1. Problem 4.1.2

**Solution:** To find the fixed points we solve \( f(\theta) = 1 + 2 \cos \theta = 0 \). The solutions are \( \theta^* = \frac{2}{3} \pi \) and \( \theta^* = \frac{4}{3} \pi \).

Using linear stability analysis we have:

\[
f'(\theta) = -2 \sin \theta \bigg|_{\theta^* = \frac{2}{3} \pi} = -2 \sin \left( \frac{2}{3} \pi \right) = -\sqrt{3} < 0 \Rightarrow \theta^* = \frac{2}{3} \pi \text{ is stable.}
\]

\[
f'(\theta) = -2 \sin \theta \bigg|_{\theta^* = \frac{4}{3} \pi} = -2 \sin \left( \frac{4}{3} \pi \right) = \sqrt{3} > 0 \Rightarrow \theta^* = \frac{4}{3} \pi \text{ is unstable.}
\]

2. Problem 4.1.5

**Solution:** To find the fixed points we solve \( f(\theta) = \sin \theta + \cos \theta = 0 \). The solutions are \( \theta^* = \frac{3}{4} \pi \) and \( \theta^* = \frac{7}{4} \pi \).

Using linear stability analysis we have:

\[
f'(\theta) = (\cos \theta - \sin \theta) \bigg|_{\theta^* = \frac{3}{4} \pi} = (\cos \left( \frac{3}{4} \pi \right) - \sin \left( \frac{3}{4} \pi \right)) = \sqrt{2} < 0 \Rightarrow \theta^* = \frac{3}{4} \pi \text{ is stable.}
\]

\[
f'(\theta) = (\cos \theta - \sin \theta) \bigg|_{\theta^* = \frac{7}{4} \pi} = (\cos \left( \frac{7}{4} \pi \right) - \sin \left( \frac{7}{4} \pi \right)) = \sqrt{2} > 0 \Rightarrow \theta^* = \frac{7}{4} \pi \text{ is unstable.}
\]

3. Problem 4.3.1

**Solution:** Let \( x = \sqrt{r} \tan \theta \) then \( dx = \sqrt{r} \sec^2 \theta d\theta \), also \( r + x^2 = r(1 + \tan^2 \theta) = r \sec^2 \theta \), substituting we get

\[
T_{\text{bottleneck}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{r + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{r} \sec^2 \theta d\theta}{r \sec^2 \theta} = \frac{1}{\sqrt{r}} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{\sqrt{r}}.
\]

4. Problem 4.3.2

**Solution:**

(a) \( u = \tan \frac{\theta}{2} \Rightarrow \theta = 2 \arctan u \Rightarrow d\theta = \frac{2}{1 + u^2} du \).

(b) \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \tan \frac{\theta}{2} \cdot \frac{1}{\sec^2 \frac{\theta}{2}} = \frac{2u}{1 + u^2} \).

(c) As \( u \to \pm \infty \), \( \arctan u \to \pm \pi/2 \) and hence by part (a) \( \theta \to \pm \pi \).

(d) With respect to \( u \) the integral becomes

\[
T = \int_{-\infty}^{\infty} \frac{d\theta}{\omega - a \sin \theta} = \int_{-\pi}^{\pi} \frac{2/(1 + u^2)}{\omega - 2au/(1 + u^2)} du = \int_{-\infty}^{\infty} \frac{2}{\omega u^2 - 2au + \omega} du
\]

(e) \( \omega u^2 - 2au + \omega = \omega (u^2 - 2\frac{a}{\omega} u + 1) = \omega \left[ (u - \frac{a}{\omega})^2 + \frac{\omega^2 - a^2}{\omega^2} \right] \) so

\[
T = \frac{2}{\omega} \int_{-\infty}^{\infty} \frac{1}{\omega^2 - a^2 + (u - \frac{a}{\omega})^2} du = \frac{2}{\omega} \int_{-\infty}^{\infty} \frac{\pi}{\sqrt{\omega^2 - a^2}} \frac{1}{\omega^2 - a^2 + (u - \frac{a}{\omega})^2} du = \frac{2\pi}{\omega \sqrt{\omega^2 - a^2}} = \frac{\pi}{\sqrt{\omega^2 - a^2}}
\]

where we used the result of problem 4.2.1 by letting \( x = u - \frac{a}{\omega} \) and \( r = \frac{\omega^2 - a^2}{\omega^2} \).
5. Problem 4.2.3

SOLUTION: If we let $\theta_h, \theta_m$ represent the positions of the hour hand and minute hand respectively then 
\[
\frac{d\theta_h}{dt} = \frac{2\pi}{12} \quad \text{and} \quad \frac{d\theta_m}{dt} = \frac{2\pi}{1} \quad \text{since it takes 12 hours for the hour hand to go around and it takes 1 hour for the minute hand to do so.}
\]
If we let $\theta = \theta_m - \theta_h$ then 
\[
\frac{d\theta}{dt} = \frac{d\theta_m}{dt} - \frac{d\theta_h}{dt} = 2\pi(1 - 1/12)
\]
So $\theta$ changes by $2\pi$ in time 
\[
\frac{2\pi}{2\pi(1 - 1/12)} = \frac{12}{11}
\]
of an hour. So the hands will be aligned at 12:00 + \frac{12}{11} hrs or 1:05:27.

6. Problem 4.3.3

SOLUTION: To find the fixed points we solve $f(\theta) = 0$. We get
\[
\mu \sin \theta - \sin 2\theta = 0
\]
\[
\sin \theta (\mu - 2 \cos \theta) = 0
\]
the fixed points are $\theta^* = 0$ and $\pi$ and $\theta^* = \arccos \frac{\mu}{2}$ which exists only for $|\mu| \leq 2$. Using linear stability analysis we get 
\[
f'(\theta) = \mu \cos \theta - 2 \cos 2\theta
\]
this implies that 
\[
f'(0) = \mu - 2 \Rightarrow \theta^* = 0 \text{ is stable for } \mu < 2 \text{ and unstable for } \mu > 2.
\]
\[
f'(\pi) = -\mu - 2 \Rightarrow \theta^* = \pi \text{ is stable for } \mu > -2 \text{ and unstable for } \mu < -2.
\]
Finally studying the fixed point $\theta^*$ such that 
\[
\cos \theta^* = \frac{\mu}{2}
\]
gives us 
\[
f'(\theta^*) = \mu \cdot \frac{\mu}{2} - 2\left(2 \cdot \frac{\mu^2}{4} - 1\right) = 2 - \frac{1}{2} \mu^2 \Rightarrow \theta^* \text{ is unstable since } |\mu| < 2.
\]
We conclude that we have a subcritical pitchfork bifurcation at $\mu = 2$: When $\mu > 2$, $\theta^* = 0$ is unstable and when $\mu < 2$, $\theta^* = 0$ becomes stable and two other unstable fixed points at $\theta^* = \arccos \frac{\mu}{2}$ and $\theta^* = -\arccos \frac{\mu}{2}$ are born.

We also have a subcritical pitchfork bifurcation at $\mu = -2$: When $\mu < -2$, $\theta^* = \pi$ is unstable and when $\mu > -2$, $\theta^* = \pi$ becomes stable and two other unstable fixed points at $\theta^* = \pi - \arccos \frac{\mu}{2}$ and $\theta^* = \pi + \arccos \frac{\mu}{2}$ are born.

The graphical analysis is given below:
7. Problem 10.1.3

**SOLUTION:** For every initial condition \( x_0 \), we have \( \lim_{n \to \infty} x_n = +\infty \).

8. Problem 10.1.6

**SOLUTION:** Very complicated dynamics. Play around with the two Maple worksheets for more details.

9. Problem 10.1.8

**SOLUTION:** For every initial condition \( x_0 \), we have \( \lim_{n \to \infty} x_n = 0 \).

10. Problem 10.1.12

**SOLUTION:**
(a) \( f(x_n) = x_n - \frac{x_n^2 - 4}{2x_n} = \frac{x_n^2}{2} + \frac{2}{x_n} \).
(b) Solve \( f(x) = x \) to get \( \frac{x}{2} + \frac{2}{x} = x \) \( \Rightarrow \) \( x^2 - 4 = 0 \) \( \Rightarrow \) \( x^* = \pm 2 \).
(c) \( f'(x) = \frac{1}{2} - \frac{x}{x^2} = f'(x^*) = f'(\pm 2) = 0 \) \( \Rightarrow \) the fixed points are superstable.
(d) \( x_1 = 2.50000, \ x_2 = 2.05000, \ x_3 = 2.00060, \ x_4 = 2.00000 \).

11. Problem 10.3.4

**SOLUTION:**
(a) \( f(x) = x \) \( \Rightarrow \) \( x^2 - x + c = 0 \) \( \Rightarrow \) \( x^*_+ = \frac{1 + \sqrt{1 - 4c}}{2} \) and \( x^*_- = \frac{1 - \sqrt{1 - 4c}}{2} \).

We will have fixed points only if \( c \leq \frac{1}{4} \). Now since \( f'(x) = 2x \) we have

\[
f'(x^*_+) = 1 + \sqrt{1 - 4c} > 1 \Rightarrow x^*_+ \text{ is repelling for all } c < \frac{1}{4}
\]

\[
f'(x^*_-) = 1 - \sqrt{1 - 4c} \text{ so } -1 < 1 - \sqrt{1 - 4c} < 1 \Rightarrow 0 < \sqrt{1 - 4c} < 2 \Rightarrow -\frac{3}{4} < c < \frac{1}{4}.
\]

So \( x^*_+ \) is attracting if \( -\frac{3}{4} < c < \frac{1}{4} \) and repelling if \( c < -\frac{3}{4} \).

(b) We have a saddle-node bifurcation at \( c = \frac{1}{4} \) since at that point two fixed points appear. We also have another bifurcation at \( c = -\frac{3}{4} \) since at that point \( x^*_- \) loses stability. It will turn out to be a period doubling bifurcation.

(c) We find the 2-cycles by solving \( (f \circ f)(x) = x \) or \( (x^2 + c)^2 + c = x \). This implies that \( x^4 + 2cx^2 - x + c^2 + c = 0 \).

We know that \( x^*_+ \) and \( x^*_- \) are roots, so to find the 2-cycles we solve

\[
\frac{x^4 + 2cx^2 - x + c^2 + c}{x^2 - x + c} = x^2 + x + c + 1 = 0
\]
We obtain
\[ p_\pm = \frac{-1 \pm \sqrt{-3 - 4c}}{2} \]

Note that \( p_\pm \) exist as real number only if \( c \leq -\frac{3}{4} \). Thus a 2-cycle appears precisely when \( c \) decreases through \( c = -\frac{3}{4} \).

To find where the 2-cycle is stable we have to find out where \( |f'(p_+)^2| < 1 \).

Thus a 2-cycle appears precisely when \( c \) decreases through \( c = -\frac{3}{4} \):

To find where the 2-cycle is stable we have to find out where \( |f'(p_+)^2| < 1 \).

The 2-cycle is superstabe when \( |f'(p_+)^2| = 0 \) and this happens for \( c = -1 \).

12. Find all fixed points and periodic points of period 2 for each of the given functions:

(a) \( F(x) = -x + 2 \)  
(b) \( F(x) = -2x - x^2 \).

**SOLUTION:**

(a) Fixed points: Solve \( F(x) = x \) to get \( x^* = 1 \).

Period 2 points: Solve \( (F \circ F)(x) = x \) to get \(-(-x + 2) + 2 = x \) and hence every point is a period 2 point.

(b) Fixed points: Solve \( F(x) = x \) to get \(-2x - x^2 = x \) and hence \( x^* = 0 \) and \( x^* = -3 \).

Period 2 points: Solve \( (F \circ F)(x) = x \) to get

\[
-2(-2x - x^2) - (-2x - x^2)^2 = x \\
4x - 2x^2 - 4x^3 - x^4 = x \\
x^4 + 4x^3 + 2x^2 - 3x = 0 \\
x(x + 3)(x^2 + x - 1) = 0
\]

So the period 2 points are the solutions of \( x^2 + x - 1 = 0 \), which are \( \frac{1}{2} \sqrt{5} - \frac{1}{2}, \frac{1}{2} \sqrt{5} - \frac{1}{2} \).

13. Describe the fate of the orbit of each of the following seeds under iteration of the function

\[
T(x) = \begin{cases} 
2x, & \text{if } x < 1/2; \\
2 - 2x, & \text{if } x \geq 1/2
\end{cases}
\]
Therefore the cycle is repelling.

14. For each of the given functions, find all fixed points and determine whether they are attracting, repelling, or neutral.

(a) \( F(x) = (\pi/2) \sin x \) \hspace{1cm} (b) \( F(x) = 3x(1-x) \).

**SOLUTION:**

(a) Fixed points: Solve \( F(x) = x \) to get \( x^* = 0, x^* = \pm \frac{\pi}{2} \). Since \( F'(0) = \frac{\pi}{2} > 1 \), \( x^* = 0 \) is repelling. Also since \( F'(\pm \frac{\pi}{2}) = 0, x^* = \pm \frac{\pi}{2} \) are attracting.

(b) Fixed points: Solve \( F(x) = x \) to get \( x^* = 0 \) and \( x^* = \frac{2}{3} \).

\( F'(0) = 3 > 1 \implies x^* = 0 \) is repelling. \( F'(2/3) = 3 - 4 = -1 \implies x^* = \frac{2}{3} \) is neutral.

15. What can you say about fixed points for \( F_c(x) = ce^x \) with \( c > 0 \)? What does the graph of \( F_c \) tell you about these fixed points?

Note that when \( c = 1/e \), \( F_c(1) = 1 \).

**SOLUTION:** Let us study the function \( f(x) = F_c(x) - x = ce^x - x \). The derivative \( f'(x) = ce^x - 1 = 0 \) when \( x = -\ln c \) and since \( f''(-\ln c) = 1 > 0 \), we conclude that \( f \) has a minimum at \( x = -\ln c \).

Case 1: If \( f(-\ln c) > 0 \), i.e., when \( 1 + \ln c > 0 \) or \( c > 1/e \) then \( F_c(x) = x = ce^x - x > 0 \) and we do not have fixed points.

Case 2: If \( f(-\ln c) = 0 \), then \( F_c(x) = x = ce^x - x \geq 0 \) and equality is true only at \( x = -\ln c \). Therefore, there is only one fixed point where the graph of \( F_c(x) = ce^x \) is tangent to \( y = x \) from above. The fixed point is neutral.

Case 3: If \( f(-\ln c) < 0 \), then \( F_c(x) = x = 0 \) at two different points. Hence there are two fixed points. Since \( F_c(x) \) is below \( y = x \) between the two points, one is attracting and the other repelling. The graphical analysis follows:

16. Consider the function

\[
T(x) = \begin{cases} 
4x, & \text{if } x < 1/2; \\
4 - 4x, & \text{if } x \geq 1/2 
\end{cases}
\]

Does \( T \) have any attracting cycles? Why or why not?

**SOLUTION:** Suppose that \( T \) has an \( n \)-cycle, \( x_0, x_1, \ldots, x_n = x_0 \) then

\[
|(T^n)'(x_0)| = |T'(x_0) \cdot T'(x_1) \cdots T'(x_{n-1})| = 4^n
\]

Therefore the cycle is repelling.
17. Each function undergoes a bifurcation of fixed points at the given parameter value. In each case use analytic or qualitative methods to identify this bifurcation as a tangent, pitchfork, or period doubling bifurcation or as none of these. Discuss the behavior of orbits near the fixed points in question at, before, and after the bifurcation.

(a) \( F_\alpha(x) = x + x^2 + \alpha, \quad \alpha = 0 \)  

(b) \( F_\alpha(x) = \alpha \sin x, \quad \alpha = 1 \).

**SOLUTION:**

(a) The fixed points are given by \( x + x^2 + \alpha = x \) or \( x^2 + \alpha = 0 \). Therefore for \( \alpha > 0 \), there are no fixed points. For \( \alpha = 0 \), there is one fixed point; and for \( \alpha < 0 \), there are two fixed points at \( x = \pm \sqrt{-\alpha} \).

Differentiation yields \( F'_\alpha(x) = 1 + 2x \) and \( F'_\alpha(\pm \sqrt{-\alpha}) = 1 \pm 2\sqrt{-\alpha} \).

Therefore for small enough \( \alpha \), \( 0 < 1 - 2\sqrt{-\alpha} < 1 \) and \( x = -\sqrt{-\alpha} \) is attracting. Since \( 1 + 2\sqrt{-\alpha} > 1 \), \( x = \sqrt{-\alpha} \) is repelling. For \( \alpha = 0 \), \( F_\alpha(x) \) is tangent to \( y = x \) from above and therefore \( x = 0 \) is neutral. The bifurcation is a saddle-node bifurcation.

(b) For \( \alpha \) slightly smaller than 1, the origin is the only fixed point and it is attracting. For \( \alpha = 1 \), \( F_\alpha(x) \) is tangent to \( y = x \) and 0 is attracting. For \( \alpha > 1 \), two more fixed points appear and they are attracting for \( \alpha \) slightly larger than 1. The origin becomes a repelling fixed point. This is a pitchfork bifurcation.