SPRING 2005 67-717 HOMEWORK SET 1 SOLUTIONS

1. Problem 2.1.4 part(a) only.

Solution: When we substitute $x_0 = \pi/4$, we obtain $t = \ln\left|\dfrac{1+\sqrt{2}}{\csc x + \cot x}\right|$, this implies that

$$\begin{aligned} e^t &= \left|\frac{1+\sqrt{2}}{\csc x + \cot x}\right| \\ \frac{e^t}{1+\sqrt{2}} &= \left|\frac{\sin x}{1+\cos x}\right| \\ \frac{e^t}{1+\sqrt{2}} &= \left|\tan\frac{x}{2}\right| \end{aligned}$$

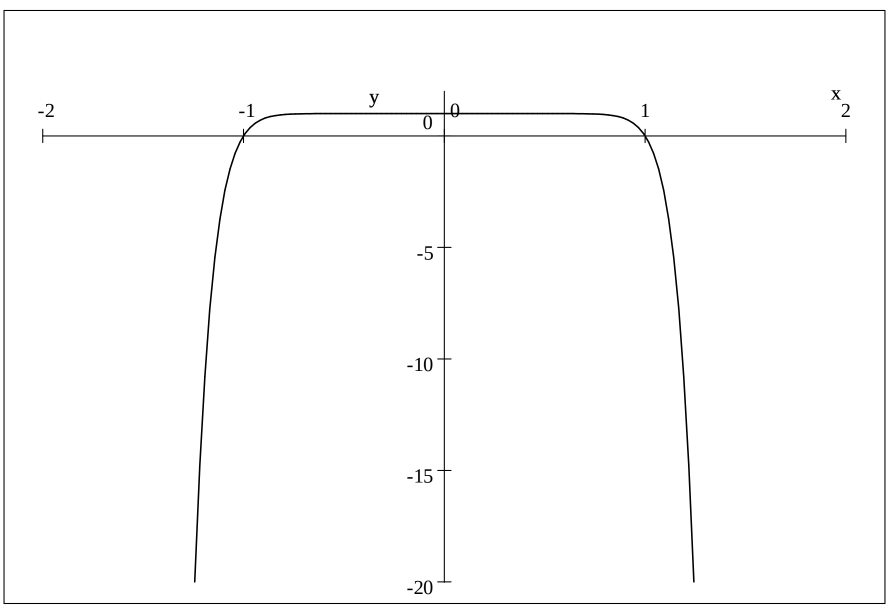
Now we know from geometric consideration,i.e., examining the flow line that if $x_0 = \pi/4$, then $\pi/4 \le x < \pi$, therefore

$$\frac{e^t}{1+\sqrt{2}} = \tan\frac{x}{2} \text{ or } x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)$$

Clearly $\lim_{t\to\infty} x(t) = 2 \cdot \lim_{t\to\infty} \tan^{-1}\left(\dfrac{e^t}{1+\sqrt{2}}\right) = 2 \cdot \dfrac{\pi}{2} = \pi.$

2. Problem 2.2.2.

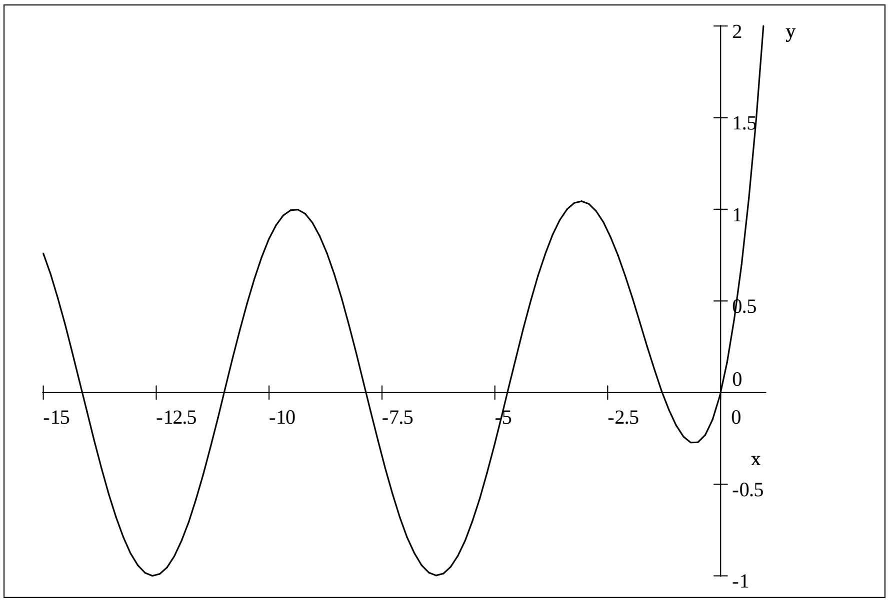
Solution: The graph is



The fixed points are at $x = -1$ and $x = 1$. The equilibrium point $x = 1$ is stable while $x = -1$ is unstable.

3. Problem 2.2.7.

Solution: The graph is $y = e^x - \cos x$



Clearly there are infinitely many fixed points and they alternate between unstable and stable starting from $x = 0$ and going in the negative direction. Note that as $x \to -\infty$, the fixed points tend to the zeros of the cosine function.

4. Problem 2.2.11.

Solution: If we let $u = \frac{V_0}{R} - \frac{Q}{RC}$ then $\frac{du}{dt} = -\frac{1}{RC}\frac{dQ}{dt}$ and $u(0) = \frac{V_0}{R} - \frac{Q(0)}{RC} = \frac{V_0}{R}$, the initial value problem becomes

$$-RC\frac{du}{dt} = u \qquad u(0) = \frac{V_0}{R}$$

Or

$$\frac{du}{dt} = -\frac{1}{RC}u \qquad u(0) = \frac{V_0}{R}$$

The solution is

$$u(t) = \frac{V_0}{R}e^{-\dfrac{t}{RC}}$$

Therefore

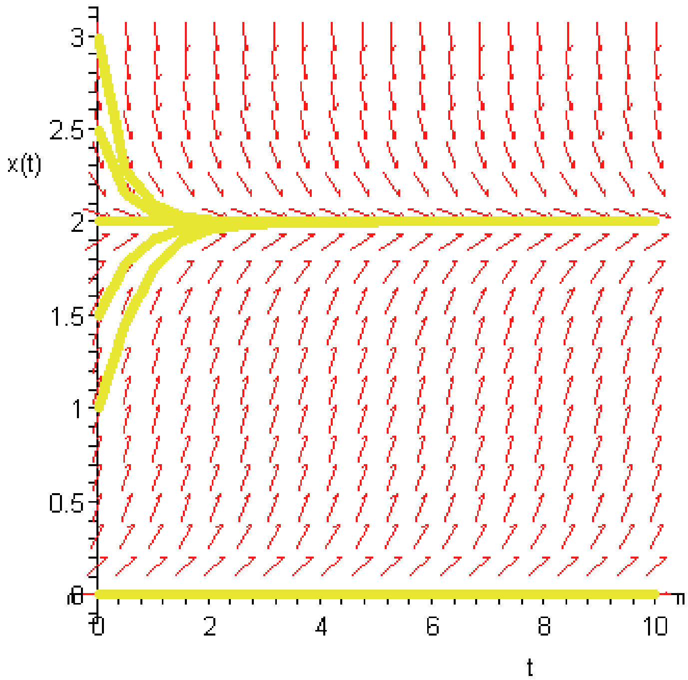
$$\frac{V_0}{R} - \frac{Q(t)}{RC} = \frac{V_0}{R}e^{-\dfrac{t}{RC}}$$

Solving for Q we get

$$Q(t) = CV_0(1 - e^{-\dfrac{t}{RC}})$$

5. Problem 2.3.2

Solution: (a) The fixed points are given by the solutions of $k_1ax - k_{-1}x^2 = 0$ so $x^* = 0$ and $x^* = \frac{k_1a}{k_{-1}}$.
Since the graph of $\dot{x}$ vs x is a parabola that opens downward, we see that $x^* = 0$ is unstable and $x^* = \frac{k_1a}{k_{-1}}$ is stable.
(b) $\dot{x} = 2x - x^2$

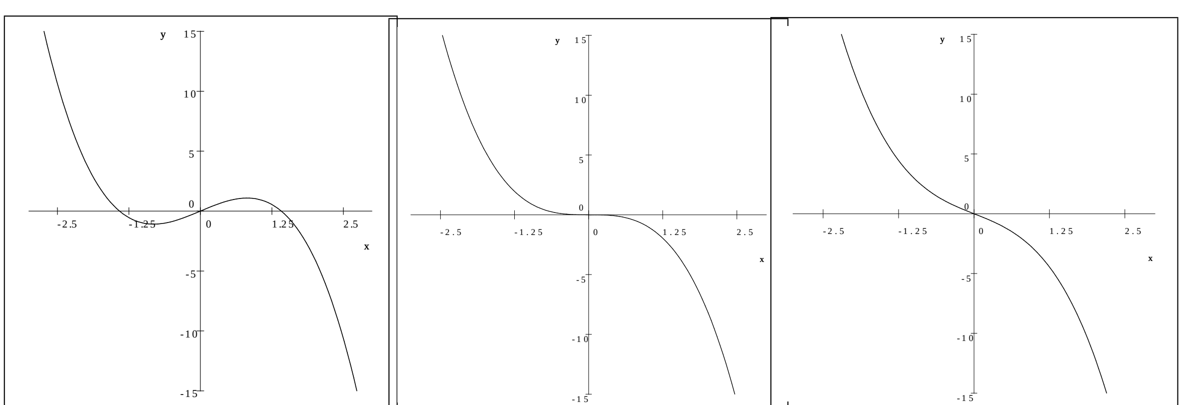


6. Problem 2.4.2.

Solution: $f(x) = x(1-x)(2-x) = 0 \Longrightarrow$The fixed points are $x^* = 0$, $x^* = 1$, and $x^* = 2$. Since $f'(x) = 3x^2 - 6x + 2$ we have $f'(0) = 2 > 0 \Longrightarrow x^* = 0$ is unstable, $f'(1) = -1 < 0 \Longrightarrow x^* = 1$ is stable and $f'(2) = 2 > 0 \Longrightarrow x^* = 2$ is unstable.

7. Problem 2.4.7.

Solution: Case 1: $a > 0$ then $f(x) = ax - x^3 = 0 \Longrightarrow x(\sqrt{a}+x)(\sqrt{a}-x) = 0 \Longrightarrow x^* = 0, \pm\sqrt{a}$ are the fixed points. Since $f'(x) = a - 3x^2$ then $f'(0) = a > 0 \Longrightarrow x^* = 0$ is unstable, $f'(\pm\sqrt{a}) = -2a < 0 \Longrightarrow x^* = \sqrt{a}$ and $x^* = -\sqrt{a}$ are stable.
Case 2: $a = 0$ then $f(x) = -x^3 \Longrightarrow x^* = 0$ is the only fixed point and graphically it is clearly stable.
Case 3: $a < 0$ then $f(x) = -x(x^2 - a) = 0 \Longrightarrow x^* = 0$ and since $f'(0) = a < 0$ it is stable.



$a > 0$ $\qquad$ $a = 0$ $\qquad$ $a < 0$

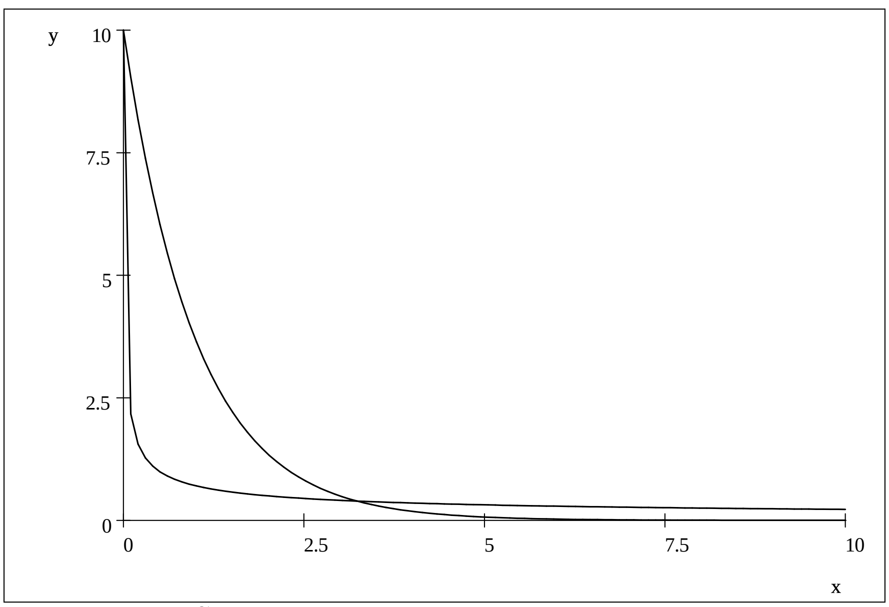
8. Problem 2.4.9.

Solution: (a) Separating the variables and integrating we get

$$\begin{aligned}
-\int_{x_0}^{x} \frac{du}{u^3} &= \int_0^t dv \\
\left.\frac{1}{2u^2}\right|_{x_0}^{x} &= t \\
\frac{1}{x^2} - \frac{1}{x_0^2} &= 2t \\
\frac{1}{x^2} &= 2t + \frac{1}{x_0^2} \\
x^2 &= \frac{x_0^2}{2x_0^2 t + 1} \\
x(t) &= \pm\frac{|x_0|}{\sqrt{2x_0^2 t + 1}} = \frac{x_0}{\sqrt{2x_0^2 t + 1}}
\end{aligned}$$

Clearly $\lim_{t\to\infty} x(t) = 0$ but it is not exponential decay.

(b) $x(t) = \dfrac{10}{\sqrt{200t+1}}$ and $x(t) = 10e^{-t}$.



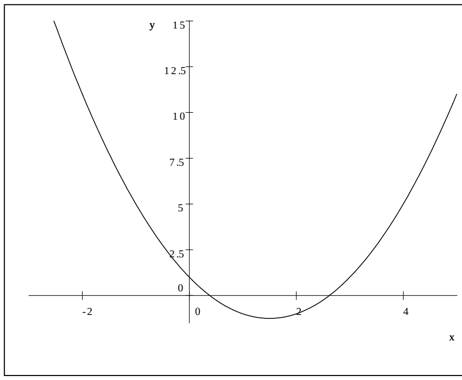
Critical slowing down vs Exponential decay

9. Problem 3.1.1

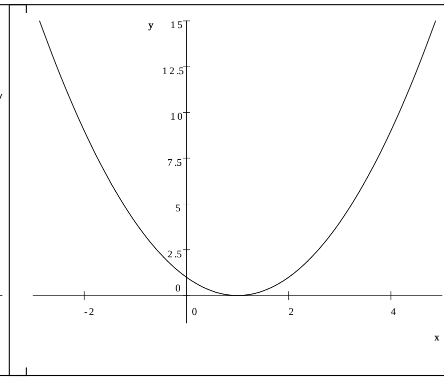
SOLUTION: We know we have a saddle-node bifurcation. Let us solve for the fixed point and parameter value at the bifurcation point. We solve $f(x,r) = 0$ and $\dfrac{\partial f}{\partial x}(x,r) = 0$ for x^* and r_c.

$$\left\{\begin{array}{c} 1 + rx + x^2 = 0 \\ r + 2x = 0 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} 1 - x^2 = 0 \\ r = -2x \end{array}\right. \Longrightarrow \left\{\begin{array}{c} x^* = \pm 1 \\ r_c = \pm 2 \end{array}\right.$$

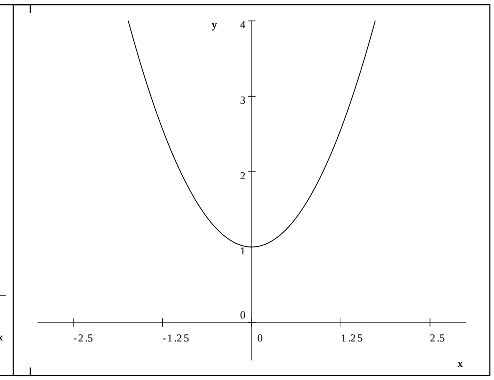
The graphical analysis is given by:



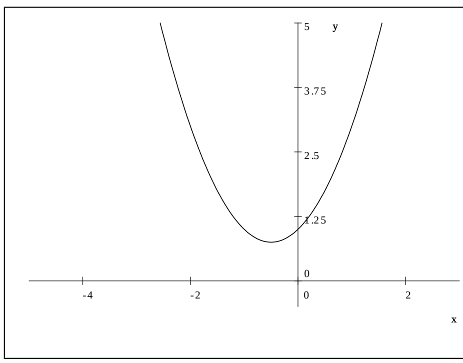
$r < -2$



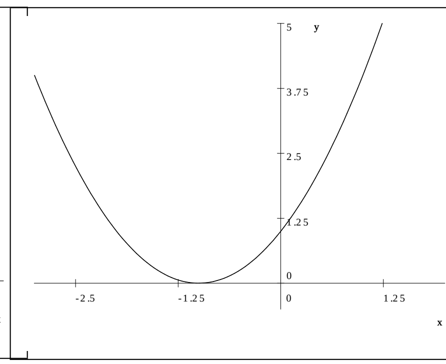
$r = -2$



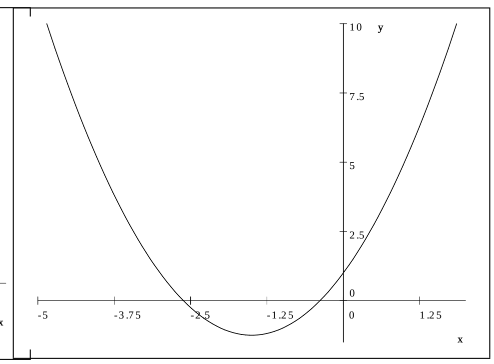
$r > -2$



$r < 2$

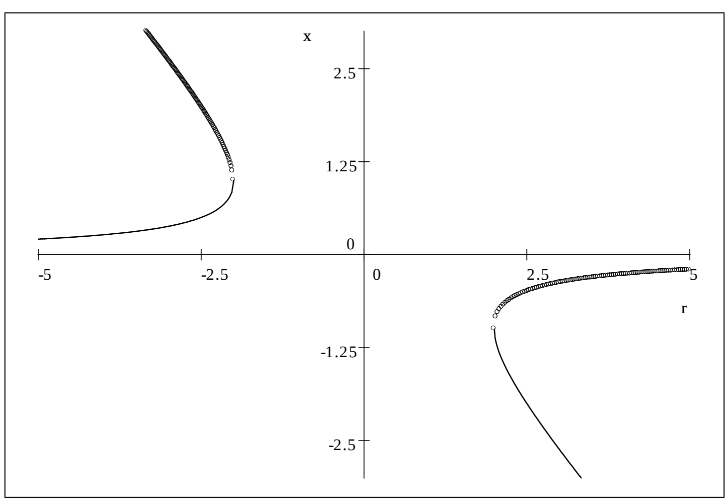


$r = 2$



$r > 2$

We can get the bifurcation diagram (in this case exactly by solving for x^* in terms of r.) When $r < -2$ we have $x^* = \dfrac{-r \pm \sqrt{r^2 - 4}}{2}$



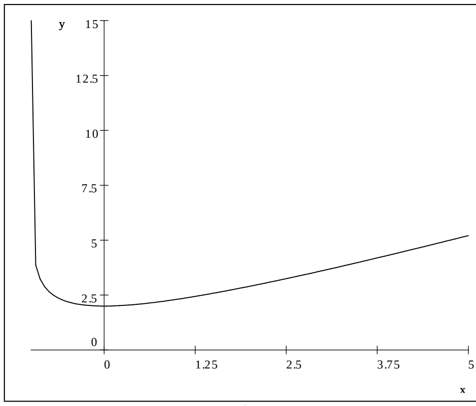
$\circ$: Unstable

10. Problem 3.1.3

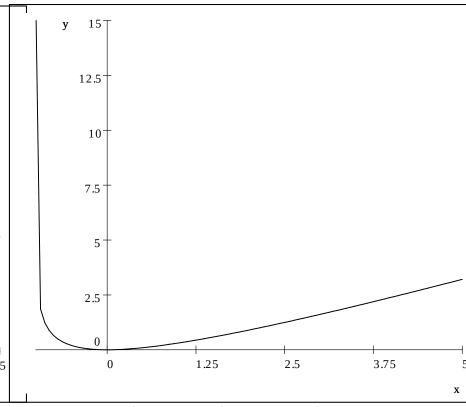
SOLUTION: Proceeding as in the previous problem, we get that

$$\begin{cases} r + x - \ln(x+1) = 0 \\ 1 - \dfrac{1}{x+1} = 0 \end{cases} \implies \begin{cases} r = \ln(1+x) - x \\ 1 + x = 1 \end{cases} \implies \begin{cases} r_c = 0 \\ x^* = 0 \end{cases}$$

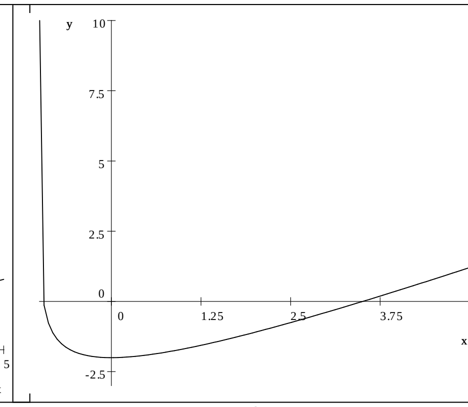
The graphical analysis is given by



$r > 0$: No fixed points

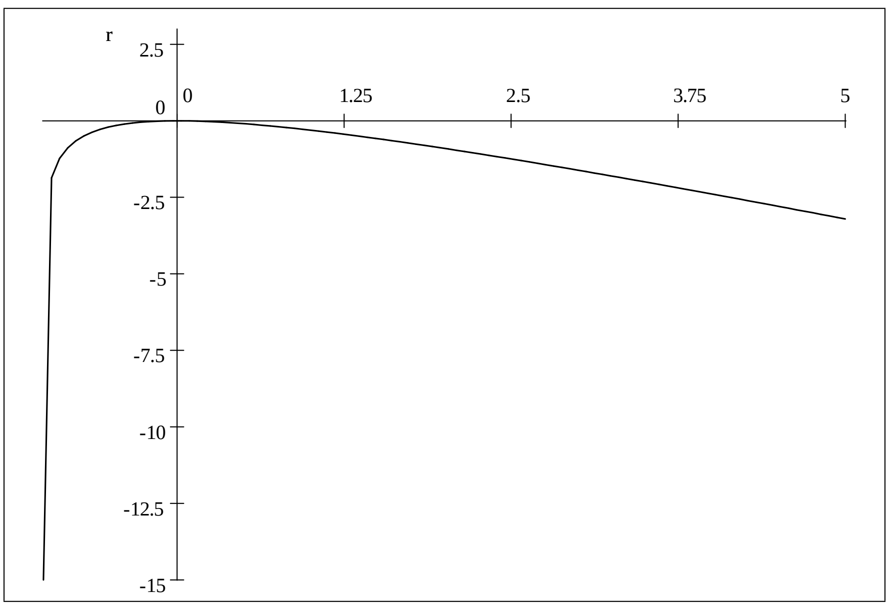


$r = 0$: One half-stable fixed point.



$r < 0$: Two fixed points.

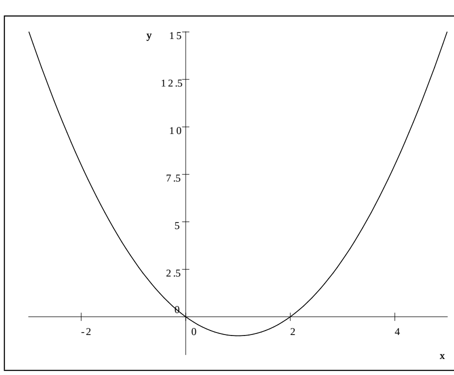
The bifurcation diagram is given by



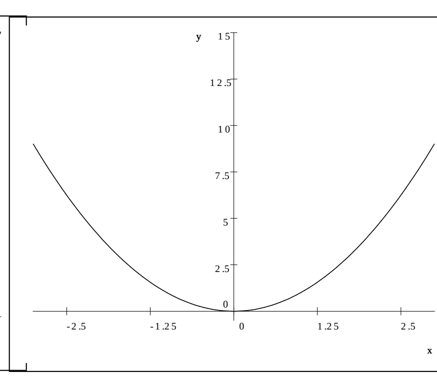
Left branch: Stable, Right branch: Unstable.

11. Problem 3.2.1.

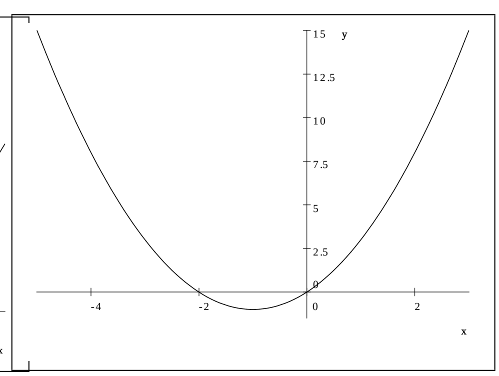
SOLUTION: This is an example of a transcritical bifurcation. Using a graphical analysis we get



$r < 0$

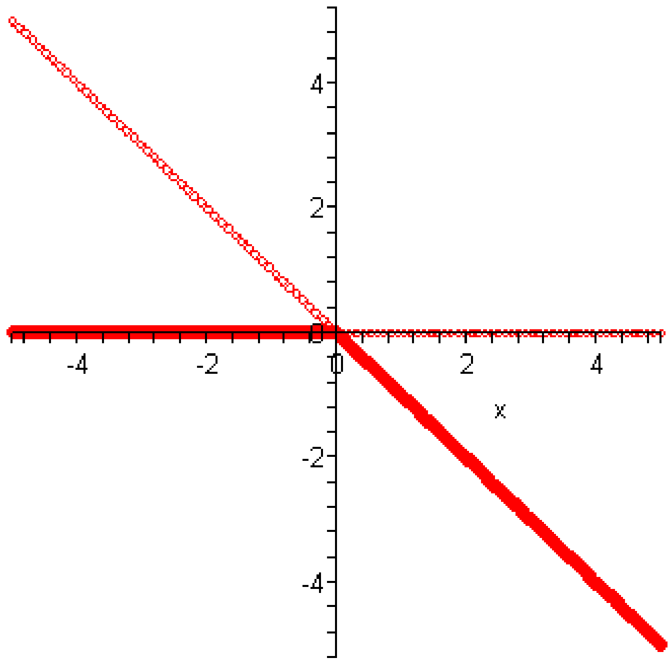


$r = 0$



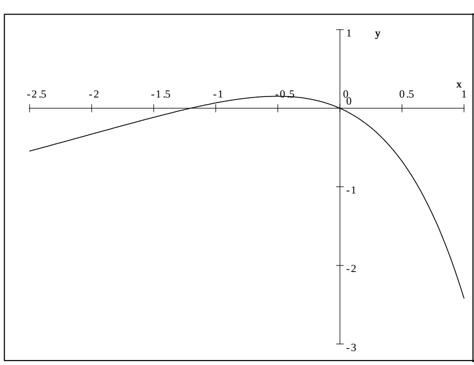
$r > 0$

This is clearly a transcritical bifurcation since $x^* = 0$ is always a fixed point and it exchanges stability with the other fixed point $x^* = -r$ as we cross the bifurcation point $r_c = 0$. The bifurcation diagram is given by

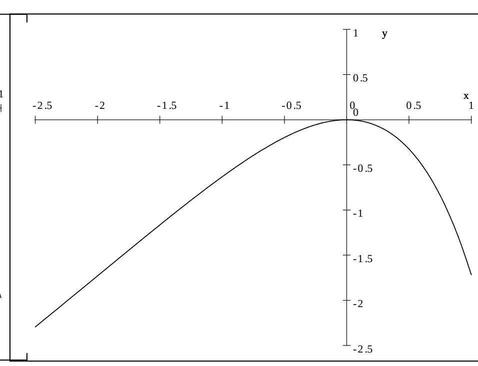


12. Problem 3.2.4.

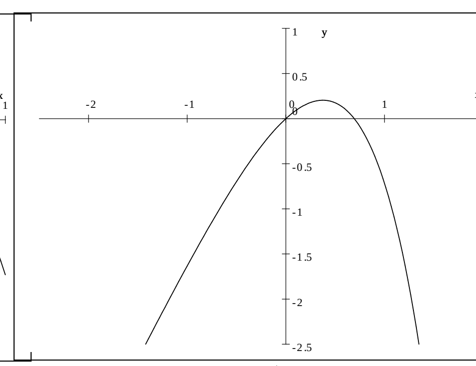
SOLUTION: By expanding we get $\dot{x} = x(r - 1 - x - O(x^2)) = (r-1)x - x^2 + O(x^2)$.
By what we know of normal forms we suspect a transcritical birfuraction at $r_c = 1$.
Let us give a graphical analysis:



$r < 1$

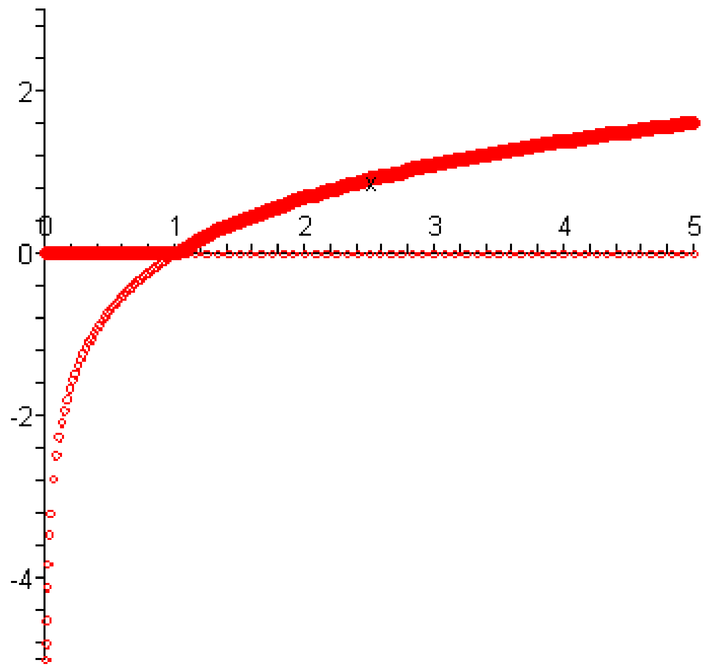


$r = 1$



$r > 1$

Again we can see that this is clearly a transcritical bifurcation since $x^* = 0$ is always a fixed point and it exchanges stability with the other fixed point $e^{x^*} = r$ or $x^* = \ln r$ as we cross the bifurcation point $r_c = 1$. The bifurcation diagram is given by



13. Problem 3.3.1 Parts (a), (b) and (c) only..

SOLUTION:(a) If we set $\dot{N} = 0$ we get from the second equation that $N = \dfrac{p}{Gn + f}$.
Substituting this value in equation 1 we get

$$\frac{dn}{dt} = \left(\frac{Gp}{f + Gn} - k\right) n = k\left(\frac{p}{k} - \frac{f}{G} - n\right)\frac{Gn}{f + Gn} = F(n)$$

(b) Since $n^* = 0$ is a fixed point, let us study its stability using linear stability analysis.by examining the sign of $\dfrac{dF}{dn}(0)$. First we compute $\dfrac{dF}{dn}$:

$$\frac{dF}{dn} = -k\frac{Gn}{f + Gn} + k\left(\frac{p}{k} - \frac{f}{G} - n\right)\frac{G(f + Gn) - Gn(G)}{(f + Gn)^2}$$

Hence:

$$\frac{dF}{dn}(0) = k\left(\frac{p}{k} - \frac{f}{G}\right)\frac{G}{f}$$

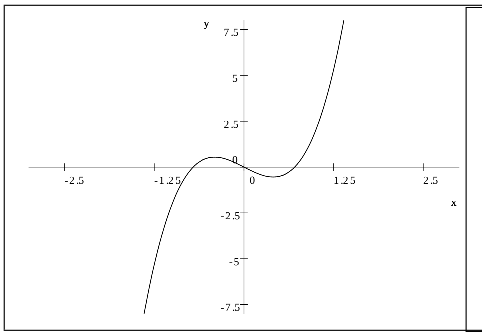
Since $G > 0, k > 0$ and $f > 0$, we conclude that

$$\frac{dF}{dn}(0) < 0 \text{ for } p < p_c = \frac{kf}{G} \quad \text{and} \quad \frac{dF}{dn}(0) > 0 \text{ for } p > p_c = \frac{kf}{G}$$

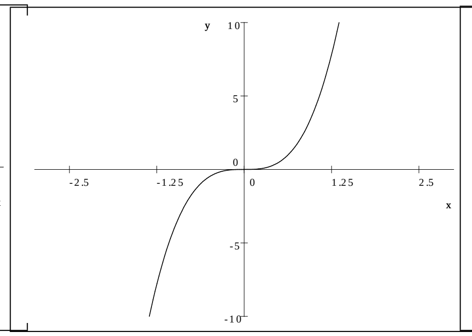
This implies that $n^* = 0$ is stable for $p < p_c$ and unstable for $p > p_c$.
(c) Clearly a bifurcation occurs at $p = p_c$. At the bifurcation point, the two fixed points $n_1^* = 0$ and $n_2^* = \dfrac{p}{k} - \dfrac{f}{G}$ switch stability and thus we have a transcritical bifurcation.

14. Problem 3.4.1

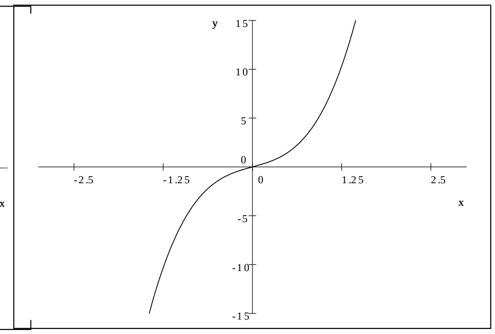
SOLUTION: By an easy change of variable we could bring the given problem into the subcritical pitchfork bifurcation normal form. The following graphical analysis confirms this fact.



$r < 0$

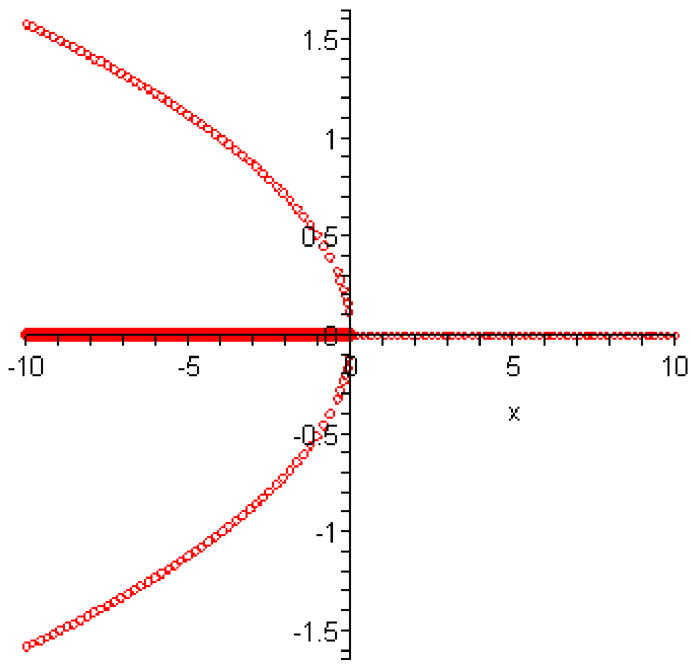


$r = 0$



$r > 0$

When $r < 0$, we have three fixed points given by $x^* = 0$ (stable), $x^* = \pm\frac{\sqrt{-r}}{2}$ (both unstable). When $r \geq 0$, we have only one fixed point $x^* = 0$ that is unstable. The bifurcation diagram is

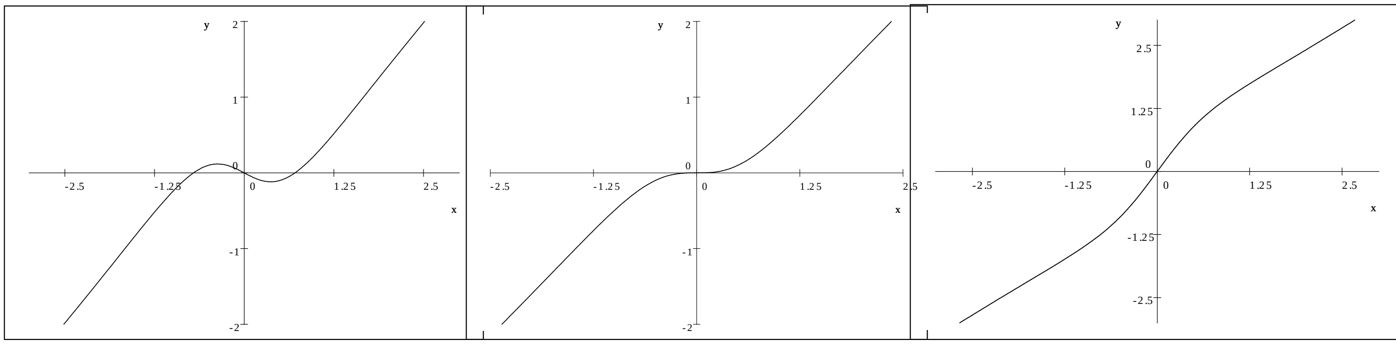


15. Problem 3.4.4.

SOLUTION: We can get an idea of what kind of pitchfork bifurcation we have in this case by expanding the right hand side:

$$\dot{x} = x + \frac{rx}{1+x^2} = x + rx(1 - x^2 + O(x^4)) = (r+1)x - rx^3 + O(x^5)$$

We expect a pitchfork bifurcation at $r_c = -1$ and it is subcritical since the cube is destabilizing (remember r is close to -1). The following graphical analysis confirms that



$r < -1$ $\quad\quad$ $r = -1$ $\quad\quad$ $r > -1$

When $r < -1$ we have three fixed points: $x^* = 0$ (stable), $x^* = \pm\sqrt{-(1+r)}$ (both unstable).
When $r \geq -1$ we have one fixed point $x^* = 0$ (unstable).
The bifurcation diagram is