1 Fixed points and Stability

Given $x_{n+1} = f(x_n)$ then $x^*$ is a fixed point if $f(x^*) = x^*$.

**STABILITY**: Consider a small perturbation of $x^*$:

$$\eta_n = x_n - x^*$$

then $x_{n+1} = f(x_n) \implies x^* + \eta_{n+1} = f(x^* + \eta_n)$

$$x^* + \eta_{n+1} = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

The linearized map near $x^*$ is

$$\eta_{n+1} = f'(x^*)\eta_n$$

The eigenvalue or multiplier is $\lambda = f'(x^*)$. So

$$\eta_1 = \lambda \eta_0, \eta_2 = \lambda^2 \eta_0, \eta_3 = \lambda^3 \eta_0, \cdots, \eta_n = \lambda^n \eta_0$$

- If $|\lambda| = |f'(x^*)| < 1 : \eta_n \to 0$ as $n \to \infty$, i.e.,
\( x_n \rightarrow x^* \). \( x^* \) is linearly stable.

- If \( |\lambda| = |f'(x^*)| > 1 \): \( |\eta_n| \rightarrow \infty \) as \( n \rightarrow \infty \), i.e., \( x^* \) is unstable.

- If \( |\lambda| = |f'(x^*)| = 1 \): marginal case

- If \( |\lambda| = |f'(x^*)| = 0 \): \( \eta_{n+1} \sim \eta_n^2 \), we get quadratic convergence, \( x^* \) is superstable.

**EXAMPLE:** \( x^2 - x - 2 = 0 \iff (x - 2)(x + 1) = 0 \), near the root \( x = 2 \)

Possible iterations:

\[
x_{n+1} = f(x_n) = x_n^2 - 2 \quad f'(x) = 2x, \quad f'(2) = 4 > 1
\]

unstable, no convergence.
\[ x_{n+1} = f(x_n) = \sqrt{x_n + 2} \quad f'(x) = \frac{1}{2\sqrt{x + 2}}, \quad f'(2) = 1/4 \]

monotonic convergence.

\[ x_{n+1} = f(x_n) = 1 + 2/x_n \quad f'(x) = -2/x^2, \quad f'(2) = -1/2 \]

oscillatory convergence.

\[ x_{n+1} = f(x_n) = x_n - \frac{x_n^2 - x_n - 2}{2x_n - 1} \]

superstable.

Cobwebbs: Example: \( x_{n+1} = \cos(x_n) \)

For any initial condition, \( x_n \rightarrow x^* \) where \( x^* = 0.739085... \)

The unique root of \( x - \cos x = 0 \).

The multiplier is \( |f'(x^*)| = |- \sin x^*| < 1 \implies \) stable

Convergence through damped oscillations.
2 Logistic Map

\[ x_{n+1} = f(x_n) = rx_n(1 - x_n) \]

The graph of \( f \) is a parabola. Since \( f'(x) = r(1 - 2x) \)
the maximum of \( f \) is at \((1/2, r/4)\).

So we restrict our attention to \( 0 \leq r \leq 4 \) : then

\[ f : [0, 1] \longrightarrow [0, 1], \text{ i.e., } f \text{ maps unit interval into itself.} \]

**FIXED POINTS:** Solve \( x = f(x) = rx(1 - x) \)

\[(r - 1)x - rx^2 = 0 \implies x^* = 0 \text{ or } x^* = 1 - \frac{1}{r} \in [0, 1] \text{ for } r \geq 1.\]

**STABILITY:** \( x^* = 0 \)

\[ f'(0) = r \implies \begin{cases} \text{stable,} & 0 \leq r < 1 \\ \text{unstable,} & r > 1 \end{cases} \]
At \( r = 1 \), \( f(x) = x - x^2 < x \) for \( 0 < x \leq 1 \) \( \implies \) \( x^* = 0 \) is stable.

\( x^* = 0 \) becomes unstable through a transcritical bifurcation at \( r = 1 \).

\[
\text{STABILITY: } x^* = 1 - \frac{1}{r}
\]
\[ f'(1 - \frac{1}{r}) = r\left(1 - 2\left(1 - \frac{1}{r}\right)\right) = 2 - r \]

So \( x^* = 1 - \frac{1}{r} \) is stable if \( |2 - r| < 1 \) or \( 1 < r < 3 \)

(note superstable when \( r = 2 \)).

and \( x^* = 1 - \frac{1}{r} \) is unstable if \( r > 3 \).

What happens for \( r > 3 \)?

For example if \( r = 3.2 \), iterates oscillate between two values \( x_1, x_2 \)

such that \( x_1 < x^* < x_2 \). We get a period 2 cycle or a 2-cycle.

We say that we have a **period doubling bifurcation** at \( r = 3 \).

For larger \( r \) : for example \( r = 3.5 \) iterates oscillate between four values, so we get a 4-cycle (we get another period doubling)

There will be further period doublings to period 8, 16, 32, 64,...
Let $r_n$ be the value of $r$ at which a $2^n$-cycle first appears

Numerically

<table>
<thead>
<tr>
<th>$r$</th>
<th>cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 = 3$</td>
<td>2</td>
</tr>
<tr>
<td>$r_2 = 1 + \sqrt{6}$</td>
<td>4</td>
</tr>
<tr>
<td>$r_3 = 3.54409...$</td>
<td>8</td>
</tr>
<tr>
<td>$r_4 = 3.5644...$</td>
<td>16</td>
</tr>
<tr>
<td>$r_5 = 3.568759...$</td>
<td>32</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$r_\infty = 3.569946...$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Successive bifurcations occur faster and faster and $r_n \rightarrow r_\infty$ as $n \rightarrow \infty$.

Convergence is geometric:

$$\delta = \lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669... \text{Feigenbaum's constant}$$

For $r > r_\infty$: **CHAOs**

- Aperiodic long-term dynamics
• Sequence $\{x_n\}$ never settles down to a fixed point or periodic orbit.

• Sensitive dependence on initial conditions: two trajectories starting close together rapidly diverge from each other.

  ($\rightarrow$ long-term prediction impossible: small uncertainties are amplified exponentially fast)

• Irregularity due to nonlinearity, not noise.

• Chaos: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Let us do some Maple to check the previous statements.

First we will graph some iterates for $r = 0.8, 1.5, 2.0, 2.8, 3.10, 3.2, 3.5, 3.55, 3.565, 3.6, 3.84, 4.0$.

Then we draw some cobweb diagrams for the logistic map.