SUMMER 2007 67-717 TEST 1 SOLUTIONS

1. Problem 1

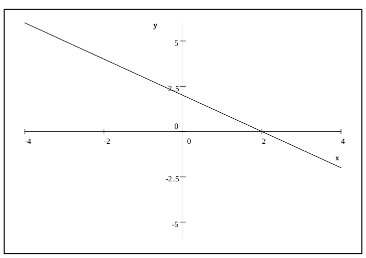
SOLUTION:
(a) The individual learns most rapidly when $\dfrac{dL}{dt}$ is maximum, i.e, at $L = 0$.
(b) At the given instant, Beth's learning rate is $= 0.2{\cdot}0.5 = 0.1$ and Eric's learning rate is $= 0.1 \cdot 0.75 = 0.075$. So Beth is learning faster.

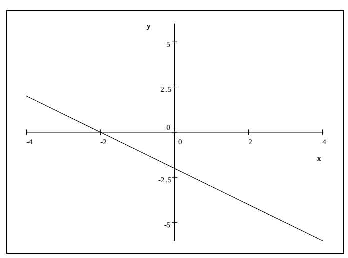
2. Problem 2

SOLUTION:
(a) The fixed points are at $p^* = 0$ and $p^* = 100$. Since $f'(0) = 2000 > 0$, $p^* = 0$ is unstable and since $f'(100) = -2000 < 0$ $p^* = 100$ is stable.
(b) The term $10l$ shifts the parabola down, the fish will not go extinct as long as the parabola is not completely negative. The bifurcation point occurs when the parabola has a double root. This happens when the discriminant is 0.
$2000^2 - 4 \cdot (-20) \cdot (-10l) = 0 \Longrightarrow l = \dfrac{2000^2}{800} = 5000$ licenses. The only equilibrium fish population will be given by
$-20p^2 + 2000p - 50000 = 0 \Longrightarrow p^* = 50$.

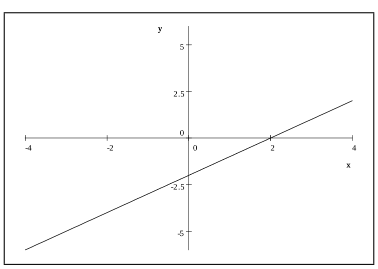
3. Problem 3

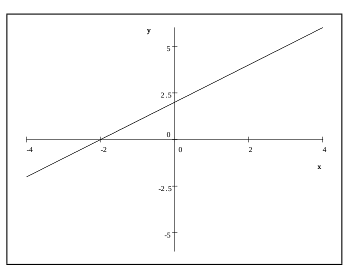
SOLUTION:
(a) $\dot{y} = -\dfrac{y}{a} + \dfrac{b}{a}$
Case 1: $a > 0$


$b > 0$


$b < 0$

One fixed point at $y^* = b$ and it is stable.
Case 2: $a < 0$


$b > 0$

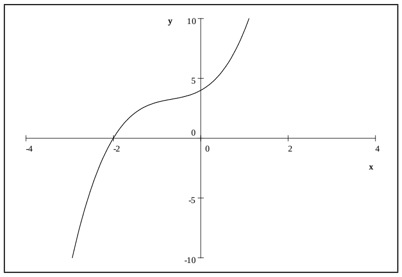

$b < 0$

One fixed point at $y^* = b$ and it is unstable.
(b) Case 1: If $a > 0$, then regardless of b any solution will tend toward b as $t \longrightarrow \infty$.
Case 2: If $a < 0$, then regardless of b any solution such that $y(0) \neq b$ will tend to $+\infty$ if $y(0) > b$ and tend to $-\infty$ if $y(0) < b$.
(c) Let $u = b - y$ then $\dot{u} = -\dot{y} = -\dfrac{b-y}{a} = -\dfrac{1}{a}u$ and hence $u(t) = u(0)e^{-\frac{t}{a}}$. Writing the solution in terms of y, we get
$b - y(t) = (b - y(0))e^{-\frac{t}{a}} \Longrightarrow y(t) = b - (b - y(0))e^{-\frac{t}{a}}$.

Clearly we can see from the solution that if $a > 0$, then $\lim_{t\to\infty} y(t) = b$ and if $a < 0$ then $\lim_{t\to\infty} y(t) = \pm\infty$ depending on whether $b - y(0) < 0$ or $b - y(0) > 0$.

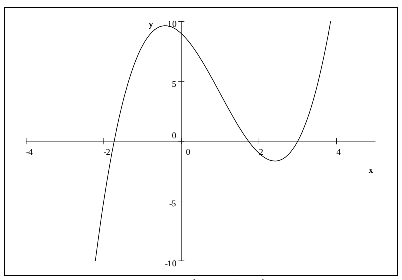
4. Problem 4

SOLUTION:
Case 1: $\lambda \leq 0$ then we have one fixed point at $x^* = \lambda$ and it is unstable. A typical graph is



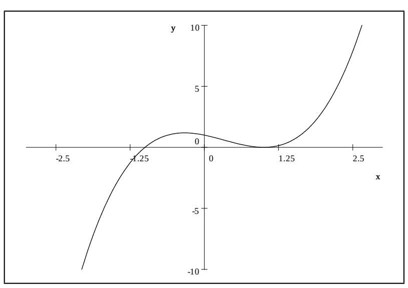
$\lambda \leq 0$

Case 2: $\lambda > 0$ and $\lambda \neq 1$ then we have three fixed points at $\lambda, \pm\sqrt{\lambda}$. A typical graph is



$\lambda > 0$ $(\lambda \neq 1)$

Case 2: $\lambda = 1 \Longrightarrow \dot{x} = (x-1)^2(x+1)$ so we have two fixed points at ± 1.



$\lambda = 1$

5. Problem 5

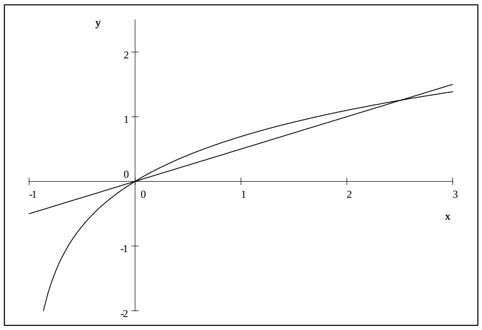
SOLUTION: $f(N) = -aN\ln(bN)$ and $f'(N) = -a(\ln(bN) + 1)$
To find fixed points we solve $f(N) = 0$ to get $N^* = 0$ and $N^* = \dfrac{1}{b}$.
$f'(0) = \lim_{N\longrightarrow 0}(-a(\ln(bN) + 1)) = +\infty \Longrightarrow N^* = 0$ is unstable.
$f'(\dfrac{1}{b}) = -a < 0 \Longrightarrow N^* = \dfrac{1}{b}$ is stable.

6. Problem 6

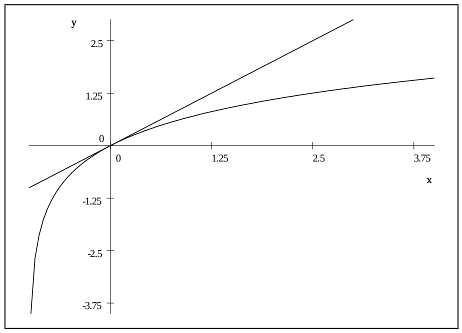
SOLUTION:

(a) See Homework set 2 solutions

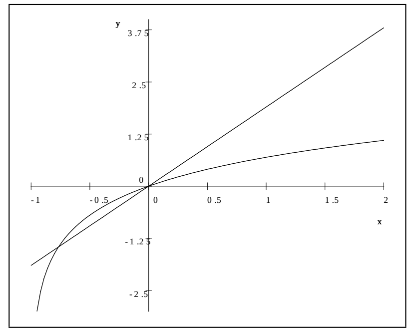
(b) Since $rx - \ln(1+x) = (r-1)x + \dfrac{1}{2}x^2 + O(x^3)$ we expect a transcritical bifurcation at $r = 1$. Let us complete the analysis graphically. We will graph both rx and $\ln(1+x)$.



$r < 1$

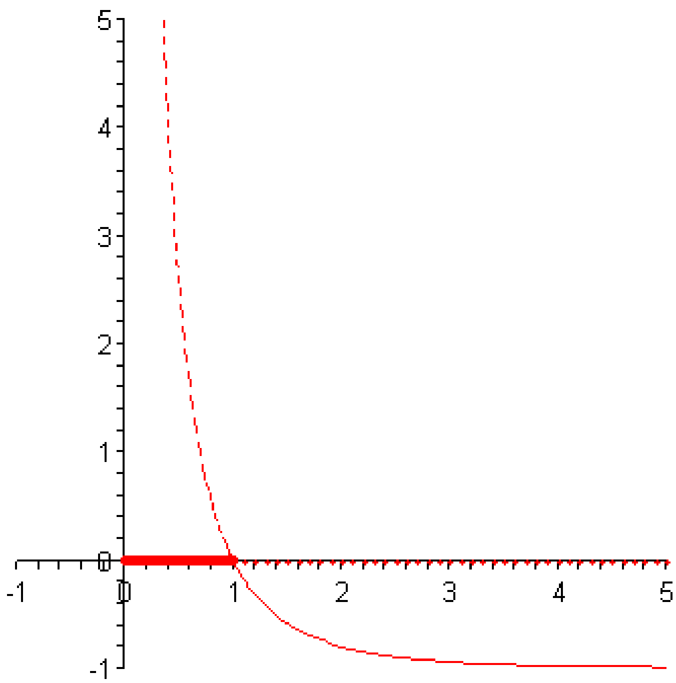


$r = 1$

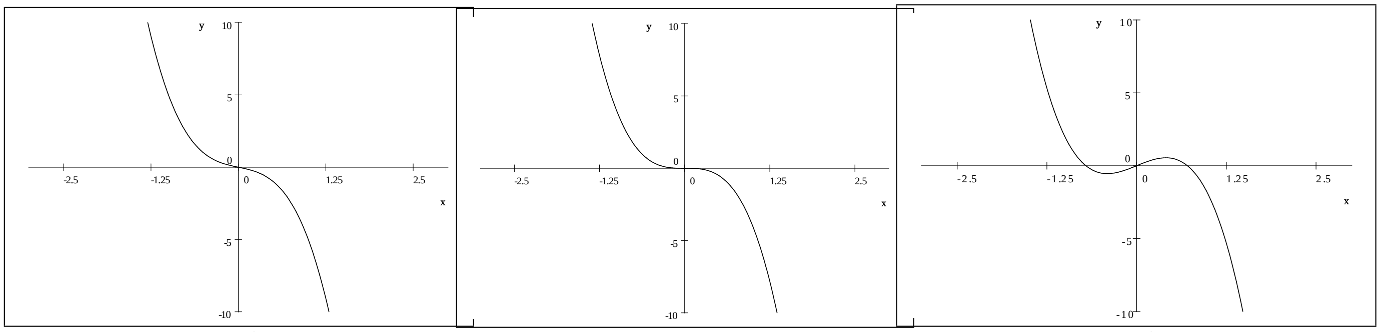


$r > 1$

We can clearly see that $x^* = 0$ is always a fixed point. When $r < 1$, $x^* = 0$ is stable and the second fixed point is unstable and when $r > 1$, $x^* = 0$ is unstable and the second fixed point is stable. This shows we do indeed have a transcritical bifurcation at $r = 1$.The bifurcation diagram is given below:



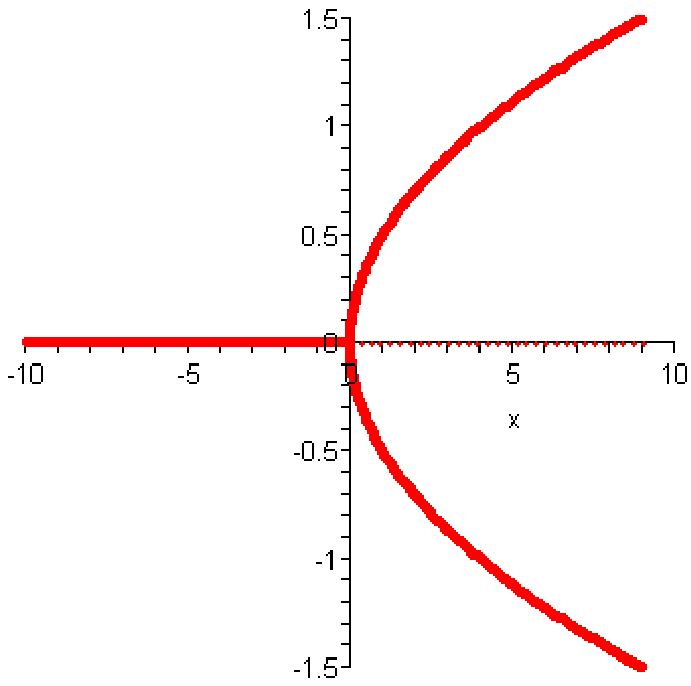
(c) By an easy change of variable we could bring the given problem into the supercritical pitchfork bifurcation normal form. The following graphical analysis confirms this fact.



$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$

When $r \leq 0$, we have one fixed point given by $x^* = 0$ (stable),

When $r > 0$, we have three fixed points $x^* = 0$ that is unstable.and two other fixed points $x^* = \pm\frac{\sqrt{r}}{2}$ that are both stable. The bifurcation diagram is



7. Problem 7

   SOLUTION: In this case the map is given by $F(x) = -2x - x^2$. To find the fixed points we solve $F(x) = x$,so

$-2x - x^2 = x \Longrightarrow x^* = 0, x^* = -3.$
To find the points of period 2 , we solve $(F \circ F)(x) = x$ or $4x + 2x^2 - (-2x - x^2)^2 = 4x - 2x^2 - 4x^3 - x^4 = x$. Simplifying we get $3x - 2x^2 - 4x^3 - x^4 = 0$. Since we know already two roots 0 and -3, to find the last two roots we solve $\dfrac{3x - 2x^2 - 4x^3 - x^4}{x(x+3)} = 0$,
We get the solutions : $-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}$.

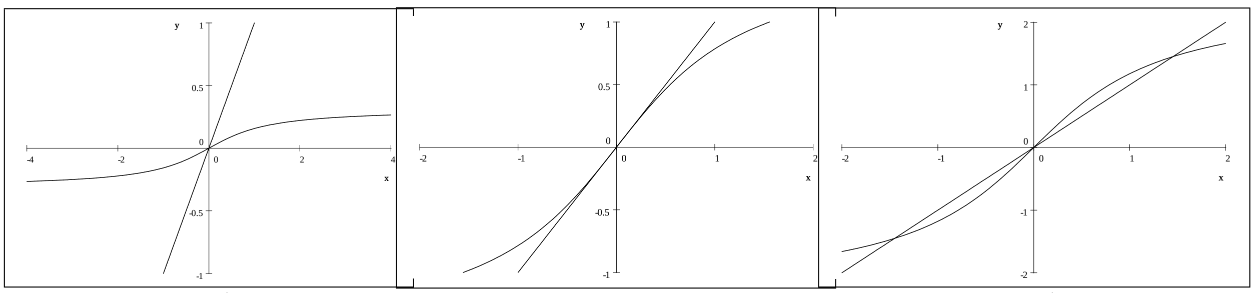
8. Problem 8

SOLUTION:
$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 4,\ f(4) = 0$, so the sequence is $0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \cdots$
We conclude that 0 is on a period 5 cycle. To determine the stability of the cycle we need to compute $(f^5)'(0)$.
We have $(f^5)'(0) = f'(0)f'(1)f'(2)f'(3)f'(4) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2 \Longrightarrow$ cycle is repelling.

9. Problem 9

SOLUTION: Clearly $\tan x = x$ has infinitely many solutions so there are infinitely many fixed points.
The fixed points at 0 is neutral since $\left.\dfrac{d}{dx}\tan x\right|_{x=0} = \sec^2 0 = 1$. An easy graphical analysis shows that it is repelling.
At all the other points $\sec^2 x^* > 1$ and hence they are repelling.

10. Problem 10.
SOLUTION:



$\alpha < 1$ $\qquad$ $\alpha = 1$ $\qquad$ $\alpha > 1$

This bifurcation is a pitchfork bifurcation. For $\alpha \leq 1$, the origin is an attracting fixed point since $F'_\alpha(0) = \alpha$. When $\alpha = 1$, the graph is tangent to the diagonal , but the graph shows that 0 is still attracting. For $\alpha > 1$, two attracting fixed points emerge and the origin becomes a repelling fixed point.