Problem 6.2.2

(a) Since f and g are clearly continuously differentiable (they are polynomial functions), the existence and uniqueness theorem is valid inside D.

(b) If $x(t) = \sin t$ and $y(t) = \cos t$ then $\dot{x}(t) = \cos t = y(t)$ and $\dot{y}(t) = -\sin t = -x(t)$.

(remember $x^2(t) + y^2(t) = 1$).So we conclude that $x(t) = \sin t$, $y(t) = \cos t$ is a solution of the system. The orbit of this solution is the unit circle.

(c) Any solution that starts inside the unit circle must stay inside since by uniqueness it cannot intersect the unit circle because the unit circle is the orbit of the solution in part (b).

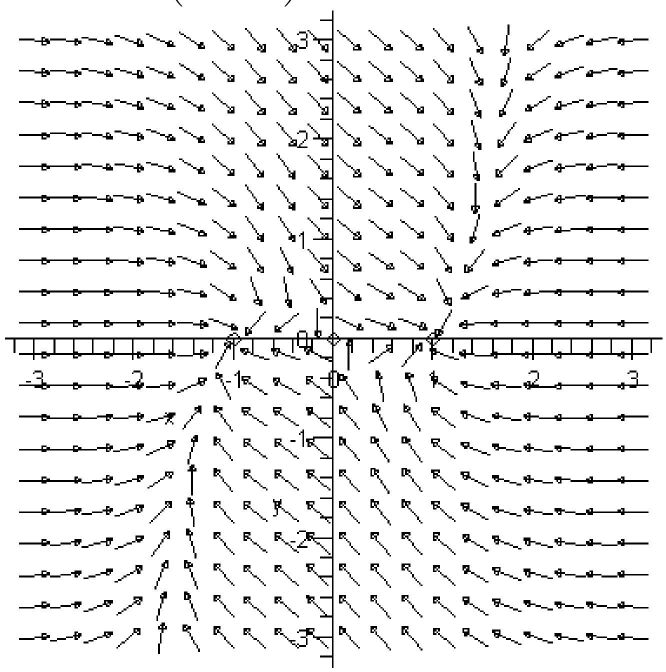
Problem 6.3.4

The fixed points are solutions to the system $y + x - x^3 = 0$ and $-y = 0$. We obtain $(\pm 1, 0)$ and (0,0).

The Jacobian matrix is $\begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}$.

At $(\pm 1, 0)$, $J = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$ so the eigenvalues are $-2, -1$ and hence we have stable nodes there.

At (0,0), $J = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the eigenvalues are 1, -1 and hence we have a saddle there.



Problem 6.3.11

(a) We can solve for r easily: $r'(t) = -r(t) \Longrightarrow r(t) = r_0 e^{-t}$. Now $\theta'(t) = \dfrac{1}{\ln r(t)} = \dfrac{1}{\ln r_0 - t}$ and this implies that $\theta(t) = \theta_0 - \ln|\ln r_0 - t| + \ln|\ln r_0|$.

(b) Clearly $\lim\limits_{t \longrightarrow \infty} r(t) = 0$ and $\lim\limits_{t \to \infty} |\theta(t)| = \infty$. This shows that the origin is a stable spiral for the nonlinear system.

(c) Let us write the system in rectangular coordinates:

$$x(t) = r(t)\cos\theta(t) \Longrightarrow x'(t) = r'(t)\cos\theta(t) - r(t)\ \theta'(t)\sin\theta(t) = -r(t)\frac{x(t)}{r(t)} - r(t)\frac{1}{\ln r(t)}\frac{y(t)}{r(t)}.$$

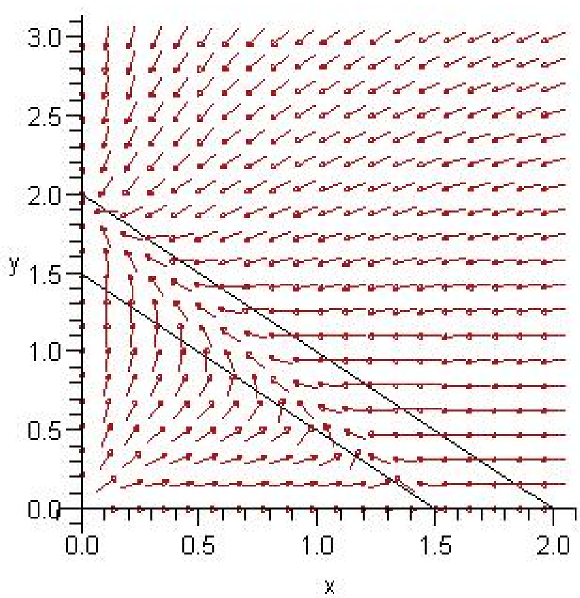
Therefore $x'(t) = -x(t) - \dfrac{y(t)}{\ln\sqrt{x^2(t) + y^2(t)}}$. Similarly

$$y(t) = r(t)\sin\theta(t) \Longrightarrow y'(t) = r'(t)\sin\theta(t) + r(t)\ \theta'(t)\cos\theta(t) = -y(t) + \frac{x(t)}{\ln\sqrt{x^2(t) + y^2(t)}}.$$

We conclude that in rectangular coordinates, the system is $x' = -x + \dfrac{2y}{\ln(x^2 + y^2)}$

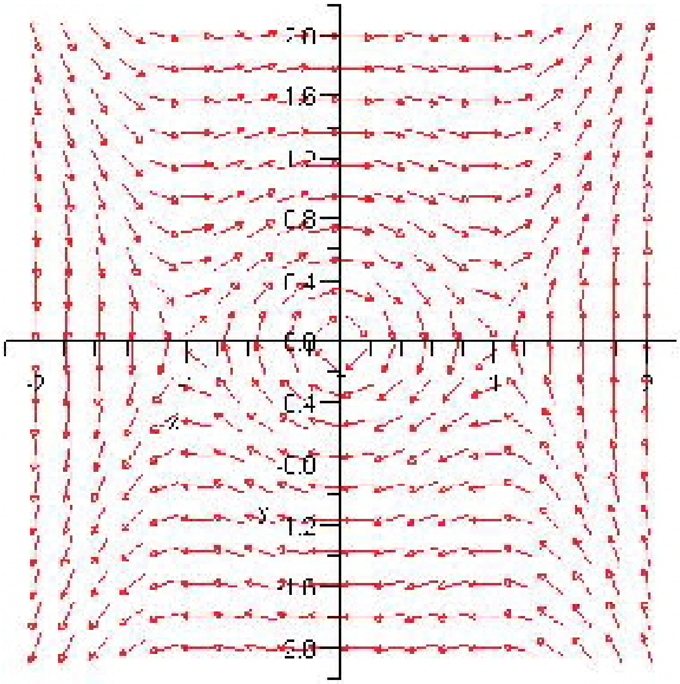
and $y' = -y + \dfrac{2x}{\ln(x^2+y^2)}$.
(d) If we linearize at the origin, the linearized system is $x' = -x,\ \ y' = -y.$. This is true because $\dfrac{2y}{\ln(x^2+y^2)}$ and $\dfrac{2x}{\ln(x^2+y^2)}$ are higher-order terms, i.e, they go to zero faster than r. Clearly $\dfrac{2y}{\ln(x^2+y^2)}\dfrac{1}{r} = \dfrac{\sin\theta}{\ln r} \longrightarrow 0$ as $r \longrightarrow 0$.and similarly for the second term.
Note that the linearized system erroneously predicts that the origin is a stable star.

6.4.3
The fixed points are solutions of $x(3-2x-2y) = 0$ and $y(2-x-y) = 0$.
These solutions are (0,0), (0,2) and
(3/2,0). The jacobian is $\begin{pmatrix} 3-4x-2y & -2x \\ -x & 2-x-2y \end{pmatrix}$.
At (0,0), $J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and hence we have an unstable node.
At (0,2), $J = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ and hence we have a stable node.
At (3/2,0), $J = \begin{pmatrix} -3 & -3 \\ -3/2 & 1/2 \end{pmatrix}$, since the determinant is negative, there is a saddle at this point.

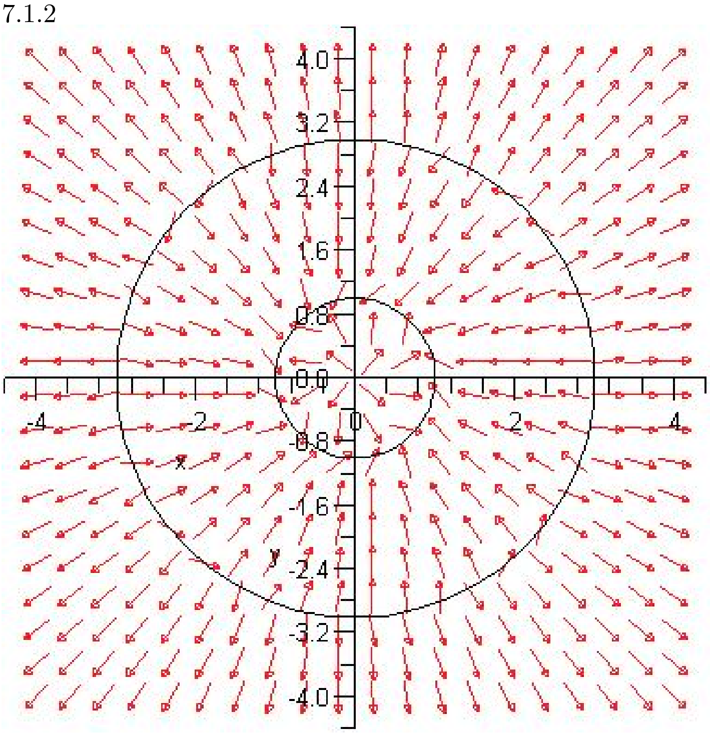


The bassin of attraction of the the node (0,2) is all the friat quadrant except the x-axis.

6.5.1
(a) The fixed points are at (0,0), $(-1,0)$ and $(1,0)$. The Jacobian is $J = \begin{pmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{pmatrix}$
At (0,0), $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ hence a center. At $(\pm 1, 0)$, $J = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ hence saddles.
(b) Clearly if we multiply by x' both sides of the differential equation, we get that $x'x'' - x'x^3 + xx' = 0$. This is the same as $\dfrac{d}{dt}(\dfrac{1}{2}x'^2 + \dfrac{1}{2}x^2 - \dfrac{1}{4}x^4) = 0$ and hence a conserved quantity is $E(x,x') = \dfrac{1}{2}x'^2 + \dfrac{1}{2}x^2 - \dfrac{1}{4}x^4$ which is the sum of kinetic and potential energies.
(c)



7.1.2



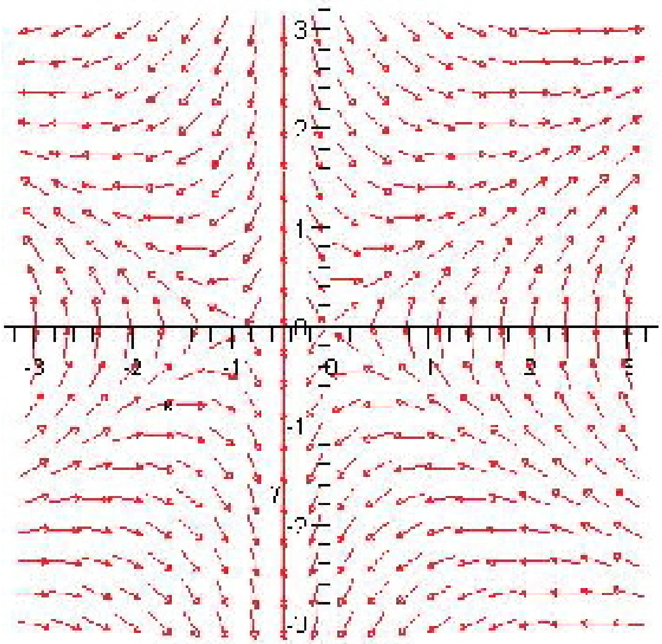
7.2.7

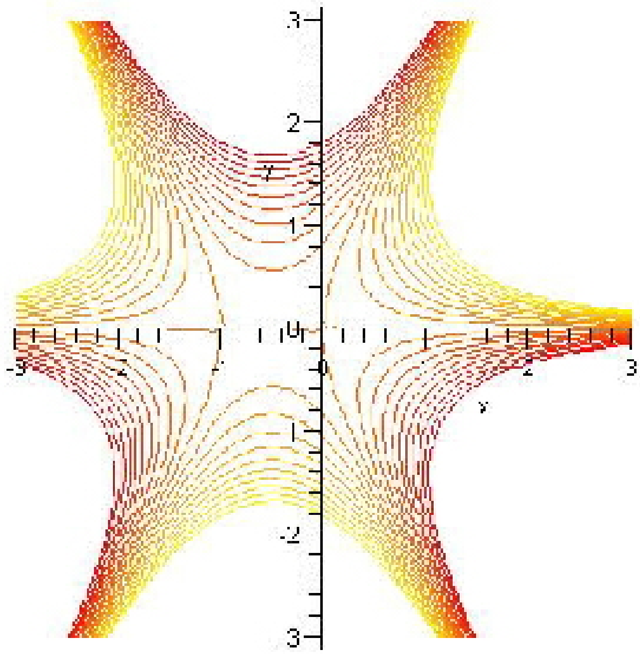
(a) $\dfrac{\partial(y+2xy)}{\partial y} = 1+2x, \quad \dfrac{\partial(x+x^2-y^2)}{\partial x} = 1+2x,$ This implies that the system is a gradient system.

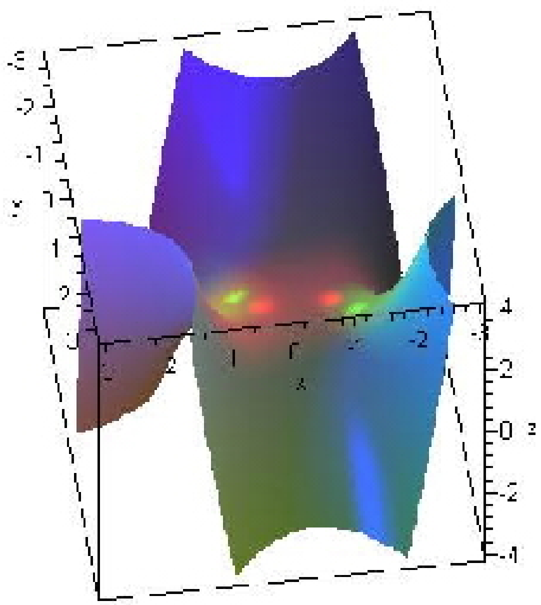
(b) $\dfrac{\partial V}{\partial x} = -y-2xy \Longrightarrow V(x,y) = -xy - x^2y + A(y),$ also $\dfrac{\partial V}{\partial y} = -x - x^2 + y^2 \Longrightarrow V = -xy - x^2y + \dfrac{y^3}{3} + B(x).$

By inspection it suffices to take $V(x,y) = -xy - x^2y + \dfrac{y^3}{3}$.

(c) The phase portrait is the gradient field of the V.







7.2.12

Clearly we need to choose a, m, n to be positive to force $V > 0$. Let us now compute $\dfrac{dV}{dt}$.

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = mx^{m-1}(-x + 2y^3 - 2y^4) + any^{n-1}(-x - y + xy) \\
&= -mx^m - any^n + 2mx^{m-1}y^3(1-y) - any^{n-1}x(1-y) \\
&= -mx^m - any^n + (1-y)(2mx^{m-1}y^3 - any^{n-1}x)
\end{aligned}
$$

If we select $n - 1 = 3$ or $n = 4$, and $m - 1 = 1$ or $m = 2$ and finally $2m - an = 0$ or $a = 1$, we get that $\dfrac{\partial V}{\partial t} = -mx^m - any^n = -2x^2 - 4y^4 < 0$.

5.1.9

(b) $\dot{x} = -y$ and $\dot{y} = -x \Longrightarrow x\dot{x} = -xy$ and $y\,\dot{y} = -xy \Longrightarrow x\dot{x} - y\,\dot{y} = 0 \Longrightarrow \frac{1}{2}(x^2 - y^2) = \text{Constant}$ $\Longrightarrow x^2 - y^2 = C$.

(c) Clearly the stable manifold is the line $y = x$ and the unstable manifold the line $y = -x$.

(d) $u = x + y \Longrightarrow \frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -(x+y) = -u \Longrightarrow u(t) = u_0 e^{-t}$.

$v = x - y \Longrightarrow \frac{dv}{dt} = \frac{dx}{dt} - \frac{dy}{dt} = x - y = v \Longrightarrow v(t) = v_0 e^t$

(e) The stable manifold is $v = 0$ and the unstable one is $u = 0$.

(f) From the given equation we have that

$$
\begin{aligned}
x(t) &= \frac{1}{2}(u(t) + v(t)) = \frac{1}{2}(x_0 + y_0)e^{-t} + \frac{1}{2}(x_0 - y_0)e^t = x_0 \cosh t - y_0 \sinh t \\
y(t) &= \frac{1}{2}(u(t) - v(t)) = \frac{1}{2}(x_0 + y_0)e^{-t} - \frac{1}{2}(x_0 - y_0)e^t = x_0 \sinh t + y_0 \cosh t
\end{aligned}
$$

Problem 5.2.1

SOLUTION:

(a) $\lambda^2 - \tau\lambda + \Delta = \lambda^2 - 5\lambda + 6 = 0 \Longrightarrow \lambda_1 = 2,\ \lambda_2 = 3$.

$(A - 2I)\vec{v}_1 = \vec{0} \Longrightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A - 3I)\vec{v}_2 = \vec{0} \Longrightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) The general solution is

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

(c) The fixed point at the origin is an unstable node.

(d) Substituting $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in the general solution, we get the system

$$
C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
$$

Clearly the solution is $C_1 = 1$ and $C_2 = 2$. So the solution is

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{2t} + 2e^{3t} \\ 2e^{2t} + 2e^{3t} \end{bmatrix}
$$

Problem 5.2.12

SOLUTION:

(a) Let $x = I$ and $y = \dot{I}$ then the equation can be rewritten as

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

(b) The eigenvalues are $\lambda_{1,2} = \dfrac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = \dfrac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}$.

Suppose that $R > 0$

Case1: If $R^2C - 4L > 0$, then clearly both eigenvalues are negative and we have stable nodes. This implies

that the origin is asymptotically stable.

Case 2: If $R^2C - 4L < 0$, then the eigenvalues have negative real part and this means we have stable spirals.

This also implies that the origin is asymptotically stable.

Case 3: If $R^2C - 4L = 0$, both eigenvalues are negative and equal to $-\frac{R}{2L}$. Hence we have a stable degenerate node.

Now suppose that $R = 0$, then the eigenvalues are purely imaginary and hence we have a center and this implies

that the origin is neutrally stable.