Meagan Mutchmor
K-8 Mathematics Consultant
Prince Charles ERC,
Phone: 788-0203 Ext.101 Fax: 772-3911
Email mmutchmor@wsd1.org

Using The Empty Numberline as a Mental Math Teaching Strategy

There has been much research done on this new approach to teaching mental math strategies in primary classrooms. Students who use the empty numberline are able to create a mental image of the strategies they are being taught and can then make the leap more easily towards mental calculations without paper.

Use of the Empty numberline also increases student's confidence in their ability to use numbers flexibly which leads to further development in their understanding of number sense.

Teachers should be encouraged to explore strategies on the empty numberline before the procedural algorithm is introduced.

- There is a growing body of research that suggests teaching of pencil and paper algorithms before fundamental part-whole thinking is established, damages students' development of number sense. That is not to say that having a functional written algorithm at some point is undesirable. Research evidence simply supports delay in teaching algorithms until appropriate part-whole understanding is fluently established with lesser numbers before written methods are applied to larger numbers. (VINCE WRIGHT 2001)
- ***http://www.nzpf.ac.nz/numeracy_framework.htm***

What is the empty numberline?

- Up to this point there is no distinction between the numberline and the empty numberline. But they differ in one central point. In contrast to the standard numberline there are neither a scale nor any other pre-given objective landmark on the empty numberline. And in the case of the empty numberline there is no rule which would require, for example, the same spatial distance between the marks which correspond to two pairs of numbers having an equal arithmetical distance. The empty numberline therefore is a reproduction of the normal numberline that is not faithful to the scale but which respects the order of numbers. Thus one can see the empty numberline as a self-made sketch that helps to elucidate important considerations about the order of numbers, and also promotes the development and

– a bit later – the reflection of halfwritten strategies for addition and subtraction (WITTMANN et. al. 1996).

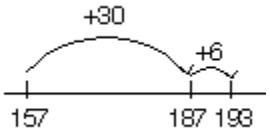
- *http://www.fmd.uni-osnabrueck.de/ebooks/gdm/PapersPdf1997/Kratzin.pdf*

Will this help me with students who are struggling?

- Yes, all students in your class will benefit from it's use in your classroom. Especially those who are having difficulty moving away from the procedural algorithm.
- This is a wonderful intervention strategy for students who struggled with Number Set E in the C.A.P. for Grade 4.
- Use of the empty numberline also allows students to see the variety of ways that the same question may be explored in attempts to find the correct answer. It is important that students "see" the numbers, "see" the strategy and explore more than one way of finding the result.

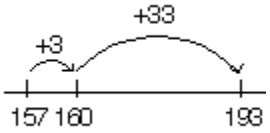
Example: 157 + 36 = (note by writing the equation horizontally you force the students to look at the numerals, whereas when written vertically students tend to immediately flop into the procedural algorithm.)

Each student can show how he or she thought about the problem by drawing and filling in an empty number line



A number line showing 30 and 6

One of the interesting things about mental calculations is that we do not all think the same way. Some people start by breaking the 36 into 33 and 3. This turns the question into the problem of adding 33 to 160.



A number line showing adding 3 and then 33

How do I introduce this technique?

The attached article is a great introduction to using the empty numberline in your classroom. It contains a step by step guide to classroom implementation.

I AM SURE THAT IF YOU TRY OUT THIS STRATEGY YOU WILL QUICKLY SEE THE POWER IT HAS TO HELP YOUR STUDENTS UNDERSTAND THE MATHEMATICS THEY ARE USING AND BECOME MORE CONFIDENT IN THEIR CLASSROOM DISCUSSIONS ABOUT THE STRATEGIES THEY USE WHEN SOLVING NUMBER SENTENCES.

Meagan

Instructional Sequence for Teaching Students Mental Computation for Adding and Subtracting Numbers Up to 100

http://peabody.vanderbilt.edu/depts/tandl/mted/faculty/Mted3250/HtmlPapers/Sally.html

Purpose of Instructional Sequence

Day to day life calls for people to be able to make mental computations. However, many people still struggle with this task as they attempt to solve problems using the standard algorithm in their head. Perhaps they will even have to pretend to write it out with their finger. People are so used to performing algorithms that they are unable to identify easier ways to perform mental computations. Therefore it is important for students to understand the mathematical tasks they are performing. This will enable them to choose methods besides using the algorithm that will make mental computations easier. The goal of this instructional sequence is to move students to the point where they will be able to mentally compute linear addition and subtraction up to one hundred in their heads by having them develop number sense and the ability to mentally estimate.

Starting point

Students should be able to count to one hundred and be able to add and subtract numbers up to twenty easily without counting. The students may be familiar with adding and subtracting up to 100, but this sequence should be done before students become fluent in using algorithms.

Phase One

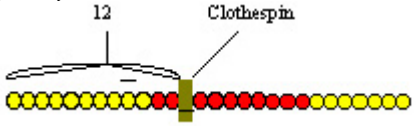
During this phase students will be using a string of one hundred beads to add and subtract. The beads alternate between two colors in groups of ten. This is to encourage students to use the going through ten strategy. The goal of this phase is for students to be able to perform addition and subtraction on the beads by using jumps and not counting the beads.

Task 1

The first task will be done as a whole class. The purpose is to explore the use of the beads. The students will gain an understanding of what they can accomplish with the beads. They will do this by:

1. Counting by tens both forwards and backwards
2. Starting at an arbitrary point and counting by ones both forwards and backwards
3. Indicating numbers on the string that they work forward from the beginning of the bead string to locate such as 12 or 21 and numbers that they work backwards using the next set of ten for reference such as 19 or 38
4. Performing jumps of ten both forwards and backwards from numbers that are not multiples of ten

Special Note: It is important the students understand the quantity indicated includes the beads up to and the given bead. For example 12 does not mean the 12th bead. It means all 12 beads in a group. The teacher can exemplify this point by using a clothes pin to mark the quantity of twelve.



Possible Discussion Questions:

1. How many beads are on the string?
2. How can we count by tens on the beads? Can you go backwards?
3. How many beads are in each color group? How can you count from 20 to 60? How can you count from 70 to 40?
4. What number is this? (Teacher points to a number that is not a multiple of ten, for example, 43.)
5. Can you count from 43 to 47? 56?
6. Can you count from 43 to 38? 33?
7. Where is the number 12? 21? 42?
8. Where is the number 19? 38? 79?
9. What number is this? *(Teacher chooses 23)*
10. How could we find 63? 13?

Anticipated Thinking and Common Misconceptions

The teacher needs to make sure that students realize there are many ways to move around the string of beads besides just counting by ones. It may not be obvious to them initially that the colors are in groups of ten. The students may also believe that the only way to be absolutely sure that the number they are trying to identify is correct is to count by ones. The teacher needs to help them realize they can correctly identify numbers using more efficient methods.

Assessment

The teacher needs to make sure that the students understand the different ways one can move around the beads. He or she does this by asking students to explain their answers to his or her questions. The teacher may continue to ask students questions that require the same type of reasoning until he or she is sure the students understand how to use the number line.

Special Note: The teacher may choose to play "Guess my Number" with the students to reinforce what they have learned about the beads and continue assessment. He or she chooses a number on the beads and the students ask questions such as is it smaller than 60 or is it larger than 10. The teacher keeps track of their range using clothespins to mark it. The students need to realize that because the number is not smaller than ten does not mean it is greater. The number could be ten.

Task 2

The next task requires students to begin adding and subtracting using the beads. The goal is to get students to the point where they are keeping track of their partial results using jumps and not counting by ones. They will do this by solving problems and verbally explaining their partial results as they go along. For missing addend problems they will work on determining the difference between the two problems. First they will do this in a whole group then they will be given a worksheet to work on independently. While they are working in a whole group, the teacher will record their jumps on the board. The teacher should tell the students that they must show the jumps they take on their worksheet. There will be beads in the classroom for the students to use while doing their worksheet, but it won't be encouraged.

Good Problems for this task:

37+26=
48+21=
24+39=
99-17=
75-26=
52-49=
54+__=78
28+__=76
25+__=100

Possible Discussion Questions:

1. Who can show us how to find 37 + 26? (Teacher has a volunteer come to the front of the classroom to demonstrate)
2. Where is your starting point? Now how are you going to add 26 to this? (During this time the teacher will interrupt a student if they are trying to count by ones. The goal is to have students perform the addition in steps while keeping track of their partial results. For example: "37+3=40", "40+3=43", "43+10=53", and "53+10=63". The student should also explain that 3+3+10+10=26. The teacher should also be recording the steps on the board, so students see they are talking about groups of beads and not a single bead.)
3. Is there an easier way you can do this without counting out 26? Who has an idea of how we could accomplish this without counting twenty-six beads?
4. How can we find 52 - 49? (The teacher has a student demonstrate his or her method to the class) Did anyone do it a different way? (The teacher wants to have students demonstrate both the taking away and finding the difference strategy). Which way is easier? (The teacher wants students to see that finding the difference is the more efficient method for this problem.)
5. Now, who can help us fill in the missing number for 55 +__=74? (The teacher takes a volunteer. The students need to realize the easiest way to do the problem is to find the two numbers and determine the difference afterwards.)

6. Where is 55? Where is 74? So how many beads do we need to add to 55 in order to get 74? (The teacher wants to encourage the student to assess the difference using jumps and not just counting the beads. This is encouraged by choosing numbers that have a great difference. The teacher should also keep track of the process or jumps the students take on the blackboard, so all the students can see visually what the student is doing. Once the students have done a number of problems together the teacher will give them the worksheet to begin independently.)

Anticipated Thinking and Common Misconceptions

The biggest challenge the teacher will face during this task is the students' desire to count by ones to solve the problem. The teacher needs to encourage them to use the beads in the other ways they did as a group, such as jumping by tens. Some students may also have difficulty explaining their partial results and the small steps they took to reach their final answer. However, they must be able to do this so the teacher can assess their understanding and other students can learn the methods.

When children are solving 27+36 the teacher wants to look for specific strategies such as adding doubles. This would occur if a student added 27+3=30, 30+30=60, 60+3=63. Students may want to stick with jumping by tens. For example: 27+10=37, 37+10=47, 47 + 10=57, 57+3=60, 60+3=63. Students may use other variations of using doubles, jumping by tens, and going through tens. It is important that students explain their reasoning during their partial steps. The teacher also needs to encourage and praise students who are using higher level thinking strategies.

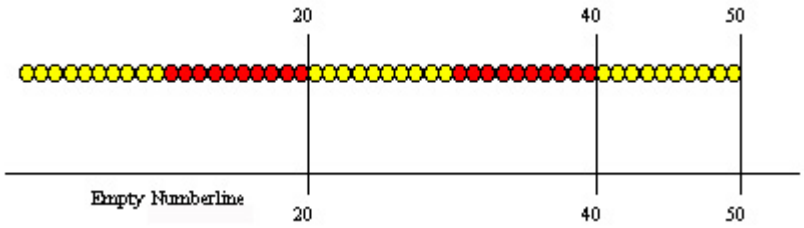
The teacher also wants to look for the two possible strategies that students can use during subtraction. Students may either look for the difference or take away. Students should be encouraged to use the method of looking for the difference. The teacher also wants students to determine when it is easier to use which method. For example when kids subtract 49 from 52 it would be much easier to look at the difference.

Assessment

Assessment will take place during the whole group discussion by making sure students explain their process for adding and subtracting. It is very important that students share their partial results while doing this. The teacher must also evaluate students' questions to see if they are on the right track. While students are working independently on the worksheet the teacher needs to be walking around monitoring the process. If the student's process is not clear on the worksheet the teacher needs to ask the student to explain his or her answer.

Phase Two

During this phase students will begin using an empty number line instead of the beads. The goal is to have them using the empty number line to add and subtract in very few jumps and understand reasonable lengths for each jump using appropriate proportions.

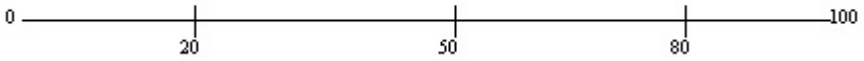


Task 1

The first task is to introduce the entire class to the empty number line. An empty number line is a straight line without numbers. This is done by displaying both the beads and the empty number line if front of the class and marking the same points on each line, so the students are allowed to compare the two. Then the students will continue to explore the empty number line by marking other numbers on it without the help of the beads.

Possible Discussion Questions:

1. Where is 40 on the beads? Where is it on the number line? How do you know that works? Do you mean you can represent numbers on the number line also? *(The teacher is pushing students to see that even though the number line does not have individual beads it can still show numbers.)*
2. Who can show me where to find 50? 80? 20? (This is done on a number line that only shows 0 and 100. The teacher should be prepared for some discourse among the students over where the numbers should go. The class should not worry about being exact. Their reasoning needs to focus on ideas such as 50 is the halfway point, so it should be in the middle and 20 is a less than halfway between 0 and 50.)



1. Who can show us where 21 is? 39? 75? (This is done on a number line where all the multiples of ten are shown.)

1. Who can show me where 50 is? 99? 36? (This is done on a number line where 0 and 100 are shown.)

Anticipated Thinking and Common Misconceptions

The teacher wants to make sure students are using logical reasoning when they are locating numbers on the number line. When students are locating 75 they should reason that it is halfway between 70 and 80. When students are locating 99 they should know that it is next to 100. The students need to be able to use and explain their strategies.

Students may feel uncomfortable estimating where a number is on the number line. They may really want to find the number exactly. The teacher needs to make sure that students do not get too concerned about locating the exact point on the empty number line. They need to focus on general reasoning and justifications such as this number should be less than halfway or it is right next to this number. The teacher needs to make sure that this type of reasoning is articulated to the rest of the class.

Assessment

The main focus of assessment is the students reasoning. If they are able to reason as outlined above then the teacher knows they are gaining an understanding of using the empty number line. If the teacher is unsure of the students' understanding from the whole group discussion he or she may choose to give the students sheets with empty number lines and have them individually locate numbers.

Task 2

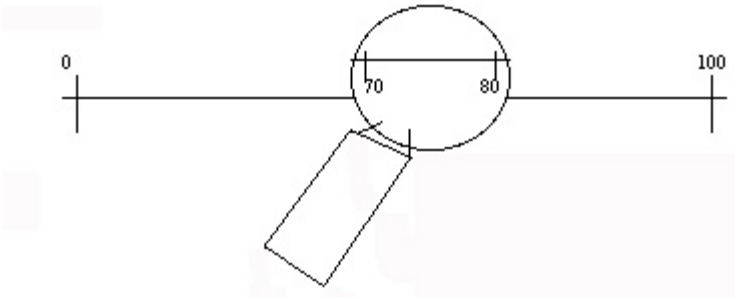
The students will begin to move along the number line from 0 to specific numbers in the least amount of jumps by one hundred, ten, and one. The teacher will tell them specifically which number they are to go to and what types of jumps they should take. Some of the strategies will require the students to go forwards and backwards. For example when they go from 0 to 69 they should take 7 jumps of ten forward and 1 jump of one backwards. The teacher should not mark the 100 on the number line because the students will become too concerned with proportionality. There needs to be a discussion about the length of the jumps. Students need to realize that all the jumps of ten should be the same size and a jump of one should be smaller. The students also need to realize that the length of the number lines can vary and that it is important to make sure you have the right proportions. This will also be emphasized by using a partial number line which is introduced by showing it pictured with a magnifying glass.

They need to realize that the same number may have different positions on the number line based on the numbers that are shown. For example two number lines may look like the same length, but one will show 20 to 40 while the other represents 20 to 80. The number 30 will be located in different positions on these two lines. They will also encounter situations that call on them to extend the number line to mark numbers.

The students will work in pairs on a worksheet that requires them to use an empty number line and locate different numbers taking into account things such as proportion.

Possible Discussion Questions:

1. How will we go from 0 to 43 in the least number of jumps of ten and one? (The class should continue working on this problem until they are able to do it in a few number of jumps. If students want to use 50 as a halfway mark and work back the teacher needs to show them they don't know the line represents 100. It may only go up to 60.)
2. How can we go from 0 to 57 in the least number of jumps of ten and one? 0 to 69? (The teacher wants students to use the strategy of going forward to 70 and then back to 69.)
3. How can we go from 0 to 88 in the least number of jumps of a hundred, ten, and one?
4. Do we know how long these lines are? Look at the one on the blackboard and the ones I've put on your worksheet. Are they the same length?
5. If they are different lengths how do we know how far a jump of ten should be? (The goal is to have students figure out that all the jumps of ten should be the same size regardless of the length of the number line.)
6. Look at this picture of a number line with a magnifying glass. Where is 75? Where is 78?



1. Where is 30 on this number line? (The number line goes from 20 to 40.)
2. Where is 30 on this number line? (The number line goes from 20 to 80, but is the same length as the one pictured for the previous question.)
3. Where would 90 be on this number line? (Students will need to extend the number line to solve this problem. If the students seem to have an understanding then the teacher puts them into pairs to work on their worksheet doing the same types of problems.)

Anticipated Thinking and Common Misconceptions

When students are told to use the least amount of jumps this does not mean they should be doing it in only one jump. They should be doing it in the least amount of types of jumps. The teacher does not want the kids trying to guess where a number is by using one jump. This should be exemplified to the students during the group work.

Students may have trouble adjusting their proportions based on the length that is represented by different number lines. This is why the teacher uses a magnifying glass to help the students understand. The teacher may have to spend more time on this idea than is outlined above.

Assessment

There are two things the teacher is assessing the students for. First of all he or she is looking to see that they are able to use the fewest amount of jumps. For example, they should be able to take one jump of one hundred and work backwards to find 88 and not have to take eight jumps of ten. The second thing the teacher is looking for is that the students are able to adjust to different lengths of number lines. He or she needs to make sure that their proportions are correct.

Phase 3

During this phase the students will use the empty number line to solve addition and subtraction problems. The goal is for students to represent their thought process on the empty number line using the least amount of jumps.

<u>Task 1</u>

Students will use the empty number line to go from one number to another. They will be told to draw their jumps to explain their thinking process. Students will no longer be restricted by using the least number of jumps or the types jumps they can use. The class will do a few problems together and then the students will do some problems individually.

Possible Discussion Questions

1. Who wants to show us how to go from 27 to 53 in a small number of jumps? Who has another way? (The teacher continues to solicit different thinking strategies and emphasizes those that are real strategies.)

2. Who can show us how to go from 63 to 45 in a small number of jumps? Who has another way? (Now that different thinking strategies have been presented to the students they may work on problems individually).

Anticipated Thinking and Common Misconceptions

Students may try to just make one jump in effort to have the least amount of jumps. One jump can be an acceptable answer if the students are able to explain their reasoning and have a real thinking strategy. The students might want to continue trying to use the least amount of jumps as was called for in the previous task. The teacher needs to discourage this type of method. He or she should encourage thinking strategies.

When going from 27 to 53 the students may do things such as jump by ten from 27 to 37 to 47 and then make six small jumps. Other students might do the same, but instead of making six small jumps they make one large jump of six or two jumps of three. Students may jump three from 27 to 30, make two jumps of ten to 50, and then jump another three. Another strategy that students might use is to make one jump of twenty and another jump of six.

Assessment

The teacher is assessing this task by looking for real thinking strategies. These would be shown through methods such as jumping by tens, working forwards and backwards, and jumping through tens. The teacher should be monitoring students while they are solving problems and asking them about their thinking strategies if they are not clear on their written work.

Phase 4

During this stage students move from using the number line for showing their thinking strategy to supporting them while solving problems. The teacher will also begin recording number sentences to show the students' thinking. This is also an opportunity for the teacher to reinforce the idea of when it is easier to use different methods such as taking away or finding the difference. The goal of the phase is to use the number line as a tool for problem solving.

<u>Task 1</u>

The students are given contextual problems to solve using the number line. The children are given the problem and an empty number line. They are asked to draw their jumps on the number line to show how they solved the problem. Examples of a problems would be:

A board 93 inches long.
I need a board 86 inches long.
How much longer is my board?

Jim had 76 dollars.
He spent 39 dollars.
How much will he have left?

The road from Madison to Adams is 47 miles.
The road from Adams to Franklin is 38 miles.

A board 84 inches long.
I need to cut the board, so that it is 49 inches long.
How much will be left over?

Sue is waiting in a store.
The person with number 28 is being served.
Mira's number is 46.
How many people are there ahead of her?

How many miles is the road from Madison to Franklin?

Ann has a book with 64 pages.
She has already read 37 pages.
How many more pages are there to be read?

Discussion Questions

(These are done after the students have worked on the problem individually)

1. Has anyone discovered any tricks that make solving problems easier? (The teacher is looking for responses such as "I added 40 and subtracted one to add 39". The teacher wants to emphasize the "tricks" or advanced thinking strategies the students are using.)

Assessment

The teacher continues to look for real thinking strategies. He or she is also checking to make sure students can translate the contextual problem correctly on the number line.

Phase 5

During this phase students move from using their thinking strategies on paper to mentally performing them. The goal is for students to be able to compute problems mentally.

Task 1

The teacher will show problems and ask students to try to solve them without writing them down. He or she should encourage them to picture the number line in their head. The ask the students to give their answer and explain how they thought about it. Some students may have to write in the beginning, but the teacher should value responses from students who can solve problems in their head. The teacher should choose contextual problems and number sentences that are similar to previous problems.

Discussion Questions

1. Try to solve these problems without writing anything down. Lisa has $50 to spend on groceries. She has already selected $37 worth of groceries. How much money does she have left? How did you solve that problem? *(The teacher should continue using problems like this and having students explain their mental processes.)*
2. How are you thinking about the problem? Are there any pictures in your head?

Anticipated Thinking and Common Misconceptions

Some students may not be able to solve the problems without writing something down. The teacher needs to keep encouraging them. They should be encouraged to visualize jumps on the number line.

Assessment

The teacher wants students to be able to solve the problems using mental computations. He or she also wants to make sure the students are using efficient thinking strategies while doing so. The only way the teacher will be able to assess this is by asking students to explain their thought process. It would be risky to have students write down their explanations too much because they might regress to using the number line too much. One way to aid the teacher in this process is to have students write their answers on a slate and hold it up. Then the teacher can ask different students to explain their thought processes.

Reference:

Gravemeijer, K. (1992). The empty numberline. Netherlands: Utrecht University. Draft Paper