1. Problem 1

**SOLUTION:**
(a) The individual learns most rapidly when \( \frac{dL}{dt} \) is maximum, i.e., at \( L = 0 \).
(b) At the given instant, Beth’s learning rate is \( = 0.2 \cdot 0.5 = 0.1 \) and Eric’s learning rate is \( = 0.1 \cdot 0.75 = 0.075 \). So Beth is learning faster.

2. Problem 2

**SOLUTION:**
(a) The fixed points are at \( p = 0 \) and \( p = 100 \). Since \( f'(0) = 2000 \) \( > 0 \), \( p = 0 \) is unstable and since \( f'(100) = -2000 < 0 \) \( p = 100 \) is stable.
(b) The term \( 10l \) shifts the parabola down, the fish will not go extinct as long as the parabola is not completely negative. The bifurcation point occurs when the discriminant is 0.
\[ 2000^2 - 4 \cdot (-20) \cdot (-10l) = 0 \implies l = \frac{2000^2}{800} = 5000 \text{ licenses.} \] The only equilibrium fish population will be given by \( -20p^2 + 2000p - 50000 = 0 \implies p^* = 50 \).

3. Problem 3

**SOLUTION:**
(a) \( y = \frac{-y}{a} + \frac{b}{a} \)

Case 1: \( a > 0 \)

Case 2: \( a < 0 \)

One fixed point at \( y^* = b \) and it is stable.

(b) Case 1: If \( a > 0 \), then regardless of \( b \) any solution will tend toward \( b \) as \( t \to \infty \).
Case 2: If \( a < 0 \), then regardless of \( b \) any solution such that \( y(0) \neq b \) will tend to \( +\infty \) if \( y(0) > b \) and tend to \( -\infty \) if \( y(0) < b \).

(c) Let \( u = b - y \) then \( \dot{u} = -y = \frac{b - y}{a} = -\frac{1}{a}u \) and hence \( u(t) = u(0)e^{\frac{-t}{a}} \). Writing the solution in terms of \( y \), we get
\[ b - y(t) = (b - y(0))e^{\frac{-t}{a}} \implies y(t) = b - (b - y(0))e^{\frac{-t}{a}}. \]
Clearly we can see from the solution that if \( a > 0 \), then \( \lim_{t \to \infty} y(t) = b \) and if \( a < 0 \) then \( \lim_{t \to \infty} y(t) = \pm \infty \) depending on whether \( b - y(0) < 0 \) or \( b - y(0) > 0 \).

4. Problem 4

**SOLUTION:**

Case 1: \( \lambda \leq 0 \) then we have one fixed point at \( x^* = \lambda \) and it is unstable. A typical graph is

![Graph](image1)

\( \lambda \leq 0 \)

Case 2: \( \lambda > 0 \) and \( \lambda \neq 1 \) then we have three fixed points at \( \lambda, \pm \sqrt{\lambda} \). A typical graph is

![Graph](image2)

\( \lambda > 0 \ (\lambda \neq 1) \)

Case 2: \( \lambda = 1 = \implies \dot{x} = (x - 1)^2(x + 1) \) so we have two fixed points at \( \pm 1 \).

![Graph](image3)

\( \lambda = 1 \)

5. Problem 5

**SOLUTION:** \( f(N) = -aN \ln(bN) \) and \( f'(N) = -a(\ln(bN) + 1) \)

To find fixed points we solve \( f(N) = 0 \) to get \( N^* = 0 \) and \( N^* = \frac{1}{b} \).

\( f'(0) = \lim_{N \to 0} (-a(\ln(bN) + 1)) = +\infty \Rightarrow N^* = 0 \) is unstable.

\( f'(\frac{1}{b}) = -a < 0 \Rightarrow N^* = \frac{1}{b} \) is stable.

6. Problem 6
SOLUTION:
(a) See Homework set 2 solutions
(b) Since \( rx - \ln(1 + x) = (r - 1)x + \frac{1}{2}x^2 + O(x^3) \) we expect a transcritical bifurcation at \( r = 1 \). Let us complete the analysis graphically. We will graph both \( rx \) and \( \ln(1 + x) \).

We can clearly see that \( x^* = 0 \) is always a fixed point. When \( r < 1 \), \( x^* = 0 \) is stable and the second fixed point is unstable and when \( r > 1 \), \( x^* = 0 \) is unstable and the second fixed point is stable. This shows we do indeed have a transcritical bifurcation at \( r = 1 \). The bifurcation diagram is given below:

(c) By an easy change of variable we could bring the given problem into the supercritical pitchfork bifurcation normal form. The following graphical analysis confirms this fact.
When \( r \leq 0 \), we have one fixed point given by \( x^* = 0 \) (stable),

When \( r > 0 \), we have three fixed points \( x^* = 0 \) that is unstable and two other fixed points \( x^* = \pm \frac{\sqrt{r}}{2} \) that are both stable. The bifurcation diagram is

7. Problem 7

**SOLUTION:** In this case the map is given by \( F(x) = -2x - x^2 \). To find the fixed points we solve \( F(x) = x \), so

\[-2x - x^2 = x \implies x^* = 0, x^* = -3.\]

To find the points of period 2, we solve \((F \circ F)(x) = x\) or

\[4x + 2x^2 - (2x - x^2)^2 = 4x - 2x^2 - 4x^3 - x^4 = x.\]

Simplifying we get

\[3x - 2x^2 - 4x^3 - x^4 = 0.\]

Since we know already two roots 0 and -3, to find the last two roots we solve

\[\frac{3x - 2x^2 - 4x^3 - x^4}{x(x + 3)} = 0,\]

We get the solutions:

\[-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}.

8. Problem 8
SOLUTION:
\[ f(0) = 1, \ f(1) = 2, \ f(2) = 3, \ f(3) = 4, \ f(4) = 0, \] so the sequence is 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \cdots
We conclude that 0 is on a period 5 cycle. To determine the stability of the cycle we need to compute \((f^5)'(0)\). We have \((f^5)'(0) = f'(0)f'(1)f'(2)f'(3)f'(4) = 1 \cdot 1 \cdot 1 \cdot 2 = 2 \implies \) cycle is repelling.

9. Problem 9

SOLUTION: Clearly \(\tan x = x\) has infinitely many solutions so there are infinitely many fixed points. The fixed points at 0 is neutral since \(\frac{d}{dx} \tan x \bigg|_{x=0} = \sec^2 0 = 1\). An easy graphical analysis shows that it is repelling. At all the other points \(\sec^2 x > 1\) and hence they are repelling.


SOLUTION:

![Graphs showing different behaviors for different values of \(\alpha\).](image)

This bifurcation is a pitchfork bifurcation. For \(\alpha \leq 1\), the origin is an attracting fixed point since \(F'_\alpha(0) = \alpha\). When \(\alpha = 1\), the graph is tangent to the diagonal, but the graph shows that 0 is still attracting. For \(\alpha > 1\), two attracting fixed points emerge and the origin becomes a repelling fixed point.