# SUMMER 2007 67-717 HOMEWORK SET 2 SOLUTIONS

1. Problem 4.1.2

**SOLUTION**: To find the fixed points we solve $f(\theta) = 1 + 2\cos\theta = 0$. The solutions are $\theta^* = \frac{2}{3}\pi$ and $\theta^* = \frac{4}{3}\pi$. Using linear stability analysis we have:

$f'(\theta)|_{\theta^*=\frac{2}{3}\pi} = -2\sin\theta|_{\theta^*=\frac{2}{3}\pi} = -\sqrt{3} < 0 \Longrightarrow \theta^* = \frac{2}{3}\pi$ is stable. and

$f'(\theta)|_{\theta^*=\frac{4}{3}\pi} = -2\sin\theta|_{\theta^*=\frac{4}{3}\pi} = \sqrt{3} > 0 \Longrightarrow \theta^* = \frac{4}{3}\pi$ is unstable.

2. Problem 4.1.5

**SOLUTION**: To find the fixed points we solve $f(\theta) = \sin\theta + \cos\theta = 0$. The solutions are $\theta^* = \frac{3\pi}{4}$ and $\theta^* = \frac{7\pi}{4}$. Using linear stability analysis we have:

$f'(\theta)|_{\theta^*=\frac{3\pi}{4}} = (\cos\theta - \sin\theta)|_{\theta^*=\frac{3\pi}{4}} = -\sqrt{2} < 0 \Longrightarrow \theta^* = \frac{3\pi}{4}$ is stable.and

$f'(\theta)|_{\theta^*=\frac{7\pi}{4}} = (\cos\theta - \sin\theta)|_{\theta^*=\frac{7\pi}{4}} = \sqrt{2} > 0 \Longrightarrow \theta^* = \frac{7\pi}{4}$ is unstable.

3. Problem 4.3.1

**SOLUTION**: Let $x = \sqrt{r}\tan\theta$ then $dx = \sqrt{r}\sec^2\theta d\theta$, also $r + x^2 = r(1 + \tan^2\theta) = r\sec^2\theta$, substituting we get

$$T_{\text{bottleneck}} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{r}\sec^2\theta d\theta}{r\sec^2\theta} = \frac{1}{\sqrt{r}}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{\sqrt{r}}.$$

4. Problem 4.3.2

**SOLUTION**:

(a) $u = \tan\frac{\theta}{2} \Longrightarrow \theta = 2\arctan u \Longrightarrow d\theta = \frac{2}{1+u^2}du.$

(b) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cdot\frac{1}{\sec^2\frac{\theta}{2}} = \frac{2u}{1+u^2}.$

(c) As $u \longrightarrow \pm\infty$, $\arctan u \longrightarrow \pm\pi/2$ and hence by part (a) $\theta \longrightarrow \pm\pi$.

(d) With respect to $u$ the integral becomes

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta} = \int_{-\infty}^{\infty} \frac{2/(1+u^2)}{\omega - 2au/(1+u^2)}du = \int_{-\infty}^{\infty} \frac{2}{\omega u^2 - 2au + \omega}du$$

(e) $\omega u^2 - 2au + \omega = \omega(u^2 - 2\frac{a}{\omega}u + 1) = \omega\left[(u - \frac{a}{\omega})^2 + \frac{\omega^2 - a^2}{\omega^2}\right]$ so

$$T = \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{1}{\frac{\omega^2 - a^2}{\omega^2} + (u - \frac{a}{\omega})^2}du = \frac{2}{\omega}\frac{\pi}{\frac{\sqrt{\omega^2 - a^2}}{\omega}} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

where we used the result of problem 4.2.1 by letting $x = u - \frac{a}{\omega}$ and $r = \frac{\omega^2 - a^2}{\omega^2}$.

5. Problem 4.2.3

**SOLUTION:** If we let $\theta_h, \theta_m$ represent the positions of the hour hand and minute hand respectively then $\frac{d\theta_h}{dt} = \frac{2\pi}{12}$ and $\frac{d\theta_m}{dt} = \frac{2\pi}{1}$ since it takes 12 hours for the hour hand to go around and it takes 1 hour for the minute hand to do so. If we let $\theta = \theta_m - \theta_h$ then

$$\frac{d\theta}{dt} = \frac{d\theta_m}{dt} - \frac{d\theta_h}{dt} = 2\pi(1 - 1/12)$$

So $\theta$ changes by $2\pi$ in time $\frac{2\pi}{2\pi(1-1/12} = \frac{12}{11}$ of an hour. So the hands will be aligned at 12:00 + $\frac{12}{11}$ hrs or $1:05:27$.

6. Problem 4.3.3

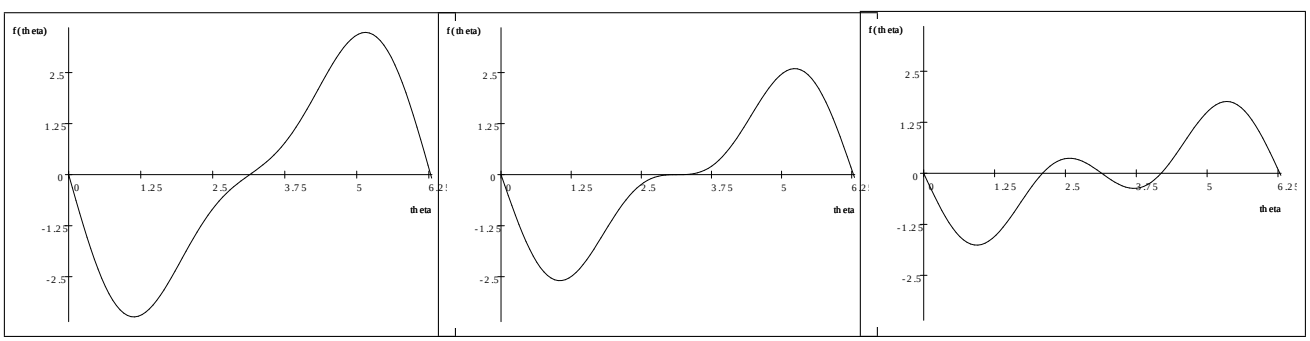
**SOLUTION:**To find the fixed points we solve $f(\theta) = 0$. We get

$$\begin{aligned} \mu \sin\theta - \sin 2\theta &= 0 \\ \sin\theta(\mu - 2\cos\theta) &= 0 \end{aligned}$$

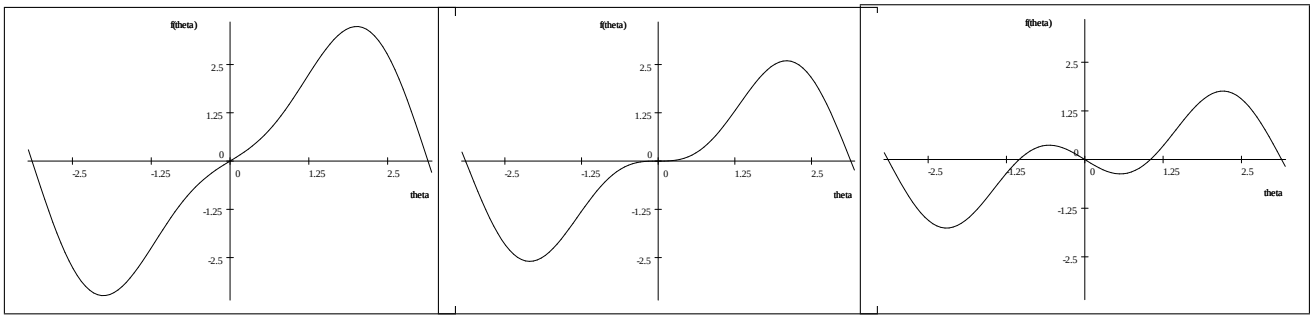
the fixed points are $\theta^* = 0$ and $\pi$ and $\theta^* = \arccos\frac{\mu}{2}$ which exists only for $|\mu| \leq 2$. Using linear stability analysis we get

$$f'(\theta) = \mu\cos\theta - 2\cos 2\theta$$

this implies that
$f'(0) = \mu - 2 \Longrightarrow \theta^* = 0$ is stable for $\mu < 2$ and unstable for $\mu > 2$.
$f'(\pi) = -\mu - 2 \Longrightarrow \theta^* = \pi$ is stable for $\mu > -2$ and unstable for $\mu < -2$.Finally studying the fixed point $\theta^*$ such that $\cos\theta^* = \frac{\mu}{2}$ gives us $f'(\theta^*) = \mu \cdot \frac{\mu}{2} - 2(2 \cdot \frac{\mu^2}{4} - 1) = 2 - \frac{1}{2}\mu^2 \Longrightarrow \theta^*$ is unstable since $|\mu| \leq 2$.
We conclude that we have a **subcritical pitchfork bifurcation** at $\mu = 2:$ When $\mu > 2$, $\theta^* = 0$ is unstable and when $\mu < 2$, $\theta^* = 0$ becomes stable and two other unstable fixed points at $\theta^* = Arc\cos\frac{\mu}{2}$ and $\theta^* = -Arc\cos\frac{\mu}{2}$ are born.
We also have a subcritical pitchfork bifurcation at $\mu = -2:$ When $\mu < -2$, $\theta^* = \pi$ is unstable and when $\mu > -2$, $\theta^* = \pi$ becomes stable and two other unstable fixed points at $\theta^* = \pi - Arc\cos\frac{|\mu|}{2}$ and $\theta^* = \pi + Arc\cos\frac{|\mu|}{2}$ are born.
The graphical analysis is given below:




$\mu < -2$ $\mu = -2$ $\mu > -2$




$\mu > 2$ $\qquad$ $\mu = 2$ $\qquad$ $\mu < 2$

7. Problem 10.1.3

**SOLUTION**: For every initial condition $x_0$, we have $\lim_{n\to\infty} x_n = +\infty$.

8. Problem 10.1.6

**SOLUTION**: Very complicated dynamics. Play around with the two Maple worksheets for more details.

9. Problem 10.1.8

**SOLUTION**: For every initial condition $x_0$, we have $\lim_{n\to\infty} x_n = 0$.

10. Problem 10.1.12

**SOLUTION:**
(a) $f(x_n) = x_n - \dfrac{x_n^2 - 4}{2x_n} = \dfrac{x_n}{2} + \dfrac{2}{x_n}$.
(b) Solve $f(x) = x$ to get $\dfrac{x}{2} + \dfrac{2}{x} = x \Longrightarrow x^2 - 4 = 0 \Longrightarrow x^* = \pm 2$.
(c) $f'(x) = \dfrac{1}{2} - \dfrac{2}{x^2} \Longrightarrow f'(x^*) = f'(\pm 2) = 0 \Longrightarrow$ the fixed points are superstable.
(d) $x_1 = 2.50000$, $x_2 = 2.05000$, $x_3 = 2.00060$, $x_4 = 2.00000$.

11. Problem 10.3.4

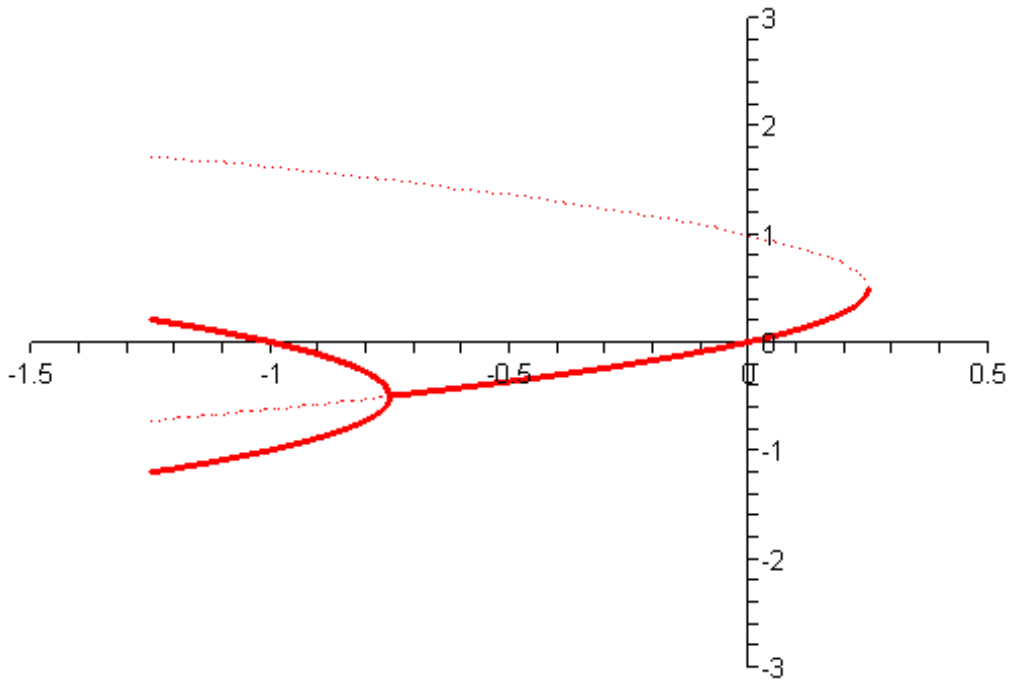
**SOLUTION**:
(a) $f(x) = x \Longrightarrow x^2 - x + c = 0 \Longrightarrow x_+^* = \dfrac{1 + \sqrt{1-4c}}{2}$ and $x_-^* = \dfrac{1 - \sqrt{1-4c}}{2}$.
We will have fixed points only if $c \le \dfrac{1}{4}$. Now since $f'(x) = 2x$ we have
$f'(x_+^*) = 1 + \sqrt{1-4c} > 1 \Longrightarrow x_+^*$ is repelling for all $c < \dfrac{1}{4}$
$f'(x_-^*) = 1 - \sqrt{1-4c}$ so $-1 < 1 - \sqrt{1-4c} < 1 \Longrightarrow 0 < \sqrt{1-4c} < 2 \Longrightarrow -\dfrac{3}{4} < c < \dfrac{1}{4}$.
So $x_-^*$ is attracting if $-\dfrac{3}{4} < c < \dfrac{1}{4}$ and repelling if $c < -\dfrac{3}{4}$.
(b) We have a saddle-node bifurcation at $c = \dfrac{1}{4}$ since at that point two fixed points appear. We also have another bifurcation at $c = \dfrac{-3}{4}$ since at that point $x_-^*$ loses stability. It will turn out to be a period doubling bifurcation.
(c) We find the 2-cycles by solving $(f \circ f)(x) = x$ or $(x^2 + c)^2 + c = x$. This implies that $x^4 + 2cx^2 - x + c^2 + c = 0$.
We know that $x_+^*$ and $x_-^*$ are roots, so to find the 2-cycles we solve

$$\frac{x^4 + 2cx^2 - x + c^2 + c}{x^2 - x + c} = x^2 + x + c + 1 = 0$$

We obtain

$$p_{\pm} = \frac{-1 \pm \sqrt{-3-4c}}{2}$$

Note that $p_{\pm}$ exist as real number only if $c \leq \frac{-3}{4}$. Thus a 2-cycle appears precisely when $c$ decreases through $c = -3/4$.
To find where the 2-cycle is stable we have to find out where $|f'(p_+)f'(p_-)| < 1$.
$|f'(p_+)f'(p_-)| < 1 = \left|(-1+\sqrt{-3-4c})(-1-\sqrt{-3-4c})\right| < 1 \Longrightarrow |4+4c| < 1 \Longrightarrow -5/4 < c < -3/4$.
The 2-cycle is superstabe when $|f'(p_+)f'(p_-)| = 0$ and this happens for $c = -1$.
(d)




12. Find all fixed points and periodic points of period 2 for each of the given functions:

(a) $F(x) = -x + 2$ (b) $F(x) = -2x - x^2$.
**SOLUTION**:
(a) Fixed points: Solve $F(x) = x$ to get $x^* = 1$.
Period 2 points: Solve $(F \circ F)(x) = x$ to get $-(-x+2)+2 = x$ and hence every point is a period 2 point.
(b) Fixed points: Solve $F(x) = x$ to get $-2x - x^2 = x$ and hence $x^* = 0$ and $x^* = -3$.
Period 2 points: Solve $(F \circ F)(x) = x$ to get

$$\begin{aligned} -2(-2x-x^2) - (-2x-x^2)^2 &= x \\ 4x - 2x^2 - 4x^3 - x^4 &= x \\ x^4 + 4x^3 + 2x^2 - 3x &= 0 \\ x(x+3)(x^2+x-1) &= 0 \end{aligned}$$

So the period 2 points are the solutions of $x^2 + x - 1 = 0$, which are $-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}$.

13. Describe the fate of the orbit of each of the following seeds under iteration of the function

$$T(x) = \begin{cases} 2x, & \text{if } x < 1/2; \\ 2-2x, & \text{if } x \geq 1/2 \end{cases}$$

(a) 2/3 (b) 1/6 (c) 2/5 (d) 1/8 (e) 1/4 (f) 1/2.

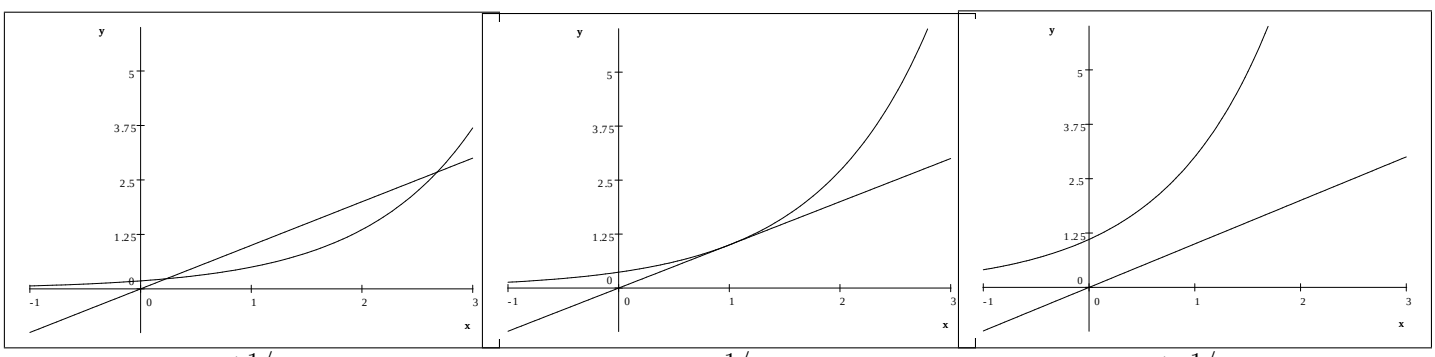
**SOLUTION**:
(a) Clearly $T(2/3) = 2/3$ so it is a fixed point.
(b) $T(1/6) = 1/3$, $T(1/3) = 2/3$, $T(2/3) = 2/3$.
(c) Period 2 cycle: $T(2/5) = 4/5$, $T(4/5) = 2/5$.
(d) $T(1/8) = 1/4$, $T(1/4) = 1/2$, $T(1/2) = 1$, $T(1) = 0$, $T(0) = 0, \cdots$
(e) (f) See (d).

14. For each of the given functions, find all fixed points and determine whether they are attracting, repelling, or neutral

(a) $F(x) = (\pi/2)\sin x$ (b) $F(x) = 3x(1-x)$.
**SOLUTION**:
(a) Fixed points: Solve $F(x) = x$ to get $x^* = 0$, $x^* = \pm\frac{\pi}{2}$ Since $F'(0) = \frac{\pi}{2} > 1$ , $x^* = 0$ is repelling.
Also since $F'(\pm\frac{\pi}{2}) = 0$, $x^* = \pm\frac{\pi}{2}$ are attracting.
(b) Fixed points: Solve $F(x) = x$ to get $x^* = 0$ and $x^* = \frac{2}{3}$.
$F'(0) = 3 > 1 \Longrightarrow x^* = 0$ is repelling. $F'(2/3) = 3 - 4 = -1 \Longrightarrow x^* = \frac{2}{3}$ is neutral.

15. What can you say about fixed points for $F_c(x) = ce^x$ with $c > 0$? What does the graph of $F_c$ tell you about these fixed points?

Note that when $c = 1/e$, $F_c(1) = 1$.
**SOLUTION**: Let us study the function $f(x) = F_c(x) - x = ce^x - x$. The derivative $f'(x) = ce^x - 1 = 0$ when $x = -\ln c$ and since $f"(-\ln c) = 1 > 0$, we conclude that $f$ has a minimum at $x = -\ln c$.
Case 1: If $f(-\ln c) > 0$ ,i.e, when $1 + \ln c > 0$ or $c > 1/e$ then $F_c(x) - x = ce^x - x > 0$ and we do not have fixed points.
Case 2: If $f(-\ln c) = 0$, then $F_c(x) - x = ce^x - x \geq 0$ and equality is true only at $x = -\ln c$. therefore, there is only one fixed point where the graph of $F_c(x) = ce^x$ is tangent to $y = x$ from above. The fixed point is neutral.
Case 3: If $f(-\ln c) < 0$, then $F_c(x) - x = 0$ at two different points. Hence there are two fixed points. Since $F_c(x)$ is below $y = x$ between the two points, one is attracting and the other repelling. The graphical analysis follows:




$c < 1/e$ $c = 1/e$ $c > 1/e$

16. Consider the function

$$T(x) = \begin{cases} 4x, & \text{if } x < 1/2; \\ 4 - 4x, & \text{if } x \geq 1/2 \end{cases}$$

Does $T$ have any attracting cycles? Why or why not?

**SOLUTION**: Suppose that $T$ has an $n$-cycle , $x_0, x_1, \cdots, x_n = x_0$ then

$$|(T^n)'(x_0)| = |T'(x_0) \cdot T'(x_1) \cdots T'(x_{n-1})| = 4^n$$

Therefore the cycle is repelling.

17. Each function undergoes a bifurcation of fixed points at the given parameter value. In each case use analytic or qualitative methods to identify this bifurcation as a tangent, pitchfork, or period doubling bifurcation or as none of these. Discuss the behavior of orbits near the fixed points in question at, before, and after the bifurcation.

(a) $F_\alpha(x) = x + x^2 + \alpha, \quad \alpha = 0$ (b) $F_\alpha(x) = \alpha \sin x, \quad \alpha = 1.$

**SOLUTION:**

(a) The fixed points are given by $x + x^2 + \alpha = x$ or $x^2 + \alpha = 0$. Therefore for $\alpha > 0$, there are no fixed points. For $\alpha = 0$, there is one fixed point; and for $\alpha < 0$, there are two fixed points at $x = \pm\sqrt{-\alpha}$. Differentiation yields $F'_\alpha(x) = 1 + 2x$ and $F'_\alpha(\pm\sqrt{-\alpha}) = 1 \pm 2\sqrt{-\alpha}$. Therefore for small enough $\alpha$, $0 < 1 - 2\sqrt{-\alpha} < 1$ and $x = -\sqrt{-\alpha}$ is attracting. Since $1 + 2\sqrt{-\alpha} > 1$, $x = \sqrt{-\alpha}$ is repelling. For $\alpha = 0$, $F_\alpha(x)$ is tangent to $y = x$ from above and therefore $x = 0$ is neutral. The bifurcation is a saddle-node bifurcation.

(b) For $\alpha$ slightly smaller than 1, the origin is the only fixed point and it is attracting. For $\alpha = 1$, $F_\alpha(x)$ is tangent to $y = x$ and 0 is attracting. For $\alpha > 1$, two more fixed points appear and they are attracting for $\alpha$ slightly larger than 1. The origin becomes a repelling fixed point. This is a pitchfork bifurcation.