



# TEACHER'S GUIDE

## **Cognitively Guided Instruction**

Cognitively Guided Instruction is an inquiry-based approach to teaching mathematics that was developed at the Wisconsin Center for Education Research (Carpenter et al, 1999). This extensively researched approach provides teachers with knowledge about the developmental stages of children's mathematical reasoning. This knowledge enables teachers to plan mathematics instruction based on their students' understanding and guide them toward greater mathematical reasoning and concept mastery.

### **Developing Mathematical Reasoning Using Word Problems**

The word problems at the end of each legend in the preceding section are based on detailed analyses of various problem situations involving number. These problems are organized within a framework that makes it possible to distinguish among problems in terms of reasoning difficulty. An important aspect of applying this knowledge when teaching is sequencing word problems from easiest to most difficult. Such sequencing allows children to develop mathematical reasoning. The following chart provides examples of 14 different types of word problems. The problems are coded for reasoning difficulty: easiest ♦, slightly more difficult ♦♥, more difficult ♥, and most difficult ♠.



**PROBLEM-SOLVING SITUATIONS**

<b>JOINING PROBLEMS</b>		
<b>Join: Result Unknown (JRU)</b>	<b>Join: Change Unknown (JCU)</b>	<b>Join: Start Unknown (JSU)</b>
<p>◆ Grandmother had 5 strawberries. Grandfather gave her 8 more strawberries. How many strawberries does Grandmother have now?</p> <p align="center"><math>5 + 8 = \square</math></p>	<p>♥ Grandmother had 5 strawberries. Grandfather gave her some more. Then Grandmother had 13 strawberries. How many strawberries did Grandfather give Grandmother?</p> <p align="center"><math>5 + \square = 13</math></p>	<p>♠ Grandmother had some strawberries, Grandfather gave her 8 more. Then she had 13 strawberries. How many strawberries did Grandmother have before Grandfather gave her any?</p> <p align="center"><math>\square + 8 = 13</math></p>
<b>SEPARATING PROBLEMS</b>		
<b>Separate: Result Unknown (SRU)</b>	<b>Separate: Change Unknown (SCU)</b>	<b>Separate: Start Unknown (SSU)</b>
<p>◆ Grandfather had 13 strawberries. He gave 5 strawberries to Grandmother. How many strawberries does Grandfather have left?</p> <p align="center"><math>13 - 5 = \square</math></p>	<p>♥, Grandfather had 13 strawberries. He gave some to Grandmother. Now he has 5 strawberries left. How many strawberries did Grandfather give Grandmother?</p> <p align="center"><math>13 - \square = 5</math></p>	<p>♠ Grandfather had some strawberries. He gave 5 to Grandmother. Now he has 8 strawberries left. How many strawberries did Grandfather have before he gave any to Grandmother?</p> <p align="center"><math>\square - 5 = 8</math></p>
<b>PART -PART -WHOLE PROBLEMS</b>		
<b>Part-Part-Whole: Whole Unknown (PPW:WU)</b>	<b>Part-Part-Whole: Part Unknown (PPW:PU)</b>	
<p>◆ Grandmother has 5 big strawberries and 8 small strawberries. How many strawberries does Grandmother have altogether?</p> <p align="center"><math>5 + 8 = \square</math></p>	<p>♥, Grandmother has 13 strawberries. Five are big and the rest are small. How many small strawberries does Grandmother have?</p> <p align="center"><math>13 - 5 = \square</math> or <math>5 + \square = 13</math></p>	
<b>COMPARE PROBLEMS</b>		
<b>Comp. Difference Unknown</b>	<b>Comp. Quantity Unknown</b>	<b>Comp. Referent Unknown</b>
<p>◆ ♥, Grandfather has 8 strawberries. Grandmother has 5 strawberries. How many more berries does Grandfather have than Grandmother?</p> <p align="center"><math>8 - 5 = \square</math> or <math>5 + \square = 8</math></p>	<p>♠ Grandmother has 5 strawberries. Grandfather has 3 more strawberries than Grandmother. How many strawberries does Grandfather have?</p> <p align="center"><math>5 + 3 = \square</math></p>	<p>♠ Grandfather has 8 strawberries. He has 3 more strawberries than Grandmother. How many strawberries does Grandmother have?</p> <p align="center"><math>8 - 3 = \square</math> or <math>\square + 3 = 8</math></p>
<b>MULTIPLICATION &amp; DIVISION PROBLEMS</b>		
<b>Multiplication</b>	<b>Measurement Division</b>	<b>Partitive Division</b>
<p>◆ Grandmother has 4 piles of strawberries. There are 3 strawberries in each pile. How many strawberries does Grandmother have?</p> <p align="center"><math>4 \times 3 = \square</math></p>	<p>◆ Grandmother had 12 strawberries. She gave them to some children. She gave each child 3 strawberries. How many children were given strawberries?</p> <p align="center"><math>12 \div 3 = \square</math></p>	<p>◆ ♥, Grandfather has 12 strawberries. He wants to give them to 3 children. If he gives the same number of strawberries to each child, how many strawberries will each child get?</p> <p align="center"><math>12 \div 3 = \square</math></p>

Problem chart based on Cognitively Guided Instruction Problem Types (Carpenter et al., 1996)



## UNDERSTANDING THE STRUCTURE OF WORD PROBLEM

### What makes a problem easy or difficult?

A goal of Cognitively Guided Instruction is that young children become independent problem solvers who are able to approach and solve word problems without having to rely on having a teacher tell them how to do it. However, a number of factors influence whether a problem is appropriate for a child to solve independently. Understanding these factors helps the teacher decide which word problems to use during instruction. These factors include the following:

#### **If the Problem Involves a Situation That the Child Can Act Out**

A problem that can be acted out is easier for a child to solve than one that cannot be acted out. For example, the first of the following two problems is easier. Here the child can actually pretend that s/he is giving strawberries away. The second problem is more difficult because it requires more thought to make sense of the question being asked.

##### **SRU (Action Direct):**

Grandfather had 8 strawberries. He gave 3 of them to Grandmother.  
How many strawberries does Grandfather have now?

##### **SRU (Action Indirect):**

Grandfather gave 3 strawberries to Grandmother. He had 8 strawberries.  
How many strawberries does Grandfather have now?

#### **If the Child is Able to Model the Problem with Counters or Drawing**

When the quantities given in a problem refer to a complete set of physical objects or amounts, the problem can be modeled directly. When a word problem can be directly modeled, that is, represented in some concrete way on fingers, with tally marks, drawings, or by manipulating counters, the problem is easier. The first of the following two problems is easier because the wording guides the child's modeling. When modeling this problem with counters, a young child might choose to set out the two quantities, lining them up side by side, and then match them to determine the difference. Solving the second problem relies on the child's ability to mentally determine the relationship between quantities within the problem.

##### **CDU direct modeling situation:**

Grandfather has 8 strawberries. Grandmother has 5 strawberries.  
How many more strawberries does Grandfather have?

##### **CRU situation that requires ability to analyze:**

Grandmother has 5 strawberries. She has 3 fewer strawberries  
than Grandfather. How many strawberries does Grandfather have?



**Multiplication and division problems can be modeled.  
Therefore, young children can multiply and divide intuitively**

Very young children can solve low number multiplication and division problems because such problems can be easily modeled. However, it is important that children first solve many problems involving joining and separating situations. These experiences will allow them to develop the ability to think about numerical quantities within the context of words and to make sense of the question being asked.

**Multiplication problem:**

Grandmother has 4 piles of strawberries. There are 3 strawberries in each pile. How many strawberries does Grandmother have?

**Measurement Division problem:**

Grandmother gave 12 strawberries to some children. She gave each child 3 strawberries. How many children got strawberries?

**Partitive Division problem:**

If Grandfather shares 12 strawberries with 3 friends, how many strawberries will each friend get?

**If a problem can be modeled or acted out in the order in which it is heard, it is easier.**

When first learning to solve word problems, young children approach them in the order in which they hear them. They do not begin at the end of the problem and work backward. The ability to use inverse thinking develops after children have had many experiences with solving problems and have developed an understanding of the relationships among the numbers within a problem - they understand that combined parts make up the total. For this reason, the first of the following two problems is easier. The wording encourages the child to set out five counters and then add eight more. However, the second problem does not provide a beginning number. The child who has not yet developed the ability to relate a part of a quantity to the total quantity will respond to the second question with, "Some. Grandmother had some strawberries."

**Joining problem that can be solved in the word order given:**

Grandmother had 5 strawberries. Grandfather gave her 8 more.  
How many strawberries does Grandmother have now?

**Joining problem that cannot be solved in the order given:**

Grandmother had some strawberries. Grandfather gave her 8 more.  
Then she had 13 strawberries.  
How many strawberries did Grandmother have before Grandfather gave her any?



## The Location of the Unknown Influences the Problem Difficulty

Because young children solve problems in the order that they hear them, problems that are worded in such a way so that the unknown quantity is located at the end (first example below) are easier to solve. Problems with the missing quantity in the middle (second example below) or at the beginning (third example below) are more difficult.

As the child's understanding of quantity and relationships among quantities develops, s/he becomes able to make sense of the entire question, represent the situation, and plan a solution. When a child is able to do these steps s/he will not need to use manipulatives. Rather, the child will use his or her own unique way of mentally manipulating quantities.

### SRU location of unknown at end of problem: $(8 - 3 = \underline{\quad})$

Grandmother had 8 strawberries. She gave 3 to Grandfather.  
How many strawberries does Grandmother have now?

### SCU location of unknown in middle of problem: $(8 - \underline{\quad} = 5)$

Grandfather had 8 strawberries. He gave some to Grandmother. Now he has 5 strawberries. How many strawberries did Grandfather give to Grandmother?

### SSU location of unknown at start of problem: $(\underline{\quad} - 3 = 5)$

Grandfather had some strawberries. He gave 3 strawberries to Grandmother. Then he had 5 strawberries left. How many strawberries did Grandfather have before sharing with Grandmother?

## Children's Intuitive Solution Strategies

Extensive research has documented the developmental thinking processes that children go through when learning to solve word problems (Carpenter et al., 1992). It is important to emphasize that these processes are **intuitive**, ones that are not taught to the student by a teacher.

To effectively promote the development of mathematical reasoning without usurping the student's intuitive thinking, a teacher must clearly understand the relationships among the different types of word problems (discussed in the previous sections) and the developmental stages of children's thinking. Detailed descriptions of how children's solutions vary depending on their developmental ability are provided in the following sections.



## Relating Solution Strategies to the Developmental Stages of Mathematical Reasoning

The following word problems are used to demonstrate how children at different developmental levels will use different strategies when solving the same problems. The strategy that the child uses indicates the child's stage or level of development.

### Join: Result Unknown (JRU)

Grandfather had 6 strawberries.  
Grandmother gave him 5 more.  
How many strawberries does  
Grandfather have now?

### Separating: Result Unknown (SRU)

Grandmother had 11  
strawberries. She gave 5 to Grandfather.  
How many strawberries does  
Grandmother have now?

## Developmental Level I *Direct Modeling*

A child using a **Direct Modeling** strategy represents each number in the problem with concrete objects. In the following examples, the child solves the Join Result Unknown (JRU) and the Separate Result Unknown (SRU) problem given above by modeling with counters.

### Child's Solution to JRU

"Grandfather had 6 strawberries.  
strawberries.  
One, two, three, four, five, six."  
*(The child sets out 6 counters.)*  
"Grandmother gave him five  
More. One, two, four, five."  
*(Child sets out 5 counters and  
then pushes both sets together and  
counts all of the counters.)*  
"Now he has 11 strawberries."

### Child's Solution to SRU

"Grandmother had 11  
One, two, three, four, five, six, seven  
eight, nine, ten, eleven."  
*(Child sets out 11 counters.)*  
"She gave 5 to Grandfather. One, two,  
Three, four, five."  
*(Child counts out  
and removes 5 counters from the  
group of 11 and counts the remaining  
counters.)* "Now she has. . . one, two,  
three, four, five, six. She has six."



## Developmental Level II

### *Counting On/Back*

A child using a **Counting On/Back** strategy is able to hold a number in her/his mind and count on or back from that number while keeping track of the quantity that is added or subtracted using fingers, tally marks, or counters.

A child at this level is able to immediately recognize groups such as the amount modeled on his/her fingers without having to recount the fingers. In the following examples the child solves the Join Result Unknown (JRU) and Separate Result Unknown (SRU) problems (problems given above)) using counting strategies.

#### **Child's Solution to JRU**

"I don't have to count the six again.  
I just have to add five to it.  
I say, 'Seven, eight,  
nine, ten, eleven'." (*Child holds  
up a finger with each count.*)  
"I have eleven."

#### **Child's Solution to SRU**

"I know Grandmother  
had eleven strawberries.  
I know she gave five away.  
So, I count five down.  
'Eleven, ten, nine, eight,  
seven.' I have six left."  
(*Child folds a finger  
down with each count.*)

## Developmental Level III

### *Deriving*

A child possessing good number sense is able to solve problems in flexible ways, often breaking numbers down and recombining them by using known facts. This child frequently visualizes the quantities and solves the problem with mental math.

#### **Child's Solution to JRU (Above)**

"I know that five and five is ten.  
I took one from the six to make five.  
But I must add the one back on.  
It's eleven."

#### **Child's Solution to SRU (Above)**

"I know that ten take away five is five,  
but I started with eleven. The answer must  
be one more. It's six."



## Matching Solution Strategies to Problem Types

The problem solving decisions that a child makes when solving a problem are determined by the problem situation posed to the child. Each of the 14 different problem types requires different reasoning processes. Examples of these processes at each developmental level are provided in this section.

### Level I

#### *Direct Modeling Strategies*

At the **Direct Modeling** level, the child concretely represents (using counters, fingers, tally marks, drawings) all numerical quantities within the problem. Below are examples of problems with direct modeling solutions.

#### **JRU Problem**

Grandfather had 3 strawberries. Grandmother gave him 5 more. How many strawberries does Grandfather have now?

##### **Solution: Joins-All**

The child constructs (with manipulatives or drawing) a set of three objects and a set of five objects. The child pushes the sets together and the union of the two combined sets is counted.

#### **JCU Problem**

Grandmother has 5 strawberries. Grandfather gave her some more strawberries. Now Grandmother has 8 strawberries.  
How many strawberries did Grandfather give her?

##### **Solution: Joins-To**

The child constructs (with manipulatives or drawing) a set of three objects. Objects are added to this set until there is a total of eight objects. The child counts the number of objects that were added to find the answer.

#### **SRU Problem**

Grandfather had 8 strawberries. He gave 3 to Grandmother.  
How many strawberries does Grandfather have now?

##### **Solution: Separates-From**

The child constructs (with manipulatives or drawings) a set of a set of eight objects. Three objects are removed. The answer is the number of remaining objects.

#### **SCU Problem**

Grandmother had 8 strawberries. She gave some to Grandfather.  
Now Grandmother has 3 strawberries.  
How many strawberries did she give to Grandfather?





**Solution: Joins-To**

A set of eight objects is counted out. Objects are removed from the set until the number of objects remaining is equal to three. The answer is the number of objects removed.

**CDU Problem**

Grandfather has 3 strawberries. Grandmother has 5 strawberries.  
How many more strawberries does Grandmother have than Grandfather?

**Solution: Matches**

A set of three objects and a set of eight objects are matched one to one until one set is used up. The answer is the number of unmatched objects remaining in the larger set.

**JSU Problem**

Grandmother had some strawberries. Grandfather gave him 3 more.  
Now he has 8 strawberries.  
How many strawberries did Grandmother have to start with?

**Solution: Trial-and-Error**

A set of objects is constructed. A set of three objects is added to or removed, and the resulting set is counted. If the final count is eight, then the number of objects in the initial set is the answer. If it is not right, then a different initial set is tried.

**Level II**

***Counting On/Back Strategies***

At the **Counting On/Back** level, the child does not have to represent all quantities in the problem concretely. S/he has learned that a number names a quantity, that is, that a number can be stated rather than represented concretely. These strategies will develop intuitively over time. If a child is not able to make sense of counting strategies, the child is not developmentally ready to use them and needs more experience modeling solutions.

**JRU Problem**

Grandfather had 3 strawberries. Grandmother gave him 5 more strawberries.  
How many strawberries does Grandfather have now?

**Solution: Counts-On-From-First Number Heard**

The child begins with 3 (the first number in the problem and continues on for 5 more counts (keeping track of counts with manipulatives, tallies, or fingers). The answer is the last number in the counting sequence.

**JRU Problem**

Grandfather had 3 strawberries. Grandmother gave him 5 more strawberries.  
How many strawberries does Grandfather have now?



**Solution: Counts-On-From-Larger**

The child begins with 5 (the larger quantity and continues on for 3 more counts (keeping track of counts with manipulatives, tallies, or fingers. The answer is the last number in the counting sequence.

**JCU Problem**

Grandmother had 3 strawberries. Grandfather gave her some more strawberries. Now Grandmother has 8 strawberries.  
How many strawberries did Grandfather give her?

**Solution: Counts-On-To**

The child uses a forward counting sequence starting from 3 and continues until 8 is reached (keeping track on fingers or tally marks). The answer is the number of counting words in the sequence.

**SRU Problem**

Grandfather had 8 strawberries. He gave 3 to Grandmother.  
How many strawberries does Grandfather have now?

**Solution: Counts-Down**

The child uses a backward counting sequence starting from eight. The sequence continues for three counts . . . eight, seven, six . . . the answer is the next number.

**SCU Problem**

Grandmother had 8 strawberries. She gave some to Grandfather.  
Now she has 3 strawberries. How many strawberries did she give to Grandfather?

**Solution: Counts Down To**

The child uses a backward counting sequence starting with 8 and continues until 3 is reached, “8, 7, 6, 5, 4 . . .” The answer is the total of number words in the counting sequence, not including the number word three.

**Level III**  
*Deriving Strategies*

At this level, the child understands relationships between numbers. S/he solves problems using number facts and derived facts (combines familiar quantities when a specific fact is not at the recall level). An example of a child using a derived fact would be, “I know that nine plus four is thirteen because nine and one is ten and three more is thirteen.”



## Solution Strategies Summary

When children begin to solve problems intuitively, they concretely represent the relationships in the problem. Over time, concrete strategies are abstracted to counting strategies, and eventually, as number facts are learned, children apply this knowledge to solve problems. This developmental approach differs from the practice of rote drill for memorization of facts. Children in drill/skill classrooms often are able to recite facts but lack understanding that a fact represents a relationship between quantities; they lack mathematical reasoning in relation to number sense. Children who have been allowed to progress through the stages of mathematical reasoning described in this manual develop both number sense and mathematical reasoning.

### SYMBOLIC PROCEDURES

Much of what has been discussed to this point has focused on children's informal or intuitive problem-solving strategies. Such strategies are often very different from the standard symbolic procedures typically taught in the elementary school. Standard procedures provide powerful problem-solving tools; however, a concern is that many children merely memorize them. They never develop an understanding of the relationships among numbers within procedures. When allowed to progress through the stages described in the preceding section, a child will develop the habit of looking for numerical relationships. When introduced to the standard procedure, this child will understand the numerical relationships and will view the procedure simply as another strategy for solving problems.



## References

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### Resource Information

To schedule a Cognitively Guided Instruction workshop, please contact:  
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A text describing CGI, [Children's Mathematics: Cognitively Guided Instruction](#), is available through the Heinemann web site: <http://www.heinemann.com>

### CGI web sites:

**Blog:** <http://mindsongmath.blogspot.com/>

<http://www.mindsongmath.com/>

<http://www.abacon.com/ie/berk/wlp452a.htm>

