## PROBLEM 1:

Use the linearization theorem to classify the fixed points of
(a) $\dot{x}=y^{2}-3 x+2, \quad \dot{y}=x^{2}-y^{2}$.
(b) $\dot{x}=\sin (x+y), \quad \dot{y}=y$.

## SOLUTION:

(a) The system $y^{2}-3 x+2=0$ and $x^{2}-y^{2}=0$ has for solutions
$(1,1),(1,-1),(2,2)$ and $(2,-2)$. The Jacobian is $\left(\begin{array}{cc}-3 & 2 y \\ 2 x & -2 y\end{array}\right)$.
At $(1,1), \mathrm{J}=\left(\begin{array}{cc}-3 & 2 \\ 2 & -2\end{array}\right), \tau=-5, \quad \Delta=2:$ stable node.
At $(1,-1), \mathrm{J}=\left(\begin{array}{cc}-3 & -2 \\ 2 & 2\end{array}\right), \tau=-1, \quad \Delta=-2$, saddle.
At (2, 2), $\mathrm{J}=\left(\begin{array}{cc}-3 & 4 \\ 4 & -4\end{array}\right), \tau=-7, \quad \Delta=-4$, saddle.
At $(2,-2), \mathrm{J}=\left(\begin{array}{cc}-3 & -4 \\ 4 & 4\end{array}\right), \tau=1, \quad \Delta=4$, unstable spiral.
(b) There are infinitely many fixed points at $(n \pi, 0), n \in \mathbb{Z}$.

The Jacobian is $\left(\begin{array}{cc}\cos (x+y) & \cos (x+y) \\ 0 & 1\end{array}\right)$.
At $(2 k \pi, 0), \mathrm{J}=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right), \tau=2, \quad \Delta=1$, unstable node.
At $((2 k+1) \pi, 0), \mathrm{J}=\left(\begin{array}{cc}-1 & -1 \\ 0 & 1\end{array}\right), \tau=0, \quad \Delta=-1$, saddle.
PROBLEM 2: Consider the system $\dot{x}=y^{3}-4 x, \dot{y}=y^{3}-y-3 x$.
(a) Show that the line $x=y$ is invariant, i.e., any trajectory that starts on it stays on it.
(b) Show that $|x(t)-y(t)| \rightarrow 0$ as $t \rightarrow \infty$ for all other trajectories. (Hint: Form a differential equation that tells us how the quantity $x(t)-y(t)$ changes with time.)
(c) Check your result by sketching a phase portrait.

SOLUTION:
(a) On $y=x$, the velocity field is $\left(x^{3}-4 x, x^{3}-4 x\right)$. So the flow is along the line $y=x$. (b) $\dot{x}-\dot{y}=-(x-y)$, this implies that $x(t)-y(t)=(x(0)-y(0)) e^{-t}$ and hence the desired result.

PROBLEM 3: Show that the system $\dot{x}=y-x^{3}, \quad \dot{y}=-x-y^{3}$ has no closed orbits, by constructing a Lyapunov function $V(x, y)=a x^{2}+b y^{2}$ with suitable $a, b$.

SOLUTION: Select $a=b=1$, then $V>0$ and $\dot{V}=-2 x^{4}-2 y^{4}<0$. Hence we have a Lyapunov function and we conclude that there are no closed orbits.

PROBLEM 4: Show that the origin is a spiral point of the system

$$
\dot{x}=-y-x \sqrt{x^{2}+y^{2}}, \quad \dot{y}=x-y \sqrt{x^{2}+y^{2}}
$$

but a center for its linear approximation.
SOLUTION: The fixed point is at the origin. The linearization is $\dot{x}=-y, \dot{y}=x$ which is a linear center. However if we convert the system to polar coordinates, we get $r \dot{r}=x \dot{x}+y \dot{y}=x(-y-x r)+y(x-y r)=-r^{3}$ or $\dot{r}=-r^{2}$ and $\dot{\theta}=\frac{x \dot{y}-y \dot{x}}{r^{2}}=\frac{x(x-y r)-y(-y-x r)}{r^{2}}=1$. Analyzing this system either geometrically or analytically, we find it is a stable spiral. For example we can solve for $r$ by separating variables and obtaining $r(t)=\frac{r_{0}}{1+t r_{0}}$. Since the trajectories rotate at a constant rate and the radius is going to zero, we obtain a stable spiral.

PROBLEM 5: A system of differential equation is called a Hamiltonian system if there exists a real-valued function $H(x, y)$ (called the Hamiltonian function) such that

$$
\dot{x}=\frac{\partial H}{\partial y} \text { and } \dot{y}=-\frac{\partial H}{\partial x} \text { for all } x, y .
$$

(a) Show that the function $H$ is a conserved quantity along the trajectories of the system.
(b) Suppose that $\left(x_{0}, y_{0}\right)$ is an equilibrium point for the Hamiltonian system, by studying the linearization at $\left(x_{0}, y_{0}\right)$, conclude that a Hamiltonian system cannot have spirals or nodes in the phase portrait.
(c) Check that system $\dot{x}=-x \sin y+2 y, \quad \dot{y}=-\cos y$ is a Hamiltonian system with Hamiltonian function $H(x, y)=x \cos y+y^{2}$.
(d) Sketch the level curves of $H$ and the phase portrait of the system. Include a description of all equilibrium points and saddle connections.

## SOLUTION:

(a) $\frac{d H}{d t}=\frac{\partial H}{\partial x} \frac{d x}{d t}+\frac{\partial H}{\partial y} \frac{d y}{d t}=\frac{\partial H}{\partial x} \frac{\partial H}{\partial y}-\frac{\partial H}{\partial y} \frac{\partial H}{\partial x}=0$, this implies that $H$ is conserved along trajectories.
(b) The Jacobian is

$$
\left(\begin{array}{cc}
\frac{\partial^{2} H}{\partial x \partial y} & \frac{\partial^{2} H}{\partial y^{2}} \\
-\frac{\partial^{2} H}{\partial x^{2}} & -\frac{\partial^{2} H}{\partial y \partial x}
\end{array}\right)=\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right) . \tau=0, \quad \Delta=-a^{2}-b c
$$

If $\Delta>0$, centers and if $\Delta<0$, saddles and if $\Delta=0,0$ is the only eigenvalue.
We conclude that it is not possible to have eigenvalues that are complex and have nonzero real parts, so solutions cannot spiral in or out of the fixed point. Similarly, we cannot have two positive or two negative eigenvalues, so no sinks or sources.
(c) Clearly $\dot{x}=\frac{\partial H}{\partial y}$ and $\dot{y}=-\frac{\partial H}{\partial x}$.
(d) The level curves of $H$ are the trajectories of the system.

PROBLEM 6: Given the system $\dot{x}=x-y-x\left(x^{2}+y^{2}\right), \quad \dot{y}=x+y-y\left(x^{2}+y^{2}\right)$
(a) Transform the system into polar coordinates.
(b) Conclude that the system has a limit cycle.
(c) Check your work by plotting the phase portrait.

SOLUTION:
(a) $r \dot{r}=x \dot{x}+y \dot{y}=x\left(x-y-x r^{2}\right)+y\left(x+y-y r^{2}\right)=r^{2}\left(1-r^{2}\right)$ implies $\dot{r}=r\left(1-r^{2}\right)$
also one finds that $\dot{\theta}=1$.
(b) This is the same system we considered in class and showed we had a stable limit cycle $r=1$.

PROBLEM 7: Given the system $\frac{d \vec{x}}{d t}=\left(\begin{array}{cc}0 & 2 \\ -2 & -1\end{array}\right) \vec{x}$ with initial condition $\vec{x}_{0}=\binom{-1}{1}$.
(a) Find the eigenvalues.
(b) Determine if the origin is a stable spiral, unstable spiral or a center.
(c) Determine the direction of oscillation in the phase plane.
(d) Sketch the $x-y$ phase portrait and the $x(t)$ and $y(t)$ graphs for the solution with the indicated initial conditions. (You have to use either Maple or pplane to do all the graphing)
SOLUTION:
(a) The eigenvalues are roots of $\lambda^{2}+\lambda+4=0$ and hence $\lambda=\frac{-1 \pm i \sqrt{15}}{2}$.
(b) Since the real parts of the eigenvalues are negative, we have a stable spiral.
(c) To determine the direction of oscillation, just evaluate the field at a convenient point such as $(1,0)$. We get the velocity vector $(0,-1)$ and hence the spiral is clockwise.

PROBLEM 8: Let $G(x, y)=x^{2}-y^{2}$.
(a) What is the gradient system with vector field given by the gradient of $G$ ?
(b) Classify the equilibrium point at the origin.
(c) Sketch the graph of $G$ and the level curves of $G$.
(d) Sketch the phase portrait of the gradient system in part (a).
(Remark: This is why saddle equilibrium points have the name saddle.)

## SOLUTION:

(a) $\dot{x}=-2 x, \dot{y}=2 y$
(b) This is a linear system with $\Delta<0$, so a saddle.
(c) Use the plot3d command to get the graph of $G$ and the contourplot command to get level curves.
(d) The phase portrait of the system is the same as the minus the gradient plot of $G$.

