1 Fixed points and Stability

Given $x_{n+1} = f(x_n)$ then x^* is a fixed point if $f(x^*) = x^*$.

STABILITY: Consider a small perturbation of x^* :

$$\eta_n = x_n - x^*$$

then $x_{n+1} = f(x_n) \Longrightarrow x^* + \eta_{n+1} = f(x^* + \eta_n)$

$$x^* + \eta_{n+1} = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

The linearized map near $x^*$is

$$\eta_{n+1} = f'(x^*)\eta_n$$

The eigenvalue or multiplier is $\lambda = f'(x^*)$. So

$\eta_1 = \lambda\eta_0$, $\eta_2 = \lambda^2\eta_0$, $\eta_3 = \lambda^3\eta_0, \cdots$, $\eta_n = \lambda^n\eta_0$

- If $|\lambda| = |f'(x^*)| < 1$: $\eta_n \longrightarrow 0$ as $n \longrightarrow \infty$, i.e.,

$x_n \longrightarrow x^*$. x^* is linearly stable.

- If $|\lambda| = |f'(x^*)| > 1$: $|\eta_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, i.e.,

x^* is unstable.

- If $|\lambda| = |f'(x^*)| = 1$: marginal case

- If $|\lambda| = |f'(x^*)| = 0$: $\eta_{n+1} \sim \eta_n^2$, we get quadratic convergence, x^* is superstable.

EXAMPLE: $x^2 - x - 2 = 0 \Longrightarrow (x-2)(x+1) = 0$, near the root $x = 2$

Possible iterations:

$$x_{n+1} = f(x_n) = x_n^2 - 2 \qquad f'(x) = 2x,\ f'(2) = 4 > 1$$

unstable, no convergence.

$$x_{n+1} = f(x_n) = \sqrt{x_n + 2} \qquad f'(x) = \frac{1}{2\sqrt{x+2}}, f'(2) = 1/4$$

monotonic convergence.

$$x_{n+1} = f(x_n) = 1 + 2/x_n \qquad f'(x) = -2/x^2, f'(2) = -1/2$$

oscillatory convergence.

$$x_{n+1} = f(x_n) = x_n - \frac{x_n^2 - x_n - 2}{2x_n - 1}$$

superstable.

Cobwebbs: Example : $x_{n+1} = \cos(x_n)$

For any initial condition, $x_n \longrightarrow x^*$ where $x^* = 0.739085...$

The unique root of $x - \cos x = 0$.

The multiplier is $|f'(x^*)| = |-\sin x^*| < 1 \Longrightarrow$ stable

Convergence through damped oscillations.

2 Logistic Map

$x_{n+1} = f(x_n) = rx_n(1 - x_n)$

The graph of f is a parabola. Since $f'(x) = r(1 - 2x)$

the maximum of f is at $(1/2, r/4)$.

So we restrict our attention to $0 \leq r \leq 4$: then

$f : [0,1] \longrightarrow [0,1]$, i.e., f maps unit interval into itself.

FIXED POINTS: Solve $x = f(x) = rx(1 - x)$

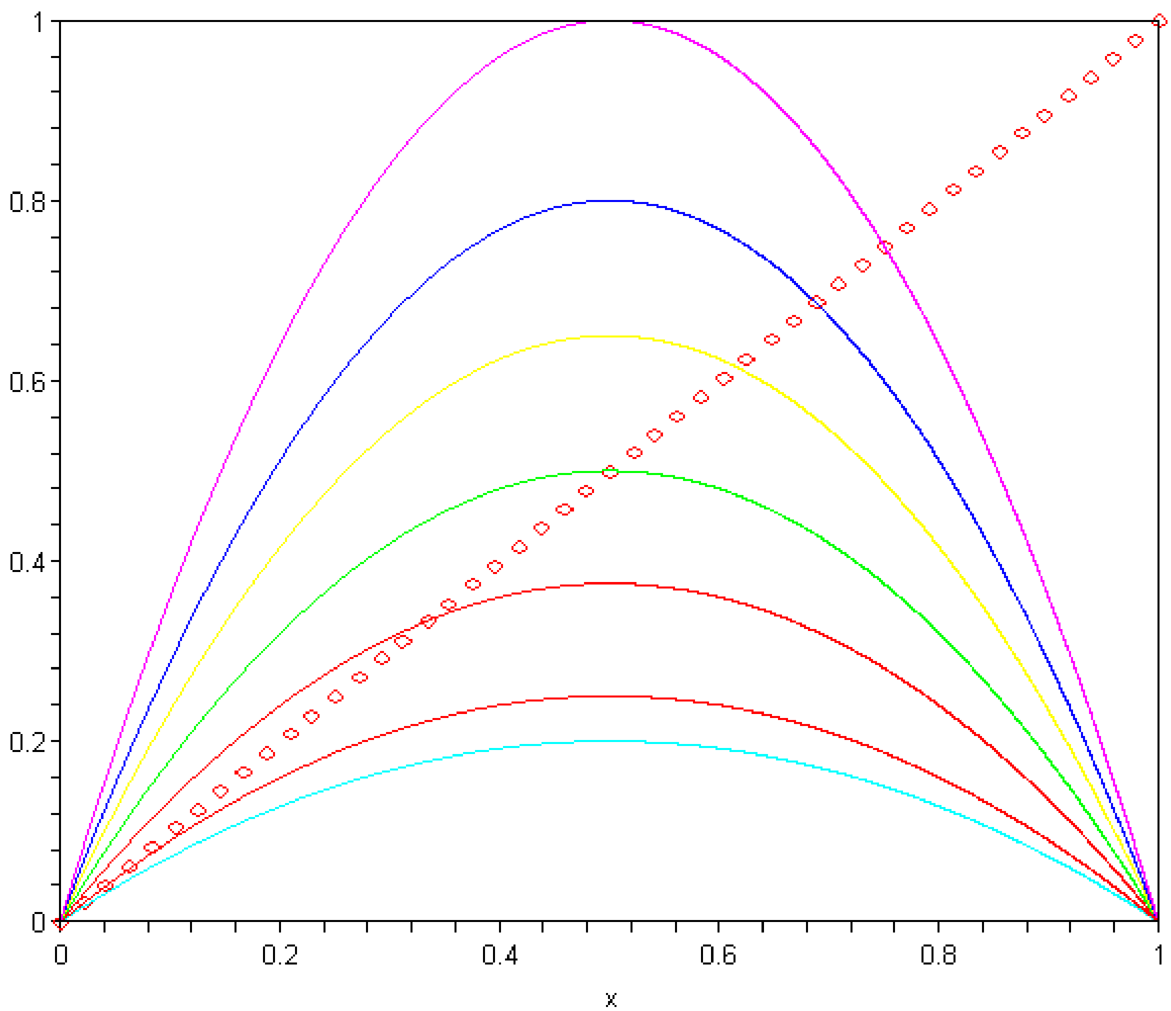
$(r-1)x - rx^2 = 0 \Longrightarrow x^* = 0$ or $x^* = 1 - \dfrac{1}{r} \in [0,1]$ for $r \geq 1$.

STABILITY: $x^* = 0$

$$f'(0) = r \Longrightarrow \begin{cases} \text{stable ,} & 0 \leq r < 1 \\ \text{unstable ,} & r > 1 \end{cases}$$

At $r = 1$, $f(x) = x - x^2 < x$ for $0 < x \leq 1 \Longrightarrow x^* = 0$ is stable.

$x^* = 0$ becomes unstable through a transcritical bifurcation at $r = 1$.



STABILITY: $x^* = 1 - \dfrac{1}{r}$

$$f'(1 - \frac{1}{r}) = r(1 - 2(1 - 1/r)) = 2 - r$$

So $x^* = 1 - \dfrac{1}{r}$ is stable if $|2 - r| < 1$ or $1 < r < 3$

(note superstable when $r = 2$).

and $x^* = 1 - \dfrac{1}{r}$ is unstable if $r > 3$.

What happens for $r > 3$?

For example if $r = 3.2$, iterates oscillate between two values x_1, x_2

such that $x_1 < x^* < x_2$. We get a period 2 cycle or a 2-cycle.

We say that we have a **period doubling bifurcation** at $r = 3$.

For larger r : for example $r = 3.5$ iterates oscillate between four values, so we get a 4-cycle (we get another period doubling)

There will be further period doublings to period 8, 16, 32, 64,...

Let r_n be the value of r at which a 2^n-cycle first appears

Numerically

r	cycle
$r_1 = 3$	2
$r_2 = 1 + \sqrt{6}$	4
$r_3 = 3.54409...$	8
$r_4 = 3.5644...$	16
$r_5 = 3.568759...$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946...$	∞

Successive bifurcations occur faster and faster and $r_n \longrightarrow r_\infty$ as $n \longrightarrow \infty$.

Convergence is geometric:

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669... \text{Feigenbaum's constant}$$

For $r > r_\infty$: **CHAOS**

- Aperiodic long-term dynamics

- Sequence $\{x_n\}$ never settles down to a fixed point or periodic orbit

- Sensitive dependence on initial conditions: two trajectories starting close together rapidly diverge from each other.

($\Longrightarrow$ long-term prediction impossible: small uncertainties are amplified exponentially fast)

- Irregularity due to nonlinearity, not noise.

- Chaos: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Let us do some Maple to check the previous statements.

First we will graph some iterates for $r = 0.8, 1.5, 2.0, 2.8, 3.10,$

$3.2, 3.5, 3.55, 3.565, 3.6, 3.84, 4.0.$

Then we draw some cobweb diagrams for the logistic map.