in calculus. Remember, though, *e* is just a number, nothing more. The importance of *e* will become more clear when we explore derivative formulas in Chapter 3.

- **Know the exponential and logarithmic properties and be able to use them to solve equations.** To solve equations for variables that appear in exponents, we need logarithmic functions. Therefore, you must know the properties. In particular, you must be comfortable using $\ln(AB) = \ln(A) + \ln(B)$ and $\ln(A^p) = p\ln(A)$. The second property is how we "rescue" a variable from the exponent.

Reading: Sections 1.7 and 1.8.

Day 5

Activity: Growth and Decay. Transformations. **Quiz 2 today.**

Doubling time in an exponential function is the length of time it takes the *y*-value to double. To find it algebraically, suppose that a function has doubled between times x_1 and x_2. So, $y_1 = ab^{x_1}$ (because it is an exponential function) and $y_2 = 2\ y_1$ because it has doubled. Putting these two expressions together gives $y_2 = ab^{x_2} = 2y_1 = 2ab^{x_1}$. If we now solve for the change in time, $x_2 - x_1$, we will have found the doubling time.

Examples of exponential functions that are quite useful in business are the Present Value and Future Value formulas on page 49. You may have had some experience with these functions in the finance section of Math 204. We will explore them briefly as examples of exponential growth or decay.

There are several hallmarks of growth functions, and you should be able to tell growth from decay just by looking at the formula. If the base of an exponential function is greater than 1, we have a growth function, and vice versa. The tricky part of checking this feature out is the case where we have negative exponents. For example, $2^{-x} = \left(2^{-1}\right)^x = \left(\frac{1}{2}\right)^x = (0.5)^x$. So at first we might think this is growth function because 2 > 1, but after the algebra we see the negative exponent shows this is a decay function, because .5 < 1.

Our other topic today is transformations, creating new functions from old. In particular, we will explore **shifts**, **stretches/compressions**, and **compositions**. When a constant is added to the *y*-value, we have a vertical shift. When a constant is added to the *x*-value, in parentheses, we have a horizontal shift. When the *y*-value is multiplied by a constant, we have a vertical stretch/compression. When the *x*-value is multiplied by a constant, we have a horizontal stretch/compression.

Composed functions are very important to understand for being able to use the chain rule later. Basically, when we have a function inside parentheses, we have a composed function. The important skill with these composed functions is identifying the "inner" and "outer" functions. See class notes for examples.

Today we will practice using exponential and logarithmic functions. Then we will explore creating new functions from linear and exponential functions. In particular we will explore **composed** functions, which are critical to understanding the chain rule of Chapter 3.

Activity 1: Doubling Times. Tripling Times. Etc.

Using a graph, explore the relationship between doubling time and the base b. Choose a value for b; by guessing and checking, determine an interval where the y-value has doubled. Calculate the doubling time by subtracting the two x-values. Repeat this calculation with a different interval where the y-value has doubled. You should notice an interesting fact.

Repeat now for tripling time. Also, try a different value for the base. Make a conjecture about the effects of the base and the multiplier (doubling, tripling, etc) on the times. Can you support your conclusions using algebra?

Activity 2: Comparing compound interest rates.

A stock has current value $150 per share and is expected to increase in value by 8% each year. In each case below, find a formula for the value of the stock t years from now and calculate the value of the stock in 10 years:

Interpret the 8% return as an annual (not continuous) rate.

Interpret the 8% return as a continuous annual rate.

Now graph both functions on the same axes. What is the effect of continuous versus annual compounding? Write a short summary.

Activity 3: Using Present Value and Future Value formulas.

Work on problem 31 page 51. Hints: Treat each year as a separate investment. For example, the bonus is put into one account at the bank. Then after each year, that year's salary is put into a separate account, etc. What is different about these accounts is the length of time they exist. Then add all the account balances together to get the total amount.

Activity 4: Using the "Rule of Four" with various composed functions.

We will use all four approaches (verbal, graphical, algebraic, and tabular) to become familiar with composed functions and transformations. Verbal: problem 35 page 56. Graphical: problems 32-34 page 56. Algebraic: problem 42 page 73. Tabular: problem 31 page 56.

Goals: Become familiar with manipulating exponential functions. Become familiar with transformations, especially composed functions.

Skills:

- **Know facts about Doubling Times.** The most important fact about doubling time is that for any exponential function, it is the same value. That is, if an exponential function doubles from time $t = 3$ to $t = 13$, it will also double between $t = 20$ to $t = 30$. From our algebraic work on Activity 1, the doubling time is $\ln 2/\ln b$.

- **Be able to use Present Value and Future Value formulas in practical settings.** The Present Value and Future Value formulas are examples of exponential functions. You should know facts about these formulas. For example, $B = P(1+r)^t$ is an exponential function in the variable t. The base is $(1 + r)$, which is greater than one, so it is a growth function. P is the y-intercept.

- **Recognize the basic functions in complicated functions, especially the shifts and stretches.** Adding and multiplying by constants create shifts and stretches. You should be able to identify the basic function being manipulated, and also the shifts and stretches taking place.

- **Be able to decompose functions into the sequential steps.** To use the chain rule to take derivatives, in Chapter 3, we need to be able to recognize the components in composed functions. The "inner" function usually is inside parentheses, and the "outer" function is the function that results if you replace the expression inside the parentheses with x.

Reading: Section 1.9.

Day 6

Activity: Power functions and polynomials. **Homework 2 due today.**

Power functions have the form $y = ax^b$. Note the apparent similarity to exponentials. It is up to you to remember which is which. My personal reminder is that x^2 is a polynomial. You should be able to deal with fractional and negative exponents. Fractional exponents are radicals like square root (an exponent of 0.5 or ½) while negative exponents are reciprocals ($x^{-1} = 1/x$).

Polynomials are several power functions (with positive integer exponents) added together. The **degree** of the polynomial is the highest power of x. An n^{th} degree polynomial can have up to $n - 1$ turning points. However, there are often fewer, such as with x^3, which has none, but is a 3rd degree polynomial.

We should also understand the asymptotic behavior of polynomials. As x gets large, only the term with the largest exponent matters. To see this, start with a polynomial that has turns and gradually increase the x-value until the graph looks like only the leading term. (See Activity 1.) Pages 92 to 94 in the text elaborate on this endpoint behavior.

We should have some extra time in class today for you to work on your presentations for next time.

Today we will play around with polynomials, a versatile class of functions. They can take on a variety of shapes, but we should understand their behavior before settling on them as final models to our data.

Activity 1: Exploring polynomial turning points.

Using trial and error, create a cubic that has 1) zero turning points 2) one turning point, and 3) two turning points. Now try the same thing for a quartic (4^{th} degree polynomial), with up to three turning points. In each case, explore the endpoint behavior by comparing the cubic or quartic to x^3 or x^4 with large x-values.

After we study Chapter 3, we will be able to better qualify when a polynomial has 0, 1, 2, etc. turning points.

Activity 2: Recognizing power functions versus exponentials.

Values of three functions are given below (the numbers have been rounded off to two decimal places). Two are power functions and one is an exponential. Classify them and find potential equations. You may find the regression functions especially helpful here. But you can also use algebra as a solution method.

x	$f(x)$	x	$g(x)$	x	$h(x)$
8.4	5.93	5	3.12	.6	3.24
9	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

Activity 3: Explore the asymptotic dominance of exponentials to polynomials.

No matter the degree, no matter the base of a growth model, an exponential function will be larger than a power function for large enough values of x. First look at problem 28 page 96. Then change the base to 1.5 and the power to 10. Zoom out sufficiently to verify that $1.5^x > x^{10}$, for large enough x. (If you are having trouble finding a window that verifies this, look at the answer on the next page, in the reading.)

Goals: Understand the features of polynomials and power functions.

Skills:

- **Know about power functions and their attributes.** Power functions have a number of features you should be aware of. Even powered functions are non-negative and symmetric about $x = 0$. Odd powered functions are symmetric about the origin. The higher the power, the quicker the function goes to infinity. Fractional powers are only defined for positive x-values. Negative powers have a vertical asymptote at $x = 0$.

- **Know the basic facts about polynomials.** Polynomials are sums of power functions with positive integer exponents. The **degree** is the largest power of *x*. An n^{th} degree polynomial can have up to $n - 1$ turning points. Endpoint behavior is determined by the term with the largest power.

- **Know the asymptotic dominance of exponentials over polynomials.** Slowly growing exponentials may be dominated by polynomials for small *x*-values. However, for large enough *x*-values, exponentials (growth models) will always exceed polynomials. We call this "endpoint behavior" and it is important in analyzing functions qualitatively.

Reading: Section 2.1. (Activity 3 window: *x*: 100 to 130 *y*: 0 to 2E21.)

Day 7

Activity: **Presentation 1.** Instantaneous Change.

> Pick one of the 50 states. (The data is at the end of these notes.) Fit a model to its population growth. You have two goals: describe the growth, and predict the 2020 census. Compare linear, exponential, and polynomial models. Your presentation should convince us that you have chosen the most appropriate descriptive model and that your estimate for 2020 is believable.

Today we begin Chapter 2, the derivative. The derivative at a point is the slope of a line that is "parallel" to the curve at that spot. We will use a variety of techniques to approximate this slope, depending on the sort of information available to us. With equations, we can use more and more precise "two point" estimates, or slopes of **secant** lines; after Chapter 3, we will use formulas instead. If we have tabled data, we will not have precise estimates, as we can only "zoom in" as much as the table allows. If we have graphs, we will have to guess using a straightedge. In any case, we're seeking the **slope** of the line, and therefore the units are a ratio, like miles per gallon, or feet per second, depending on the units used for the two variables.

I have two activities today to explore instantaneous change, or derivative. Both relate to the fact that if we zoom in close enough on any continuously differentiable (or smoothly curving) function, the function will resemble a straight line. This phenomenon is called **local linearity**.

Activity 1: Exploring Local Linearity. Using Tangent on the TI-83.

Graph the function $y = 5\left(x^3 - x\right)$ on the standard window. Zoom in on what you think is the curviest spot. Keep zooming in, say 8 times. Using two points on the "line", estimate the equation of the line this zoomed in function is close to. Graph your candidate in the same window.

Now, at your selected *x*-value, use the Tangent function to get an equation of the line. Compare to your estimate from the "two point" method above. Note the Tangent function reports the entire equation of the tangent line; often we are only interested in the slope.

Activity 2: Estimating the derivative at a point using secant lines.

The derivative at a point can be approximated with an appropriately chosen secant line, that is a line between two well-chosen points on the curve. The following exercise should help you see what the calculator is doing when it calculates **Tangent**.

Fill in the table, using $x = 7$, and $f(x) = \sin(x)$. Compare your answers with the others in your group. You may be getting different answers. If so, explain whose values are "correct". Note that the two y-values forming the numerator of the secant slope are either $y = f(x-h)$ or $y = f(x+h)$. The two x-values are $x - h$ and $x + h$.

h	$f(x-h)$	$f(x+h)$	Secant slope
.1			
.01			
.001			

Now graph $f(x) = \sin(x)$, making sure that your window includes the point where $x = 7$. Use the **DRAW-Tangent** feature and draw a tangent line on your window. Now, use the **dy/dx** key on the **CALC** menu. How do these two techniques compare numerically? Graphically? Is one preferable over the other?

Compare the definition of the derivative (page 135) with your calculations when filling out the table. Observe how the calculator computes derivative values with **dy/dx**. However, sometimes we cannot use our calculators (perhaps a parameter in the equation has an unknown or variable value) and we must use our algebra skills. Specifically, notice how our authors do algebraic derivatives on page 137. Don't fear, though, you won't be able to use this method for all problems, so we will need other tools (theorems) to help us, and when we actually calculate derivatives, we will use rules, not this definition.

Zoom In (**ZOOM 2**) allows us to make the window "closer" by a factor of four. To use it, press **ZOOM 2**, then move the cursor to where you want the new window to be centered, then press **ENTER**.

The **CALC** menu (found by pressing **2nd TRACE**) is most useful to calculus. The functions in this menu will allow us to find minimum and maximum values, find roots of equations, and perform the differentiation and integration activities of calculus. We will explore the syntax of these commands as we use them. Today we used **dy/dx**, which gives the slope of the tangent line at that point.

The **DRAW** menu (found by pressing **2nd PRGM**) will allow you to draw various lines and shapes on your window. In particular, we will want to draw "tangent" lines to curves. These tangent lines are straight lines that just touch a curve at a point, and are in some sense "parallel" to the curve at that point. **DRAW - Tangent** can be used in two ways: from an existing graph, or from the calculation screen. To use it for an existing graph in the graphing window, make sure you have the point of interest on-screen. Then press **DRAW - Tangent**. Select the curve you want using up or down arrow, if you have more than one curve graphed. Choose the *x*-value you want by using right or left arrow or by typing the *x*-value of interest. Finally press **ENTER**. The command syntax from the calculation screen is: **DRAW – Tangent(Y#, x)**, where **Y#** is the curve of interest (such as **Y1**, or **Y2**, etc.) and **x** is the point at which you want to have the tangent line drawn.

Goals: Understand that most functions we look at are "locally linear". Understand slopes of secant lines as approximations for the slope of the tangent line.

Skills:

- **Understand the definition of derivative as the slope of the tangent line.** The tangent line just touches a curve at the point of interest, and is in a loose sense "parallel" to the line. The **slope** of this line is the derivative at that point. Because it is the slope of a straight line, we know much about its features: it is a rate of change (rise over run), it is important to know the sign, etc.

- **Evaluate derivatives numerically.** If your calculator can produce numerical values for a function (whether from a formula or just from some calculation), and the input values can be arbitrarily close together (that is what *h* approaching zero means), then you can calculate a derivative numerically. You must calculate the slopes of some **secant** lines, and should evaluate several such slopes, making sure the limit in fact does exist. You must also realize you may have the only **estimated** the value of the derivative, and the exact value may only be **close** to the value you have. For more exact values, either use the algebraic approach, or look ahead to the theorems we will encounter in Chapter 3.

- **Evaluate derivatives graphically.** If you can phrase a function in the form of an equation, then your graphing calculator can help you calculate a derivative at specific input values. The TI-83 can draw tangent lines at various places on a curve, and can calculate derivatives numerically as well, displayed on the graphing window.

- **Understand the definition of the derivative.** You should be comfortable with the notion of a limit of slopes of secant line. You should also be comfortable with the equations $\frac{\Delta y}{\Delta t} = \frac{f(b)-f(a)}{b-a}$ and $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. (Note that the second equation is precisely the limit of slopes of secant line. See page 135.) This last expression differs slightly from Activity 2 today; I personally think it makes more sense to **center** the secant line on the *x*-value instead of favoring the right side. It should make no difference in the limit, but practically we can only make *h* so small using our TI-83.

- **Know several methods of estimating the derivative at a point.** If we have a formula, we can use successively narrower intervals and use the "two point" form for a line to estimate a slope at a point. After Chapter 3, we may be able to use a formula approach. If we have tabular data, we can only estimate roughly the slope of the tangent line, using secant lines. If we have a graph, we can estimate slopes using a straightedge.

Reading: Chapter 1.

Day 8

Activity: **Exam 1.**

This first exam will cover the elementary functions of Chapter 1. Some of the questions may be multiple choice or T/F. Others may require you to show your worked out solution.

Reading: Sections 2.2 and 2.3.

Day 9

Activity: Sketching the derivative function. Interpreting the derivative function.

The derivative is a slope of a function at a particular point. If we evaluate the derivative at many such x-values, and graph the result, we have the derivative **function**. This is a graph, just like the original function, but with different interpretations, as the y-values are now the **slopes** at each x-value, instead of the original functional values. Today we will begin by estimating the derivative function from tabular data. Then we will estimate functional values by knowing the derivative at a point.

Our second and third activities today involve a graphical exercise. After the activities, we will look at a handy function on the calculator that will approximate the derivative at all x-values in the graphing window. See calculator commands below.

Comment on notations: There are two main notations mathematicians have used to designate derivatives. I will use them interchangeably, without thinking, as it is second nature to me. These notations are:

1) Prime notation. $f'(x)$.

2) Leibniz notation. This notation reminds us that derivative is a ratio of differences, a slope. Either we use $\Delta y/\Delta x$ or dy/dx. One advantage of Leibniz notation is that we get to see the actual variables involved. Many times with the "f-prime" notation we just say "f-prime". This isn't very illuminative. What are the variables! Unfortunately, though, Leibniz notation doesn't allow us to specify **which** x-value we're talking about. In fact, to designate which x-value we're using becomes quite cumbersome. Page 113 shows you the messiness.

Activity 1: Estimating the derivative using tabular data.

Using the following half-marathon times, find the estimated derivative function. Note it will be difficult to estimate the slope at the beginning and end. You don't have the luxury of points before **and** after. Discuss with your group members what is reasonable.

Mile	Time on Clock	Mile	Time on Clock
1	7:36	7	55:07
2	15:29	8	1:02:50
3	23:25	9	1:10:29
4	31:23	10	1:18:20

5	39:20	11	1:26:08
6	47:18	13.1	1:42:58

Activity 2: Estimating the derivative using a graph, and translating back.

Each of you will sketch an arbitrary function on a piece of paper, labeling it "Original Curve" and putting your name on it. You will then pass your graph to someone else; they will graph the derivative function on a separate sheet of paper, labeled with "Derivative Curve for <insert name here>". The person drawing the derivative will have to carefully estimate the slopes, so a scale is needed. I will show you in class the method I use to estimate these slopes. It involves placing a straight edge tangent to the curve, and finding the rise over run for that angle. This is repeated for a number of x-values.

After sketching the derivative, the second person will pass the derivative graph to a third person (keep the original aside to compare with later); the third person will attempt to redraw the original graph based solely on the information from the derivative graph. Caution: this last part is tricky, as the starting location is not unique. You need to arbitrarily pick a y-intercept to get started. From there, the derivative graph shows you how steep the graph needs to be at that point, so draw a little line segment with that slope. Move over slightly, and repeat the process.

I will show you an example in class before you attempt this activity. If everyone has done the estimates correctly, the graph the third person draws should match the "Original Curve" graph. If there are discrepancies, the two sketchers should resolve them. It might be that the person drawing the derivative made poor estimates, or it may be that the third person didn't translate the information well.

Activity 3: Estimating using local linearity.

Work on problem 2, parts d and e, on page 133. These parts are about predicting new values using local linearity (or in this case **extrapolating** as 7 feet is beyond the available data).

nDeriv((**MATH 8**) will produce an estimate for the derivative at a point. The syntax is **nDeriv(expression, variable, value)**. **expression** is the formula for the function. I will often use **Y#**, having already stored the function in a **Y** variable. **variable** is generally **x**, but you have some flexibility here in case you want another letter to be the variable. **value** is whatever number you're interested in. When using **nDeriv(** to graph the entire derivative function in the graphing window, use **x** here instead. Example: **nDeriv(Y1, x, x)**.

Goals: Realize that the derivative can be viewed as a function.

Skills:

- **Evaluate derivatives from tabular data.** When information is available in tabular form, we cannot "zoom in" to get a limit of secant slopes. We have only a few choices to estimate the derivative at each *x*-value. Generally, the best option is to average the secant slope before the point with the secant slope after the point. This is algebraically equivalent to finding the secant slope for the two points before and after.

- **Interpret the derivative verbally.** For problems with a real-world setting, you should be able to use the value of the derivative at a point in an English sentence. For example, you may say, "At a production level of 1,000 car seats, we can expect profits to rise $10 for every additional car seat produced." If you are having trouble with this verbal description of the derivative, one thing that may help is to pay close attention to the units involved, for instance dollars, or number of car seats produced. The examples in Section 2.3 should help you understand this verbal phrasing and interpretations of the derivative.

- **Understand local linearity and how to use it estimate new values.** If we are close enough to a point where we know the tangent slope, we can **project** the tangent line a short way and use it to **estimate** the value of the function at that new point. Caution: if the line is very "curvy" at this spot, our tangent line will poorly represent the function, so it is important to only use this method **very close** to the known derivative value.

- **Know how to use the TI-83 to produce a graph of the estimated derivative of a formula.** The command **nDeriv(** will estimate the derivative numerically with a small secant line. If we use this in the **Y=** window, we can graph the entire derivative function on the graphing window. The syntax for this is **nDeriv(Y#, x, x)**.

Reading: Sections 2.4 and 2.5.

Day 10

Activity: Introduction to the Second Derivative. Economic Examples.

After discovering that the derivative is a function just like the original curve, there is no reason we cannot take the derivative of the derivative. This is called the **second derivative**, and often reflects useful information in real world problems. It is the **change** in the **change** of a function. The second derivative also can be thought of as the curvature of a function. You have probably seen this idea already in terms of **concavity**. In particular, if the second derivative is positive, we say we have **positive concavity**, and the other way around for negative values.

When we look at the information from the first and second derivatives, there are four main situations of interest. The first derivative can be either positive or negative (or zero, but we will address that situation later) and the second derivative can be either positive or negative.

1) Positive first derivative, positive second derivative: an increasing curve that is getting steeper.

2) Positive first, negative second: an increasing curve that is leveling off, approaching a peak.

3) Negative first, positive second: a decreasing curve that is leveling off, approaching a minimum.

4) Negative first, negative second: a decreasing curve that is falling faster.

The Leibniz notation gets a little messy for second derivatives. The best way to phrase them is to use the prime notation, adding another prime for the second derivative. We usually say "*f* double prime", referring to $f''(x)$. Note the messy Leibniz notation on page 119.

One important application of the derivative is the idea of marginal analysis. In fact, the term **marginal** is synonymous with derivative. If either the cost function or the revenue function is a straight line, then the marginal cost or revenue is simply the slope of that line. We will look at this topic again in Section 4.4 (Day 15), after we explore the shortcut formulas to differentiation.

In Activity 3 we revisit the concepts of marginal cost and marginal revenue. The important notion is that we are talking about the cost or revenue of the next item only. Because quantity is our independent variable, this marginal cost is the same as the derivative of cost, expressed in units of dollars per item. Usually, quantity can only be expressed as integers, so to calculate marginal cost or revenue directly, we subtract two sequential values, for example the difference between the cost to make 10 items and 11 items represents the marginal cost of the 10^{th} item (or the 11^{th} item). We can also approximate marginal cost using the derivative.

Activity 1: Comparing a function to its first and second derivatives.

Enter $y = xe^{-x^2}$, along with its first and second derivatives, in the Y= window. (See calculator commands below.) Select **only** the second derivative and use the window -2 < *x* < 2 and -2 < *y* < 2. Make statements about the original function given what you see about the second derivative. Repeat using **just** the first derivative. Before graphing the original function, make a sketch that satisfies your statements. Then compare and see how close you were. If you are off in any of your statements, closely examine where you went wrong.

Activity 2: Interpreting derivatives in a real world setting.

Problem 30 page 132. Parts c and d are especially important; you **must** be able to convert the mathematical info into real world uses. In this case, the context of declining graduation rates is very important to school officials.

Activity 3: Marginal cost and revenue.

Problem 10 page 129. Estimate values for the marginal cost and revenue at both 50 and 90. Use these figures in your answers.

There isn't a separate command on the TI-83 for the second derivative; it is simply the derivative of the first derivative. The easiest way to get the calculator to estimate the second derivative function is to use these two **Y=** functions. Put your formula in **Y1**. In **Y2**, put **nDeriv(Y1, x, x)**. In **Y3**, put **nDeriv(Y2, x, x)**.

Goals: Investigate the properties of the second derivative.

Skills:

- **Be able to graph the second derivative on the TI-83.** Using **nDeriv(** will produce a numerical derivative of a formula. If we repeat the command on the new formula, we will approximate the second derivative. I recommend keeping these two commands in **Y2** and **Y3** for the rest of the semester. Put the formula you want to analyze in **Y1**. Use **Y4** to **Y0** for any other functions you want to graph.

- **Understand what the second derivative says about the concavity of a function.** The second derivative measures the concavity of a function. When it is positive, we know the original function is bowl-shaped (concave up); when it is negative, the original function is humped (concave down). When the second derivative is zero, it is neither bowl-shaped nor humped; rather it is very nearly linear at that point. Earlier we talked about local linearity; when the second derivative is zero, we might think of that point of the curve being even **more** locally linear!

- **Be able to convert second derivative facts into everyday English.** Because the second derivative is a change in the first derivative, when we convert to an English description, we have to talk about the rate of change in the rate of change. For example, the speed of the car is increasing. Sometimes we have special words for these derivatives. With the motion of an object, the first derivative is speed and the second derivative is acceleration.

- **Realize that marginal costs/revenues/etc are simply derivatives.** Marginal costs, revenues, profit, etc are important ideas in economics. Because the marginal cost is the cost of the **next** item, we are just talking about the slope of the tangent line, which is the derivative. Similarly for revenue, the derivative is the marginal revenue. We will explore these ideas more in Section 4.4.

Reading: Sections 3.1 and 3.2.

Day 11

Activity: Using Polynomial and Exponential derivative formulas. **Quiz 3 today.**

A calculator approximation for the derivative function is convenient, but there will be times when we would rather have an exact formula. Fortunately, there are theorems (shortcuts) we can use. We won't **prove** many of these results, but we will use them to produce formulas. Chapter 3, therefore, is only concerned with the algebraic point of view. When we have tabular data, graphs, or verbal descriptions, we cannot use these theorems.

Several of the theorems apply to any function. Others are specific to particular forms. The general rules are the additive constant rule, the multiplicative constant rule, the addition/subtraction rule, the product rule, the quotient rule, and the chain rule. The specific functions are the power rule, the exponential rule, and the logarithmic rule.

Additive constant rule: For this rule, we can make a quick argument to see the answer. What happens to the slope of a curve when we add a constant to it? Adding the same constant to every value simply lifts or lowers the entire curve that much, but doesn't change the shape at all. Thus, the additive constant rule is that there is no change to the derivative. Using notation: $\frac{d\left(f(x)+a\right)}{dx} = f'(x)$.

Multiplicative constant rule: It is a little harder to verbally prove this rule, but we can see for straight lines that multiplying by a constant increases the slope by that constant. With algebra, and the definition of derivative on page 135, we can discover that the derivative of a multiplied function is multiplied by the same amount. Using notation: $\frac{d\left(af(x)\right)}{dx} = af'(x)$.

Addition/subtraction rule: Again, using algebra is the easiest way to prove this rule, but we will accept the result on faith. (If you would like to see the algebra, see me after class.) Basically, the derivative of a sum is the sum of the derivatives. Using notation: $\frac{d\left(f(x)+g(x)\right)}{dx} = f'(x)+g'(x)$.

Power rule: To prove the power rule, we need the binomial theorem, and lots of algebra. Again, we will accept this result on faith. $\frac{d\left(x^n\right)}{dx} = nx^{n-1}$. When we combine this rule with the multiplicative constant rule, we get the most common rule we'll use: $\frac{d\left(ax^n\right)}{dx} = anx^{n-1}$. We need to use this rule for reciprocals and radicals, as they can be written as exponents. This means **you** will have to recognize that square roots, and reciprocals, are power functions. We will do some examples in class.

Exponential rule: The exponential class of functions is quite unique. They **are** their own derivatives, multiplied by a constant. Activity 2 below will hopefully convince you of this. The constant is the natural logarithm of the base. Using notation: $\frac{d\left(a^x\right)}{dx} = \ln(a)a^x$. When the base is *e*, ln(*e*) = 1, so the rule is even simpler: $\frac{d\left(e^x\right)}{dx} = e^x$ One additional note: functions of the form e^{kx} can be thought of as $\left(e^k\right)^x$ so their derivatives have *k* in front as a multiplier ($\ln(e^k) = k$). Example: $\frac{d\left(e^{4x}\right)}{dx} = 4e^{4x}$.

Logarithmic rule: The logarithmic rule is very simple: $\frac{d(\ln(x))}{dx} = \frac{1}{x}$. I will show a simple proof of this in class based on the exponential rule.

Activity 1: Try some basic expressions.

For each of the following functions, plot the function in **Y1**, its **nDeriv(** in **Y2**, and your candidate answer in **Y4**. Using trace, check to see if your answer is right. (Compare **Y2** to **Y4**.) (Note in problem 5 you will have to make up values for *k* and *a*. This sort of problem is why knowing algebra is still important.)

1) $y = 3x^4$

2) $y = 3 - x^{2.7}$

3) $y = x^3 - 2x + e$

4) $y = \frac{1}{x^2} + \sqrt{x} - \sqrt[3]{4}$

5) $y = ke^{-ax}$

6) $y = 3e^{4x} + 3\ln(x)$

Activity 2: Discovering the unique character of the exponential functions.

Graph $y = 2^x$ and its derivative in the same window. What is the doubling time for $y = 2^x$? What is the doubling time for its derivative? These two doubling times imply an important result. Use this result to deduce the formula for the derivative of $y = 2^x$.

Goals: Learn and use the basic rules for differentiation shortcuts.

Skills:

- **Know the Rule for Sums.** $\frac{d(f(x)+g(x))}{dx} = f'(x) + g'(x)$.
- **Know the Rule for Powers.** $\frac{d(x^n)}{dx} = nx^{n-1}$. Note that n can be any number, including fractions and negatives.
- **Know the Rules for Exponential Functions.** $\frac{d(a^x)}{dx} = \ln(a)a^x$. This rule is even simpler than the power rule, because exponential functions **are** their own derivatives, multiplied by a constant. With the power rule, you must decrease the power, which is more complicated.

- **Know the Rule for the Natural Logarithmic Function.** $\frac{d(\ln(x))}{dx} = \frac{1}{x}$.

- **Realize that your nDeriv(function will verify that you have a correct derivative.** By graphing the numerical derivative on your calculator (nDeriv), along with what you think the answer is, you can verify if your answer is correct. You can either compare the values for a few haphazardly chosen values, or you can graph their difference on a separate window. If they are the same, the difference should be zero (or very close but not exact due to rounding).

Reading: Sections 3.3 and 3.4.

Day 12

Activity: Practicing the Chain, Product, and Quotient Rules. **Homework 3 due today.**

Today we practice formulas. I will show you how the product, quotient, and chain rules work. Then we will spend time practicing.

The chain rule is used in composed functions. The idea is that you first must calculate one function's result before you can finish the calculation. This first function, often called the "inner" function, can be substituted with another letter. But the key is that the "outer" function (what we do to the inner function's result) must be evaluated at the inner function's result. If we use the symbol x for the input to the inner function, then we can't use x as the input to the outer function. We still do the derivative rules the same, we just evaluate the outer function at the inside function's result, not at x. Hopefully, our class discussion will make this clearer.

A note that seems to help people understand how to use the chain rule: ultimately, the chain rule is a product of the derivatives of the inner and outer functions. To use the rule effectively, you must be able to decompose the function into its parts. This is why understanding composed functions from Chapter 1 is so critical to success in this section of the material.

Activity 1: Practicing the Product, Quotient, and Chain Rules.

Calculate the derivatives of the following functions. Be sure to first decide whether the function requires the product rule, the chain rule, the addition rule, etc. Then check your answers on your calculator using nDeriv.

1) $f(x) = (x+3)(x^3 - 3x^2)$. (Use the product rule.)

2) $f(x) = (x+3)(x^3 - 3x^2)$. (Expand first, then use the power rules.)

3) $f(x) = \frac{(x^2 + x + 1)}{3x - 2}$.

4) $f(x) = \dfrac{\left(x^2 + x + 1\right)}{(3x - 2)(x - 1)}$.

5) $f(x) = \left(x^3 - 3x^2\right)^2$. (Expand first, then use the power rule.)

6) $f(x) = \left(x^3 - 3x^2\right)^2$. (Use the chain rule.)

7) $f(x) = e^{\left(3x + x^2\right)}$.

8) $f(x) = \ln\left(\sqrt{x + 2}\right)$. (This should require two substitutions, or two uses of the chain rule.)

Goals: Become familiar with the product, quotient, and chain rules.

Skills:

- **Identify the particular derivative rule needed for a problem.** For many functions, only one of the derivative rules we have learned is actually used. (Of course, for some functions, more than one type of rule might be present.) Your task, then, is to be able to identify which particular rule or rules are needed. This skill will come with practice. It is up to you to put in the time so that you have the experience to choose the proper rules. There are a lot of problems on page 173 for you to practice on.

- **Know the Rule for Products.** $\dfrac{d\left(f(x)g(x)\right)}{dx} = f(x)g'(x) + f'(x)g(x)$. It is important to note that the product rule is **definitely** not the product of the derivatives. That is actually closer to what the chain rule says.

- **Know the Rule for Quotients.** $\dfrac{d\left(\dfrac{f(x)}{g(x)}\right)}{dx} = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$. To remember this rule, I've memorized the little ditty "Low dee high, less high dee low, square the bottom down below". I've never forgotten the quotient rule because of it!

- **Know the Rule for Compositions (Chain Rule).** $\dfrac{d\left(f\left(g(x)\right)\right)}{dx} = g'(x)f'\left(g(x)\right)$. You **must** understand composed functions to use this rule correctly. If you cannot identify what $g(x)$ is, you can't get the correct derivative in front. You also need to be able to "replace" $g(x)$ with x in the f function to get the proper derivative there, then again "replace" x with $g(x)$.

- **Identify the particular derivative rule needed for a problem.** For many functions, only one of the derivative rules we have learned is actually used. (Of course, for some functions, more than one type of rule might be present.) Your task, then, is to be able to identify which particular rule or rules are needed. This skill will come with practice. It is up to you to put in the time so that you have the experience to choose the proper rules. There are a lot of problems on page 173 for you to practice on.

Reading: Section 4.1.

Day 13

Activity: Exploring Local Extrema. **Quiz 4 today.**

Now that we have learned some formulas, we can make use of this information algebraically to find interesting places on curves. In particular, we can find peaks and valleys, or more formally, local maxima and minima (together called **extrema**). You may think at first with our powerful calculator that we don't need algebra any more. While our machines help us in many circumstances, there is still a use for analytical results. For example, the calculator will help us find the turns in a polynomial, but **only** if we have a suitable window already. Algebraic results help us find the proper window. We will explore this in Activity 1.

Another circumstance where algebra is necessary is when the parameters of a model are unspecified. Activity 2 today addresses this situation.

Activity 1: Analyzing polynomial turns.

Without doing calculus, try to find the part of this cubic where the critical points are. $f(x)=x^3-63x^2+1320x$. Now, find the critical points algebraically. Use the first derivative test to classify the critical points. Use the second derivative test to classify the critical points.

Activity 2: Finding the conditions on a cubic so it has two turns.

The general form for a cubic polynomial is $f(x)=ax^3+bx^2+cx+d$. However, we know that some cubics have no extrema, such as $f(x)=x^3$. What conditions on the parameters cause a cubic to have none, one, or two critical points? Hint: you will need to use the quadratic formula and note where the discriminant is negative, zero, or positive.

Activity 3: Critical points of a non-polynomial.

Without doing calculus, try to find the part of this function where the critical points are. $f(x)=xe^{-x}$. Now, find the critical points algebraically. Use the first derivative test to classify the critical points. Use the second derivative test to classify the critical points.

Activity 4: Using a table of derivative values to find the maximum and minimum.

Problem 20 page 181.

Goals: Understand critical points, and how to classify them.

Skills:

- **Know the definition of a Critical Point.** Locations on the graph of a function where the derivative is either zero, or undefined, are **critical points**. Places with zero slope might be maxima, minima, or neither. Important examples to keep in mind are the cubic power function, which has a critical point that is neither a maximum nor a minimum, and the absolute value function, which has a critical point with an undefined derivative.

- **Be able to use the First and Second Derivative Tests for classifying extrema.** The second derivative test is useful for determining whether a critical point is a maximum or a minimum. Simply evaluate the second derivative at the candidate point, and classify it as a maximum, a minimum, as it is negative or positive. If the second derivative is zero, we must resort to the first derivative test. To perform this test, we assess whether the first derivative is positive or negative around the critical point. If it is negative to the left and positive to the right, we have a minimum. If it is positive to the left and negative to the right, we have a maximum. If it is positive on both sides of a critical point or if it is negative on both sides of the critical point, then we have a "saddle point", which is neither a maximum nor a minimum.

Reading: Sections 4.2 and 4.3.

Day 14

Activity: Exploring Inflection Points and identifying Global Extrema. **Homework 4 due today.**

We have seen that the critical points of a function describe that functions extrema, if any exist. The critical points of the **derivative** function represent places where the concavity of the original function changes sign. These points are called **inflection points**. We discover them in just the same way we found critical points, but working with the **second** derivative instead of the first derivative. Remember that it is possible for the second derivative to be zero and yet the concavity doesn't change. The fourth degree power function is one example.

In addition to determining inflection points and critical points, we also want to determine **global** extrema. We have already talked about **relative** (or local) extrema. The overall maximum (or minimum) must be either one of the critical points, unbounded (such as with a vertical asymptote), **or** one of the endpoints (if the region is bounded). When you look for global extrema, I recommend making a list of the critical points, and endpoints. Then, after looking at the graph for places where the graph goes off to infinity in either direction, choose the largest for the maximum, and the smallest for the minimum. If the graph **does** go to infinity, the best phrase to use is "There is no global maximum (or minimum)."

Asymptotes are straight lines that a graph approaches. $y = 1/x$ is an example with two asymptotes; there is a horizontal asymptote on the x-axis and a vertical asymptote on the y-axis. We will mostly be concerned with only horizontal and vertical asymptotes, but they could also be oblique (diagonal). Generally, when the denominator of a rational function is zero, we have a possible vertical asymptote. While these are not critical points, they **are** important to identify, as a function is unbounded at an asymptote.

Activity 1: Describing the interesting points in a function.

For each of the following functions, find all the interesting points/features, including critical points, extrema, inflection points, asymptotes, increasing and decreasing intervals, and positive and negative concavity.

1) $f(x)=(x+\pi)(x-2)(x^2-9)$ [Hint: Use the TI-83.]

2) $f(x)=\ln\left(x^2+2e^x\right)$ [Hint: The second derivative algebraically is tough; persevere, or use the TI-83.]

3) $f(x)=\dfrac{x^2+x+1}{3x-2}$ [Hint: The second derivative algebraically is tough; persevere, or use the TI-83.]

4) $f(x)=x\left(x^2+x-2\right)$

5) $f(x)=x\left(x^2+x-2\right),\ x\geq 0$

Goals: Be able to find and interpret points of inflection. Understand the difference between **relative** extrema and **global** extrema.

Skills:

- **Know the definition of Inflection Points.** Points on a graph where the concavity changes sign are **inflection points**. We normally detect these points by examining where the second derivative is zero. However, we still must check on each side of such candidates, as simply equaling zero is not the same as it changing sign.

- **Understand the overall strategy for analyzing and sketching functions.** Our overall strategy is to find the critical points, the asymptotes (both vertical and horizontal), inflection points, intercepts, and any other easy-to-find points. You should be able to make both quantitative and qualitative descriptions of functions/graphs/equations.

- **Sketch graphs for equations with unspecified parameters.** Using the skills acquired in classifying critical points, and using the skills for finding asymptotes, you should be able to sketch curves with unspecified parameters. You may need to be told whether the values of the coefficients are positive or negative, or over what range the coefficients can have values. For example, for parabolas we know the vertex has an x-coordinate of $-b/2a$, where $y=ax^2+bx+c$. It may be difficult or impossible to graph the family of curves in general, as different coefficients may yield differently shaped curves, but in most cases you can construct an effective sketch.

- **Be able to find global extrema.** To find the overall extrema (**global** extrema), we examine all the critical points, as well as any endpoints of the domain or any points with undefined derivative. Caution: only looking at the critical points will not be sufficient, as many functions have no global maximum or minimum due to the function's values approaching infinity.

Reading: Sections 4.4 and 4.5.

Day 15

Activity: Economic Examples. **Quiz 5 today.**

Today we return to economic applications. Recall that profit is the difference between revenue and cost. The first thing to notice is that the quantity that maximizes revenue is not always the same quantity that maximizes the profit. We have several approaches to solving the maximum profit problem. We can simply calculate profit at all quantities and then choose the maximum. This can be tedious and time consuming. It might be much easier to use the calculus rules we have learned. Specifically, we know that when a derivative is zero, the function has a relative maximum or minimum. Because profit is a difference ($\pi = R - C$), we can use the formulas from Chapter 3 to show that $\pi' = R' - C'$. Now, if we set the derivative to zero and solve, we find $R' - C' = 0 \Rightarrow R' = C'$. We use this strategy now to solve problems with only a graph, or a table of marginal values: find where marginal cost and marginal revenue are equal. We will practice with all three approaches (tabular, graphical, algebraic) in Activity 1.

Another example using derivatives in economics is average cost. By dividing the cost function by quantity, we have the formula for average cost. Using the quotient rule (which I will do in class), we discover that the minimum cost occurs where average cost equals marginal cost. If we have the formulas, this will just be an algebra problem. If we have graphs, it will be easier, as there is a handy geometric solution (see page 204 figure 4.60).

Activity 1: Profit maximization.

We will maximize profit using three sets of information: tabular, graphical, and algebraic.

Tabular: problem 8 page 200.

Graphical: problem 13 page 200.

Algebraic: problem 16 page 201.

Activity 2: Exploring Average Cost.

Using the graphical and algebraic info from Activity 1, find the minimum average cost.

Goals: Understand some uses of the derivative in economics and business.

Skills:

- **Know that maximum (or minimum) profit occurs where marginal cost equals marginal revenue.** Because profit is cost subtracted from revenue, and because maximum profit occurs when its derivative is zero, we can conclude that profit is a maximum when marginal cost equals marginal revenue. In equation form: $\pi = R - C$, $\pi' = R' - C'$, $\pi' = 0 \Rightarrow R' - C' = 0 \Rightarrow R' = C'$. We don't have a guarantee that such spots are maxima; we must check to make sure using the first derivative test, for example.

- **Know that average cost is a minimum when average cost equals marginal cost.** By using the quotient rule to find the derivative of the average cost, we find that average cost is minimized when average cost equals marginal cost.

Reading: Sections 4.7 and 4.8.

Day 16

Activity: **Presentation 2.** Logistic Growth, Surge Functions. **Homework 5 due today.**

Pick one of these functions (first come, first served): 1) $y = e^{-x} - e^{-2x}$, 2) $y = x + 1/x$, 3) $y = x\ln x + x$, 4) $y = \ln(1+x^2)$, 5) $y = 2^x + 2^{-x}$, 6) $y = e^{-x^2}$, 7) $y = x^2 - 1/x$. Completely describe the interesting behavior, without graphing. Be sure to include critical points, inflection points, global extremes, endpoint behaviors, etc. After your description, show us a graph with an **appropriate** window that demonstrates the correctness of your analysis.

We saw that some state populations grow nearly exponentially for periods of time. However, we also know that this exponential growth cannot occur forever, due to real world constraints, such as available space and resources. A more realistic model would account for this eventual upper bound. The **logistic function** is such a model. Today we will explore this function, by taking its derivatives, finding its interesting points, and sketching graphs for its various parameters.

The **surge function** is often used to model drug concentration problems. We will explore this function today also. I will work out the derivatives and the graph during class; then you will practice yourself.

Activity 1: Revisiting state populations.

Pick one of the 50 states and fit a logistic regression curve using the TI-83. You will find the function in the STAT CALC menu of the calculator, at the bottom of the menu. There are some data sets for which the TI-83 will fail to find a good fit. I haven't figured out when it will and will not work; it may have to do with the shape of the data not looking "logistic" enough.

Activity 2: Surge function example.

Problem 8, page 226. In addition to answering the questions asked, try to come up with estimates of the formulas.

Goals: Examine two further examples of derivatives, the logistic function in population growth and the surge function in drug concentrations.

Skills:

- **Know the form of the logistic growth function.** One formulation of the logistic function is $P(t) = \dfrac{L}{1+Ce^{-kt}}$. This curve models population growth realistically. The domain is all real numbers, and the range is 0 to L.
- **Know facts about the logistic growth function.** Through our calculus results, we find that there are no critical points, but there is an inflection point where $P = L / 2$, also called the

point of diminishing returns. L is the **carrying capacity**, or the value of the horizontal asymptote as x approaches infinity.

- **Know the form of the surge function.** The surge function is $y = ate^{-bt}$. The domain is all positive real numbers, and the range is 0 to 1 / *be*.
- **Know facts about the surge function.** The surge function begins at the origin, increases to a peak at $x = 1 / b$, then decreases to a horizontal asymptote at zero. The curve is often used to model drug concentration curves.

Reading: Sections 5.1 and 5.2.

Day 17

Activity: Introduction to Definite Integrals, using horse speeds.

We will base our initial discussion on the formula "Distance equals speed times time". In many cases, we will not know the speed at **any** arbitrary time, but at fixed intervals. Thus we must **guess** the values in between. We usually assume smoothness, and therefore pretend our functions are **monotonic**, or either **only** increasing or **only** decreasing. So, for each interval, we will have an upper and lower estimate of the distance covered, depending on whether we use the speed before or after the current time period.

In the following data, we have the time of a horse race, and the speed of the horse at that moment. Use this information to estimate the total distance the horse has traveled.

Time (sec)	0	30	60	90	120
Speed (mph)	0	40	38	35	37

How could we improve this estimate of distance? The most important conclusion we will make today is that the idea of distance turns out to be an **area**, not a length. It is critical that you understand this point in the upcoming material.

We will typically talk about left and right sums, but these represent the lower and upper estimates **only** on monotone intervals. If the speed bounces up and down (as in the horse race example) then we will have to be careful about which estimate is the lower one and which is the upper one.

To find the value of the **definite integral**, we take smaller and smaller intervals (if we can) and eventually the **limit** as this interval width approaches zero. These **Riemann sums** are mostly a conceptual notion; in practice we will use a different approach (antiderivatives in Section 7.1, Day 22).

Activity 1: Did they hit the skunk?

Jan and Pat are driving along a country road at 45 miles per hour (about 66 ft/sec). As the car rounds a curve, Jan sees a skunk in the middle of the road about 100 feet ahead. Jan immediately applies the brakes, and Pat notices that the speed of the car drops from 66 ft/sec to 51 ft/sec to 34 ft/sec to 0 ft/sec over the next three seconds. (Pat is a bit strange.) Does the car hit the skunk?

Goals: Understand how distance can be estimated by knowing speed. Explore Reimann sums and definite integrals

Skills:

- **Be able to estimate distance given speed.** By knowing that "Distance equals speed times time", we can calculate distance traveled over an interval with knowledge of the speed. This fact is the basis for all of our distance calculations, even for speeds that are not constant, as we shall see in the upcoming material.

- **Know there are upper and lower bounds for the distance estimate.** Because speed changes over an interval, and we do not know the values in between two time points, we must make assumptions about how speed varies. Generally we will assume that the speed does not go above or below the two values that bracket a time interval. This leads to two estimates of distance in one time interval, an "upper" and a "lower" estimate. We add all the lower estimates and all the upper estimates over an entire set of intervals to find the accumulated distance traveled.

- **Realize the distance estimate can be viewed as an area under a curve.** A very important observation to make about our distance calculations is that these distances can be thought of as **areas** under the curve of the speed values. In general, when we have a rate function, and are interested in the cumulative change in the "distance" function for that rate, we will calculate an area.

- **Know that the definite integral is a limit of converging upper and lower estimates.** If we have the luxury of "refining" our intervals (that is, making them narrower), then we can force the lower and upper estimates to converge to the true value of the distance traveled. The value to which the estimates converge is called the **definite integral**.

- **Realize that if the function isn't monotone, the upper and lower estimates won't be identical to right and left sums.** If we use a graph and carefully keep track of which rectangle represents the lower estimate and which represents the upper estimate, then we see that "upper" and "lower" are also "right" and "left" **only** on an interval that is **monotone** (either always increasing or always decreasing).

Reading: Chapters 2, 3, and 4.

Day 18

Activity: **Exam 2.**

This second exam is on Derivatives and Applications, Chapters 2, 3, and 4. Some of the questions may be multiple choice. Others may require you to show your worked out solution.

Reading: Section 5.3.

Day 19

Activity: Exploring Areas and Integrals.

Obviously using smaller and smaller intervals is tedious work by hand. Fortunately we have a calculator command that saves us. **fnInt(** (**MATH 9**) accomplishes the task for us. Keep in mind that this command calculates the **integral**, not necessarily the **area** (due to the sign on the *y*-values). Note the implications: There **is** a difference between positive and negative values on the integral. If we want **area** we must keep track separately of regions above and below the *x*-axis.

Activity 1: Finding areas under curves.

For each of the following functions, find the area indicated.

1) The area bounded between $f(x) = x(x-1)^3 + 1$, $y = 0$, $x = 0$, and $x = 2$.

2) The area enclosed between $g(x) = xe^{-x}$ and $h(x) = -xe^{-x}$, between $x = 0$ and $x = 5$.

3) The area between the *x*-axis and $f(x) = \dfrac{1}{(x-3)(x+2)}$ between $x = 0$ and $x = 2$. Note: make sure you know how to enter this formula correctly into the calculator. The point $\left(1, -\frac{1}{6}\right)$ should be on the graph, not $(1, -1.5)$.

Activity 2: Heart pumping rate.

If $r(t)$ represents the rate at which the heart is pumping blood, in liters per second, and t is time in seconds, give the units and meaning of the following integral: $\int_0^{10} r(t)dt$. Even though we don't know the equation of $r(t)$, graph a generic sketch of this integral.

Activity 3: Growth of a population.

Assume $f(t) = 60\sqrt{t}$ gives the rate of change of the population of a city, in people per year, at time t years since 2000. If the population of the city is 5,000 people in 2000, what is the population in 2009?

In the same menu as **nDeriv(** is our chief tool for integration: **fnInt(** (**MATH 9**). The syntax is **fnInt(expression, variable, start, end)**. **expression** is the formula for the derivative that we want the (signed) area underneath, **variable** is usually **x**, just as in **nDeriv(**, and **start** and **end** are the boundaries of the interval we want. We can also access this function from the **CALC** menu while on the graphing screen (**CALC 7** or **∫f(x)dx**).

Goals: Know the graphical interpretation of the definite integral.

Skills:

- **Be able to use integrals to find areas bounded by curves.** Areas can be calculated using integrals. However, you must be aware that integrals can be negative, if the function is negative. So to find areas, we must ensure that all functions are positive. If we have to, we multiply by -1 to make a function positive. This amounts to adding a minus sign to an integral to find the corresponding area. If we are dealing with the area between two curves, we subtract the lower curve from the higher curve, and the resulting integral is the area between them. If they cross and therefore switch roles, we reverse the subtraction.

- **Know the calculator commands to find areas.** We can calculate definite integrals (or areas under curves) with **fnInt(** or **∫f(x)dx**. **fnInt(** requires proper syntax while **∫f(x)dx** requires the area be currently on the graphing window.

Reading: Sections 5.4 and 5.5.

Day 20

Activity: Interpret the Fundamental Theorem of Calculus in real world settings. **Quiz 6 today.**

The Fundamental Theorem of Calculus lets us talk about accumulated change of a function using its derivative information. This is pretty much what we have been doing the last few sessions. Today we will work on some examples where we make sure we're putting the information in context. I will start with a hypothetical bicycle trip (Problem 26 page 267). Then you will work on several problems yourself.

The Fundamental Theorem of Calculus relates derivatives and integrals. To find the accumulated area under a curve, we can find the values directly with Riemann sums. The difference between two accumulated sums can be interpreted as the definite integral over an interval.

Activity 1: Bicycle trip.

Problem 30 page 261.

Activity 2: Balloon flight.

Problem 38 page 262.

Activity 3: Theater line.

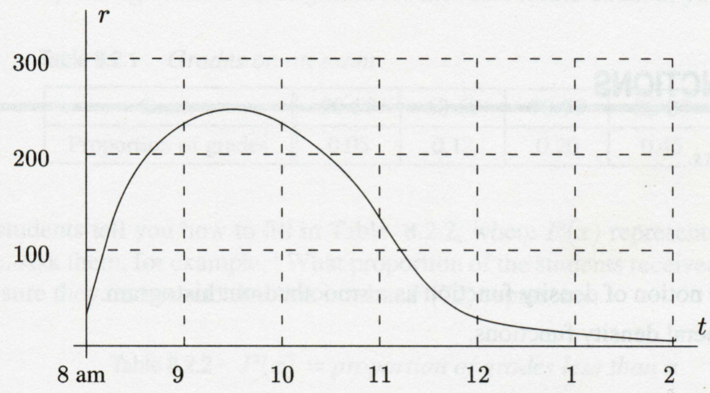

Figure 7.4.1: Rate of arrival at theater

Here is the graph of the rate (in arrivals per hour) at which patrons arrive at the theater to get rush seats for the evening performance. The first people arrive at 8 a.m. and the ticket windows open at 9 a.m. Suppose that once the windows open, people can be served at an average rate of 200 per hour. Use the graph to approximate:

1) The length of the line at 9 a.m. when the windows open.

2) The length of the line at 10 a.m. and 11 a.m.

3) The rate at which the line is growing at 10 a.m.

4) The time when the line is longest.

5) The length of time a person who arrives at 9 a.m. has to stand in line.

6) The time the line disappears.

7) Suppose you were given a formula for r in terms of t. Explain how you would answer the above.

Goals: Using the Fundamental Theorem of Calculus in real world settings.

Skills:

- **Know the Fundamental Theorem of Calculus.** The Fundamental Theorem of Calculus relates integral and derivative as inverses. To find the integral, we use the derivative, but for a function we might not yet know. Fortunately, the integral can be interpreted as an area, so we don't need to know the original function explicitly if we can approximate it using areas.

- **Be able to approximate areas under curves using graphs.** In the examples today we calculated areas given graphs. This is usually best accomplished with a suitable grid on graph paper, and counting boxes. But if other approximations work (like triangles) you should use them. The important part is being able to find a good answer for how much area is bounded by the curve. Later (Chapter 7) we will focus more on formulas.

Reading: Sections 6.1 to 6.3.

Day 21

Activity: Examples using integrals. **Homework 6 due today.**

We will look at two examples today, one a general use of integrals and one from economics.

If we could replace a function over an interval with a constant, so that the areas are equal, then we would have the **average value** over that interval. The key idea is that the areas are equal. Because a constant function makes a rectangular area, all we need to calculate average area is the width of the interval and the area (the definite integral). In Activity 1 you will practice this rephrasing.

The second examples are in economics. **Consumer surplus** is the amount of money **not** spent that would have been spent at higher prices. This is different for each consumer, as there are many different “demand” levels. So, for each price level, we determine how much money was “saved” from the actual price versus the willing price, as determined by the demand curve, and total this over all prices (down to the current price). Similarly, we can figure a **producer surplus**, but using the supply curve. Again the reasoning is that if the price were lower, fewer items would be made, and therefore sold. It is important to note that at equilibrium, both producers **and** consumers are “gaining” from the transaction.

Geometrically, the consumer surplus is the area bounded by the price (horizontal line) and the demand curve (integral area if the demand curve isn’t linear). The producer surplus is the area below the price line bounded by the supply curve.

The interesting work comes when we (perhaps the government) impose non-equilibrium prices. What effect does this have on the economic interpretations? We will explore this idea in Activity 2. In class, I will work on Problem 10 page 285.

Our last economic example today concerns income streams. Any process that can be summed can be approximated with an integral. The idea behind the Reimann sums that form our integrals is the shrinking rectangle widths. An analogous situation is the compounding period for interest in annuities. We have looked at continuous compounding, and this corresponds to the definite integral. Today we will look at adding up a continuous income stream.

Activity 1: Average Value.

Is the average of the maximum and minimum over an interval equal to the average value over the interval? Work Problems 12 and 18 on page 279, which address this.

Activity 2: Consumer and Producer surplus.

Problem 9, page 285. I recommend graphing the curves in addition to using **fnInt(**.

Now suppose a price **greater** than equilibrium is imposed. (Invent one.) Calculate the change in the two surpluses.

Now suppose a price **lower** than equilibrium is imposed. (Invent one.) Calculate the change in the two surpluses.

Activity 3: Understanding continuous income streams.

Compare several income streams, adding up the total income each time. Then calculate the definite integral, using the continuous compound interest formula.

Goals: Use integrals in economics settings.

Skills:

- **Know the Average Value of a function over an interval.** Graphically, we can interpret the definite integral as the area of a rectangle over an interval. The height of this rectangle represents the **average value** of the function over the interval.

- **Understand the Consumer and Produce Surplus examples.** Equilibrium price is lower than many consumers are willing to pay. The difference between what they **would** have paid and what they **are** paying is called the consumer surplus. Similarly, the equilibrium price is higher than many producers are willing to produce. The difference in the equilibrium price and the supplier's willing price is the producer surplus.

- **Know how to calculate the total income given an income stream.** Because the definite integral is akin to a sum, we can use it to find sums with continuous compound interest problems.

Reading: Sections 7.1 and 7.2.

Day 22

Activity: Antiderivatives. Integration by Substitution. **Quiz 7 today.**

We have explored how to interpret definite integrals. The techniques we've been using involve estimating areas under curves. The Fundamental Theorem of Calculus guided us, but it also shows us another approach to the solution, if we have a formula for the rate function (the derivative). The FTC says all we have to do is come up with a formula whose derivative is the formula we've started with. This sounds easier than it often is.

However, when such a formula **does** exist, the solution to a definite integral is then simply the difference of two values in this new function, which, because it is an inverse function, is called an **antiderivative**. It is important to note right away that antiderivatives are not unique functions. We know from Chapter 3 that when we add a constant to a function we don't change the derivative at all. So there are **many** antiderivatives to any problem, but they only differ by adding a different constant.

To find antiderivatives we need to recognize a few features of the rules we learned in Chapter 3. First, derivatives add together, so we can work on each part in a sum separately. Second, derivatives of power functions are themselves power functions. So we just need to work backwards. Exponential functions are also their own derivatives. Composed functions are another matter. They may or may not have simple solutions. We will use substitution to see if we can discover the answers to them.

Composed functions do not have simple antiderivatives. We will use substitution to see if we can discover their antiderivatives. But substitution doesn't always work; we must have functions that match the chain rule exactly. Substitution gives us a chance at least. It might require trial and error to find the right substitution to make. The strategy is to try w as an "inner" function; then $dw = w'dx$. Replacing what we can, we see if we have made the problem into something simpler and solvable. I will work problems in class to show you the gist of it. Then you will practice with both the simple functions (Activity 1) and the composed functions (Activity 2).

Activity 1: Working "backwards".

Work as many of the problems on page 304 as you can.

Activity 2: Integration by Substitution.

Work as many of the problems on page 308 as you can.

Goals: Realize that antiderivatives are the inverses of derivatives. Know how to do integration by substitution.

Skills:

- **Realize that an antiderivative is a function whose derivative is the original expression.** The Fundamental Theorem of Calculus shows us that accumulated change in a function is an area under the derivative curve. Conversely, if we know the original function's formula, we can simply subtract two values to find the definite integral.
- **Know how to find antiderivatives of simple functions.** Power functions, exponential functions, constants, and the reciprocal function $1/x$ are all simple functions that have simple antiderivatives. Section 7.1 enumerates them in the various boxed formulas.
- **Know how to find antiderivatives using substitution.** The antiderivatives for composed functions can sometimes be found using substitution. This technique only works if the derivative of the proposed substitution appears in the formula in just the right way. If the substitution is chosen well, then the problem after substitution will be of a simpler nature.

Reading: Sections 7.3 and 7.4.

Day 23

Activity: Analyzing Antiderivatives. **Homework 7 due today.**

Definite integrals, as we have seen, are specific areas under a curve. If we have a formula for the derivative that we can find an antiderivative for, we can use the Fundamental Theorem of Calculus to find the area exactly. Today we will do work very similar to yesterday's work, but we will move on to do the actual subtractions specified by the FTC.

In addition to finite integrals, we can also try our hand at **improper integrals**, or those with infinity in either integrand. These problems will be solved with limits, and therefore may be tricky to conclude convergence with (recall the harmonic series of Day 1). However, the FTC saves us, if we are able to evaluate the antiderivative as x approaches infinity.

We will also revisit an activity from Chapter 2 (Day 9), and notice (hopefully) how much simpler the last part is now that we know about integration.

Activity 1: Evaluating definite integrals exactly using the FTC.

For each of the following functions, find an antiderivative. Then, evaluate the definite integral using your calculator (`fnInt`), and by using the Fundamental Theorem of Calculus. Compare answers. Which one is "right" and which one is only an approximation?

1) $\int x(x+1)^{10}\,dx$. Integrate from 0 to 10.

2) $\int_1^{\infty} \frac{1}{\sqrt{x}}dx$. Note: you will have to decide how to "con" your calculator into doing infinity.

3) $\int_1^{\infty} \frac{1}{x}dx$. This integral is one way to prove that the harmonic series diverges.

Activity 2: Estimating the derivative using a graph, and translating back.

Our next activity is a repeat of what we did on Day 9. It will take less time than before, hopefully.

Each of you will sketch an arbitrary function on a piece of paper, labeling it "Original Curve" and putting your name on it. You will then pass your graph to someone else; they will graph the derivative function on a separate sheet of paper, labeled with "Derivative Curve for <insert name here>". The person drawing the derivative will have to carefully estimate the slopes, so a scale is needed. Finally, the second person will pass the derivative graph to a third person (keep the original aside to compare with later); the third person will attempt to redraw the original graph based solely on the information from the derivative graph. Caution: this last part is tricky.

Note that the last part isn't nearly as tricky now as it was on Day 9; we now have the FTC to guide us in exactly how much to make the original graph rise or fall.

Goals: Calculate and interpret integrals using antiderivatives.

Skills:

- **Evaluate definite integrals using both antiderivatives and the TI-83.** The definite integral can be calculated with antiderivatives (using the Fundamental Theorem of Calculus) or by numerical methods (using **fnInt** on the TI-83). You can also use **CALC 7** (**∫f(x)dx**) on the graphing window.

- **Be able to draw an antiderivative given a derivative graph.** On Day 9 it was difficult during our exercise to reconstruct the original function from the derivative graph, because we didn't know how much to increase or decrease the function just knowing the derivative. Now, after studying the FTC, we know the **area** is the important missing factor. With this knowledge, you should now be able to draw accurate **antiderivatives**, as this is really what they are.

- **Be able to calculate an improper integral.** An improper integral involves infinity as one of the integrands. Therefore, to evaluate an improper integral exactly, we must use the FTC and some limit ideas.

Reading: Sections 9.1 and 9.2.

Day 24

Activity: Introduction to Multivariate Functions. **Quiz 8 today.**

The real world is rarely explained by simple one-variable functions. Everything depends on everything else. The complexity is sometimes daunting. However, we can try to model things with mathematical formulas, and these often prove useful. For example, we know that the amount of money in a bank account can be represented by the formula $B = Pe^{rt}$. We can view B as a function of three variables: P, r, and t. Of course, in the real world the account balance **won't** always be predicted by this formula unless the account is left completely alone, and the bank doesn't close the account. In Activity 1, we will explore how to describe a multi-variable function with a table.

Graphically, we view multivariate functions by holding all but two variables constant, and then graphing the remaining two variables using plotting techniques we already know. Because we view the dependent variable differently than the independent variables, the techniques fall into two basic types.

Cross-sections occur when the dependent variable is one of the two variables we graph. In a three-dimensional setting, we can imagine we have "sliced" the surface vertically and are looking at the surface from the side, in a cross-section. If we line up a series of cross-sections, we may be able to visualize the three-dimensional surface accurately. Cross-sections can be done from any dimension, as long as the dependent variable is on the vertical axis. Activity 2 today will give us some practice with graphing cross-sections.

Contours occur in a three-dimensional surface when the dependent variable is held constant, and the other two variables are graphed. Due to the nature of functions, cross-sections will always create graphable formulas, but contours may result in something quite difficult to create. For example, it's not at all clear when we begin what values to use for the dependent variable. Common uses for contours are maps. You have seen weather maps that highlight temperatures. Instead of simply showing the isotherms (lines of equal temperatures) color is commonly used. In Activity 3 today we will explore contours in more detail.

After you work on the three activities today, I will explore $z = xy^3 + x^2$. This function is tougher than the one you're working on, but we should see all the same issues.

Activity 1: Describing a multivariate function with a table.

In your groups, create tables of values for this two-variable function: $B = Pe^{.02t}$. The goal is to convey to a reader what the various values of B might be. I will let each group decide how to make the table; we will compare among groups to see if you chose similar methods.

Is B an increasing or a decreasing function?

Activity 2: Describing a multivariate function with cross-sections.

Using $B = Pe^{.02t}$ hold P constant (choose some values) and draw the resulting B vs. t graphs. Then repeat holding t constant and drawing the B vs. P graphs. Do they give you the same impression of the surface? Is it the same as the impression you got in Activity 1?

Activity 3: Describing a multivariate function with contours.

In practice, you will most likely not be producing contours. More often you will interpret them. But we want to be able to produce contours for simpler functions. Again, using $B = Pe^{.02t}$, create some contours. You will need to choose some values of B to make the contours for. It is not always clear what values will make the most sense. Trial and error may be in order. Does this contour graph give you the same impression that you got in Activities 1 and 2?

When graphing cross-sections or contours, we may want to graph a whole series of values for x, or z. In our calculators, we can replace the variable with a list that will accomplish this for us, saving a lot of typing. For example, if we wanted to graph $y = z^2 - x - 3$ for z = -20, -10, 0, 10, and 20, we can enter this: **Y1={-20,-10,0,10,20}^2-x-3**. The calculator will graph first $y = 400 - x - 3$, then $y = 100 - x - 3$, etc. One drawback to this approach is that when you **TRACE**, you won't know which value in the list the curve represents.

Goals: Introduce multivariate functions. Explore tables, cross-sections, and contours as ways to view multivariate functions.

Skills:

- **Understand how to represent a multivariate function with a table of values.** Tables can describe multivariate functions, but they are not as good as graphs. On the other hand, graphs can be difficult to produce or interpret, and sometimes having the raw numbers is better. The best approach is to have a formula, but many real world settings don't yield known formulas (daily highs across the country is one example).

- **Be able to produce cross-sections for a multivariate function.** To make a cross-section of a multivariate function, hold all but one of the independent variables constant; then graph the dependent variable versus that last independent variable. Naturally, if there are many independent variables held constant, it will be difficult to visualize the entire surface. In the three-dimensional case, we can think of this approach as vertical "slices" of the surface, viewed from the side.

- **Be able to read and interpret contours for a multivariate function.** Contour diagrams are views from above, basically. Imagine looking down on the surface, in the case of three dimensions. Contours represent horizontal "slices". Contours may be difficult to produce, as the curves traced out may not be functions at all (for example: circles at a relative maximum).

Reading: Sections 9.3 and 9.4.

Day 25

Activity: Calculating Partial Derivatives. **Homework 8 due today.**

Just as for one-variable functions, we can talk about derivatives with multivariate functions. Basically, we will let one of the variables remain constant and explore how the other variable changes. This technique is called **partial derivatives**. All that we know about derivatives from earlier chapters apply here. One new aspect is that there are **several** possible derivatives. We also use different notations (see page 361).

There is another new idea about derivatives that we haven't encountered before: the mixed second partial derivative. The regular second partial derivatives measure concavity, the same as the one-variable second derivatives. But when we take the mixed second partial derivative (see page 370) we are really estimating how the change in one direction changes as we move in the **other** direction. We are more estimating a kind of "twisting" in the surface. We will make more use of this on Day 26 when we classify the extrema.

Activity 1: Calculate partial derivatives from tabular data.

Problem 8 page 366.

Activity 2: Calculate partial derivatives from graphs, both cross-sections and contours.

Using your graphs from $B = Pe^{.02t}$ from Activities 2 and 3 on Day 24, estimate some values of the partial derivatives.

Activity 3: Calculate partial derivative formulas.

Verify your answers in Activity 2 using algebra. Then work on a few of problems 26 to 37 on page 372.

Goals: Calculate partial derivatives from tabular data, from graphs (both cross-sections and contours), and from formulas.

Skills:

- **Be able to estimate partial derivatives from tabular data.** Calculating a derivative from a table in two or more dimensions is no different than it was in Chapter 2. We use slopes of secant lines, and due to the nature of tabular data, we can only "zoom in" so much. The only trick is to pay attention to which variable is being held constant.

- **Be able to estimate partial derivatives from graphs.** From cross-section graphs, we can estimate the partial derivative for that variable in just the same way as in Chapters 2 and 3. For the contour graphs, we must use a different approach. Typically, we will estimate the difference between two contours, and express the ratio of the change in contours to distance between contours as the derivative in that direction.

- **Be able to calculate partial derivatives from formulas.** Using the formulas from Chapter 4, we can calculate partial derivatives exactly. The only difficulty is keeping track of which variable is allowed to vary; our notation is intended to remind us of this (see page 361).

Reading: Section 9.5.

Day 26

Activity: Multivariate Optimization. **Quiz 9 today.**

To maximize or minimize a multivariate function, we use the same criteria we did for one-variable function: critical points and the second derivative test. The details are slightly different for the second derivative test, and we will do several problems today practicing this technique. First we solve the first derivatives jointly by setting them to zero. This will give us candidates for extrema. Now we use the second derivative test (page 376) to help classify the candidates as maxima, minima, or neither. It is also possible the test is inconclusive. In those situations, we must use some other approach, perhaps something akin to the first derivative test, although using that approach is a bit trickier in multiple dimensions. Note: solving the first derivative formulas simultaneously for all variables present may be very difficult. One special case is when all the derivatives are **linear**. Then you can use techniques from MATH 204 (the linear algebra/matrix results).

In addition to using formulas, make sure you can use graphs also to find extrema. Problems 1 and 2, and 14 to 16 on page 377 are good practice. A new sort of critical point occurs in multiple dimensions called a **saddle point**. You can think of a mountain pass as one example; in one direction (going over the pass) the function is a maximum but in the other direction (going from one mountain to the other through the pass) the function is a minimum. The second derivative test will classify these saddle points as "neither".

Activity 1: Optimizing multivariate functions.

I will work problems 3 and 6 in class. Try as many of the others as you can. Problems 3 through 12 page 377.

Goals: Understand how the derivatives can be used to find the extrema in multivariate functions.

Skills:

- **Be able to find the extrema using a contour graph.** Extrema on contour graphs are represented with closed loops. To find whether they are maxima or minima entails paying attention to the values of the contours around the points.

- **Be able to find extrema using algebra.** Using the second derivative test, you should be able to classify the extrema as maxima, minima, or neither. In some cases, the second derivative test is inconclusive.

- **Understand the saddle point in multivariate functions.** In one direction, a saddle point is a maximum, but in another direction it is a minimum. If we are trying to optimize a function, it is critical to know if our critical points are maxima, minima, saddle points, ridges, etc.

Reading: Chapters 5, 6, 7, and 9.

Day 27

Activity: **Presentation 3. Homework 9 due today.**

> Pick one of these functions (first come, first served): 1) $z = x^2 + 3y^2 - 4x + 6y + 10$, 2) $z = x^3 - 3x + y^2$, 3) $z = xy^2e^{-x}$, 4) $z = x^3 + y + xy$, 5) $z = (x - y)^3$, 6) $z = 1/\left(x^2 + y^2 + 1\right)$, 7) $z = x(1 + y) + y^2$. Graph some cross-sections, a contour graph, and classify the critical points. Show how your critical points appear in each of your displays.

Reading: Chapters 5, 6, 7, and 9.

Day 28

Activity: **Exam 3.**

> This last exam covers integrals, including antiderivatives, and multivariate functions, Chapters 5, 6, 7, and 9. Some of the questions may be multiple choice. Others may require you to show your worked out solution.

Populations for the 50 states, DC, and the USA, by decade.
(in thousands)

	AL	AK	AZ	AR	CA	CO	CT	DE	DC	FL	GA	HI	ID	IL	IN	IA	KS	KY
1790							238	59			83							74
1800	1						251	64	8		163				6			221
1810	9			1			262	73	16		252			12	25			407
1820	128			14			275	73	23		341			55	147			564
1830	310			30			298	77	30	35	517			157	343			688
1840	591			98			310	78	34	54	691			476	686	43		780
1850	772			210	93		371	92	52	87	906			851	988	192		982
1860	964			435	380	34	460	112	75	140	1057			1712	1350	675	107	1156
1870	997		10	484	560	40	537	125	132	188	1184		15	2540	1680	1194	364	1321
1880	1263	33	40	803	865	194	623	147	178	269	1542		33	3078	1978	1625	996	1649

1890	1513	32	88	1128	1213	413	746	168	230	391	1837		89	3826	2192	1912	1428	1859
1900	1829	64	123	1312	1485	540	908	185	279	529	2216	154	162	4822	2516	2232	1470	2147
1910	2138	64	204	1574	2378	799	1115	202	331	753	2609	192	326	5639	2701	2225	1691	2290
1920	2348	55	334	1752	3427	940	1381	223	438	968	2896	256	432	6485	2930	2404	1769	2417
1930	2646	59	436	1854	5677	1036	1607	238	487	1468	2909	368	445	7631	3239	2471	1881	2615
1940	2833	73	499	1949	6907	1123	1709	267	663	1897	3124	423	525	7897	3428	2538	1801	2846
1950	3062	129	750	1910	10586	1325	2007	318	802	2771	3445	500	589	8712	3934	2621	1905	2945
1960	3267	226	1302	1786	15717	1754	2535	446	764	4952	3943	633	667	10081	4662	2758	2179	3038
1970	3444	303	1775	1923	19971	2210	3032	548	757	6791	4588	770	713	11110	5195	2825	2249	3221
1980	3894	402	2717	2286	23668	2890	3108	594	638	9747	5463	965	944	11427	5490	2914	2364	3660
1990	4040	550	3665	2351	29760	3294	3287	666	607	12938	6478	1108	1007	11430	5544	2777	2478	3685
2000	4447	627	5131	2673	33872	4301	3406	784	572	15982	8186	1212	1294	12419	6080	2926	2688	4042
2010	4780	710	6392	2916	37254	5029	3574	898	602	18801	9688	1360	1568	12831	6484	3046	2853	4339

	LA	ME	MD	MA	MI	MN	MS	MO	MT	NE	NV	NH	NJ	NM	NY	NC	ND	OH
1790		97	320	379								142	184		340	394		
1800		152	342	423			8					184	211		589	478		45
1810	77	229	381	472	5		31	20				214	246		959	556		231
1820	153	298	407	523	9		75	67				244	278		1373	639		581
1830	216	399	447	610	32		137	140				269	321		1919	736		938
1840	352	502	470	738	212		376	384				285	373		2429	753		1519
1850	518	583	583	995	398	6	607	682				318	490	62	3097	869		1980
1860	708	628	687	1231	749	172	791	1182		29	7	326	672	94	3881	993		2340
1870	727	627	781	1457	1184	440	828	1721	21	123	42	318	906	92	4383	1071	2	2665
1880	940	649	935	1783	1637	781	1132	2168	39	452	62	347	1131	120	5083	1400	37	3198
1890	1119	661	1042	2239	2094	1310	1290	2679	143	1063	47	377	1445	160	6003	1618	191	3672
1900	1382	694	1188	2805	2421	1751	1551	3107	243	1066	42	412	1884	195	7269	1894	319	4158
1910	1656	742	1295	3366	2810	2076	1797	3293	376	1192	82	431	2537	327	9114	2206	577	4767
1920	1799	768	1450	3852	3668	2387	1791	3404	549	1296	77	443	3156	360	10385	2559	647	5759
1930	2102	797	1632	4250	4842	2564	2010	3629	538	1378	91	465	4041	423	12588	3170	681	6647
1940	2364	847	1821	4317	5256	2792	2184	3785	559	1316	110	492	4160	532	13479	3572	642	6908
1950	2684	914	2343	4691	6372	2982	2179	3955	591	1326	160	533	4835	681	14830	4062	620	7947
1960	3257	969	3101	5149	7823	3414	2178	4320	675	1411	285	607	6067	951	16782	4556	632	9706
1970	3645	994	3924	5689	8882	3806	2217	4678	694	1485	489	738	7171	1017	18241	5084	618	10657
1980	4206	1125	4217	5737	9262	4076	2521	4917	787	1570	801	921	7365	1303	17558	5880	653	10798
1990	4220	1228	4781	6016	9295	4375	2573	5117	799	1578	1202	1109	7730	1515	17990	6629	639	10847
2000	4469	1275	5296	6349	9938	4919	2845	5595	902	1711	1998	1236	8414	1819	18976	8049	642	11353
2010	4533	1328	5774	6548	9884	5304	2967	5989	989	1826	2701	1316	8792	2059	19378	9535	673	11537

	OK	OR	PA	RI	SC	SD	TN	TX	UT	VT	VA	WA	WV	WI	WY	USA
1790			434	69	249		36			85	692		56			3929
1800			602	69	346		106			154	808		79			5308
1810			810	77	415		262			218	878		105			7240
1820			1049	83	503		423			236	938		137			9638
1830			1348	97	581		682			281	1044		177			12866
1840			1724	109	594		829			292	1025		225	31		17069
1850		12	2312	148	669		1003	213	11	314	1119	1	302	305		23192
1860		52	2906	175	704	5	1110	604	40	315	1220	12	377	776		31443
1870		91	3522	217	706	12	1259	819	87	331	1225	24	442	1055	9	38558
1880		175	4283	277	996	98	1542	1592	144	332	1513	75	618	1315	21	50189
1890	259	318	5258	346	1151	349	1768	2236	211	332	1656	357	763	1693	63	62980
1900	790	414	6302	429	1340	402	2021	3049	277	344	1854	518	959	2069	93	76212
1910	1657	673	7665	543	1515	584	2185	3897	373	356	2062	1142	1221	2334	146	92228
1920	2028	783	8720	604	1684	637	2338	4663	449	352	2309	1357	1464	2632	194	106022
1930	2396	954	9631	687	1739	693	2617	5825	508	360	2422	1563	1729	2939	226	123203
1940	2336	1090	9900	713	1900	643	2916	6415	550	359	2678	1736	1902	3138	251	132165
1950	2233	1521	10498	792	2117	653	3292	7711	689	378	3319	2379	2006	3435	291	151326
1960	2328	1769	11319	859	2383	681	3567	9580	891	390	3967	2853	1860	3952	330	179323
1970	2559	2092	11801	950	2591	666	3926	11199	1059	445	4651	3413	1744	4418	332	203302
1980	3025	2633	11865	947	3121	691	4591	14226	1461	511	5347	4132	1950	4706	470	226542
1990	3146	2842	11882	1003	3487	696	4877	16987	1723	563	6187	4867	1793	4892	454	248710
2000	3451	3421	12281	1048	4012	755	5689	20852	2233	609	7079	5894	1808	5364	494	281422
2010	3751	3831	12702	1053	4625	814	6346	25146	2764	626	8001	6725	1853	5687	564	308746

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