# Developing Base Ten Understanding: Working with Tens, The Difference Between Numbers, Doubling, Tripling…, Splitting, Sharing & Scaling Up

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A child's understanding of place value and the base ten system is built upon a broad conceptual framework of number size, number relations, flexibly decomposing and reconfiguring numbers, and thinking multiplicatively. Combine this view of place value and base ten understanding with the notion of mathematical proficiency (National Research Council (NRC), 2001). Mathematical proficiency is defined as being based upon five interwoven components:

> *Conceptual understanding* – comprehension of mathematical concepts, operations, and relations
> *Procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
> *Strategic competence* – ability to formulate, represent, and solve mathematical problems
> *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification
> *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001, p. 116).

This article looks explicitly at how to help students develop mathematical proficiency by moving agilely up and down the number system around landmarks of ten and to begin to think in scale. Place value and the base ten system is an early and easy entry point for students to begin to explore this agility. Without this level of flexibility and fluency, students are limited to inefficient strategies or are overly dependent upon tactical procedures they know only through rote application.

### How to Use These Activities

The activities presented in this article are not isolated performance activities or assessments. They represent a series of activities *the maturing responses to which* form an assessment analysis. In turn, the activities become targeted approaches to develop mathematical proficiency in number. These can be done singly with a child, in small groups, or as warm-up with the entire class. The fluid response to such activities forms the basis for successful navigation of the base ten system moving either forwards or backwards. The number sense nurtured here reflects the core skills to add, subtract, multiply, or divide and form key algebraic ideas around how numbers can be operated upon. Fluency in these areas allows progressively more efficient use of strategies such as Tens & Ones, Incremental, Compensation, and Difference Between in addition and subtraction as well the Distributive and Associative strategies in multiplication and division.

> **[These activities] represent a series of activities *the maturing responses to which* form an assessment analysis and targeted instructional approaches to develop mathematical proficiency in number.**

### Goals & Objectives

The developmental foundation of these activities is to monitor the progress of a child to:

- Fluidly know the combinations of ten including in reverse from ten and any decade of ten
- Fluidly increment by ten from any number
- Span differences between numbers in as few increments as possible
- Move either forward or backwards through the number system fluidly in the fewest increments
- Decompose and reconfigure numbers fluidly in multiplying and partitioning numbers
- See place value as a multiplicative relationship
- Transition from additive to multiplicative thinking
- Think in scale

### As Assessment Tools

These activities can be used over a range of a student's school years from first through fifth grade. The *number range* and the *questions posed* are what expand over time and increase the complexity of the relations being explored. As an assessment, they are analyzing a combination of a student's:

- Conceptual understanding
- Strategic competence
- Procedural fluency (accuracy, fluency)
- Adaptive reasoning (efficiency[1] and logical thought, reflection, explanation, and justification)
- Place value knowledge & thinking in scale

**Conceptual Understanding:** Specific to the topic of base ten understanding, students need to know that numbers can be

[1] Differing from the National Research Council's definition, where efficiency is placed within the category of Procedural Fluency, efficiency is being defined in this article as a matter of making judgments and therefore more a reflection of adaptive reasoning than procedural thought.

*decomposed and reconfigured* to make equivalent combinations. This ability is foundational to the exploration of the various algebraic properties and how the various operations are unique and intertwined. This includes understanding properties of zero, and the identity, associative, and distributive properties. Conceptual understanding also includes the development of relational thinking.

**Strategic Competency:** Student strategies range from *direct modeling, various counting strategies, flexible and derived strategies,* and *abstract number strategies.* It is the progressive maturation of these strategies that is being monitored and described.

Since many of the activities in this article focus on developing a sense of *the difference between* numbers, a careful understanding of the possible solution strategies is required. In many texts, the 'difference between' is often solely represented using a subtraction 'take-away' strategy: $23 - 10 = x$. However, research has shown students have three strategies that are equally legitimate. Those strategies include not only $23 - 10 = x$ but also $10 + x = 23$, and $23 - x = 10$. Students should be allowed to use the strategy that works efficiently for them given the number combinations present, and not be narrowly guided to choose one over the other.

**Procedural Fluency:**

*Accuracy:* Accuracy is a constant goal. Accuracy is possible with the full range of strategies from direct modeling all the way through to the abstract number solutions.

*Fluency:* A child's fluency is described by determining if:

- The response is automatic or calculated
- The *length of the span used* in the solution
- The presence of *relational thinking* in the strategy used to determine the solution

Fluency matures over time. It is the ease at which any of the levels of strategy is utilized. It is a reflection of the number of steps a child needs to take to find the solution. While fluency may be reflected in the speed at which a student works, fluency is not synonymous with efficiency. One can be very fast with counting on by ones but it is not considered an efficient strategy among older elementary students.

**Adaptive Reasoning:**

*Efficiency:* Efficiency emerges from being able to make *judgments* about which *strategy* to use given the *numbers* and the *context* of the problem as well as how *fluidly* one moves through that selection and solution process. To make these judgments, a student needs to be familiar with and fluid in using more than one strategy. To find how much it takes to get from 2 to 20, while an addition context, may be most efficiently solved as $20 - 2$. However, how much to get from 12 to 20 may be best left as $12 + x = 20$ or $20 - x = 12$. While learning a new, more efficient strategy a student often has to slow down to work more methodically with the new, more sophisticated concepts involved. Students need to be reminded that this is a normal occurrence and another reason to disassociate efficiency from fluency.

*Student Reflection*: The teacher has opportunities to ask students to reflect upon their thinking processes at various points in the learning cycle. The questions or prompts used as a student is first considering a task are potentially different than those asked during the process of solving the task as well as in reflecting back upon the task once solved. Questions and prompts can range from being able to visualize the task, asking questions about the forms of representation being used, comparing and contrasting with other solutions, and reflecting on where efficiencies can be found (Jacobs & Ambrose, 2008). The following are questions and prompts to consider while using the activities that follow:

- To scaffold a more advanced strategy, e.g., *I know you can count on [by ones; skip count one ten at a time] but can you save yourself time and get yourself to [the next ten; next hundred] in as few steps as possible?*
- The actions used while solving the problem, e.g., *Can you use what you know about getting to then next ten to help you with getting to the next one hundred? Is there a way to break the numbers apart to make them easier to work with?*
- If further efficiencies are possible, e.g. *"Now that you know the answer, looking back could you have solved that problem using fewer steps?"*
- The algebraic implications that come from generalizing the action of this problem to others, e.g., *Can you do that all the time with all numbers?*

**Place Value Knowledge & Capacity to Think in Scale:**

*Number relations:* The numbers selected for a task can help a student explore the relationships in how a number is composed and can be decomposed in terms of its place value. Example: *You have 20 cookies, I give you 3 more. How many do you have now? There were 17 cookies on a plate. Seven get eaten. How many are left? You have 12 cookies and I only have 10. How many more do you have than me?* With each of these number combinations (20, 3) (17, 7), (12, 10), placed within specific problem types (Join, result unknown; Separate, result unknown; Compare, difference unknown) causes the student to consider place value ideas.

*Thinking in Scale:* Place value is a multiplicative relation. One can think of 42 as 40 + 2, but also as 4 x 10 + 2. To multiply 40 x 30 without merely relying on the surface pattern of "counting the zeros," one can decompose each number by factoring the power of 10. Thus:

$$\begin{aligned} 40 \times 30 &= 4 \times 10 \times 3 \times 10 \\ &= 4 \times 3 \times 10 \times 10 \\ &= 12 \times 100 \\ &= 1200 \end{aligned}$$

Place value is based upon maintaining a 1:10 ratio, otherwise known as the "Rate of Ten." This means that for every one composite ten that is made, a simultaneous accumulation of 10 ones is also acquired. If I want 4 times as many tens (4 x 10) I simultaneously need to coordinate getting 40 times as many ones (40 x 1). To have a robust understanding of place value requires the capacity to think in scale.

### Summary

The following pages outline individual tasks and how the same task can be used at different grade levels. Moving forward, however, requires following these important ideas:

- Language Matters – Speak in value, not in digits; the underlying algebraic ideas can be traced better when values are spoken rather than through digit manipulation. So *watch your language!*
- Decomposition of Number – Numbers are made up of various subunits. So, *if you don't like the numbers that you have, break them apart or change them to make them easier to work with.*
- Think Relationally – *Use what you know to figure out what you don't know.*
- Keep it Equal – If you do break numbers apart, just make sure the values are equal to the original quantities.
- Make the Math Visible – Find ways to represent the mathematics in formats where all can see and understand why the strategies work the way they do.

## Descriptions and Objectives of Tasks: *Addition & Subtraction*

**Materials needed:** A deck of playing cards with the picture cards removed; The 0 - 100 cards from a hundreds pocket charts or from a cut up hundreds chart; paper and pencils as needed to make student thinking visible.

**Notation:** Initially when playing Getting From Any Number to the Next Ten, or other similar activities, making a visual recording often helps a child organize his or her thinking. Notations also allow the child to compare one strategy to another thereby reflecting upon similarities, differences, and efficiencies. A recommended form of notation would be to use an *"empty number line"* format or the following arrow notation system:

Example: *How much to get from 34 to 100?*

$34 \xrightarrow{+6} 40 \xrightarrow{+10} 50 \xrightarrow{+50} 100$ The answer is 66

or

$34 \xrightarrow{+6} 40 \xrightarrow{\quad +60 \quad} 100$ The answer is still 66

Spending time recording visually how the distance is spanned provides a mental map for students. Eventually, with a number to get to 100, 1,000 or 10,000 the increments become more mentally tracked.

## *Addition*

**Get to 10** – Use a deck of regular playing cards. Aces are worth one. Draw a card and show it to the child. Say, *"How much to get to 10?"* **Objective:** To fluid learn the combinations of 10. Ten Frame Dot Images are also an excellent tool to help children gain accuracy and fluidness with these combinations.

**Get to 20, 30, 40 or 50** – Use a deck of regular playing cards. Aces are worth one. Draw a card and show it to the child. Say, *"How much to get to 20 (or 30 – 50)?"* **Objective:** To use the combinations to get to a ten plus add a ten or tens. Get to 20 helps reinforce that the teen numbers are a 10 + *a* combination. Get to 30, or beyond, reinforces incrementing by ten from the start of a decade and that multidigit numbers are composed in tens and ones components, e.g., 20 + 3 = 23.

$7 \xrightarrow{+3} 10 \xrightarrow{+10} 20$ The answer is 13

or

$7 \xrightarrow{+10} 17 \xrightarrow{+3} 20$ The answer is 13

**Get from Any Number to the Next 10** – Use the cards from the 100-card deck. Draw a card and show it to the child and ask, *"How much to get to the next 10?"* **Objective:** To reinforce the combinations of ten and to extend the application of that knowledge to multidigit numbers. Example: 4 + 6 = 10; 54 + 6 = 60. **Variations** include asking *"How many to get to [the second/third decade]?* Example: Draw a 32 and ask, *How many to get to 50?* Or, draw 46 and ask, *How many to get to 70?*

**100 Facts** – Use the decade cards from the 100-card deck. Draw a card and show it to the child and ask, *"How much to get to 100?"* **Objective:** To directly relate the combinations to ten to the combinations of 100. Extends the knowledge of the combinations to tens to multidigit numbers, e.g., 3 + 7 = 10, 30 + 70 = 100.

**Get to 100** – Use the cards from the 100-card deck. Draw a card and show it to the child and ask, *"How much to get to 100?"* **Objective:** To use either a large increment of ten to get as close to 100 as possible and then add the small piece to 100, or add to get to the next ten and then increment the remaining span as one group of ten.

**Get to 1,000/10,000** - Play as above except just write a number of your own choosing on a piece of paper.

**Get from One Number to Another Number (Under 100)** – Use the cards from the 100-card deck or just write two numbers of your choosing down on a piece of paper. Draw two cards and show them to the child and ask, *"How much to get to from x to y?"* Example: *"How much to get from 23 to 54?"* **Objective**: To span the difference between two numbers using key landmarks of ten. Strengthens the number sense of the child as he or she considers just how far apart the two numbers are. **Variation**: Present the cards as drawn. If 54 were drawn first, then the previous example would be presented as, *"How much to get from 54 to 23?"* It is up to the child to decide whether or not to add (23 + $x$ = 54; a flexible strategy) or to subtract (54 – $x$ = 23 or 54 – 23 = $x$; also flexible strategies)

**Get from One Number to Another Number (Above 100)** – Play as above except just write two numbers of your own choosing on a piece of paper. As capacity is built, the numbers selected can become quite sophisticated. Example: 2,674, how much to get to 4,128. Fluidly done, it takes 326 to get to 3000, and 1,128 to get to 4,128. The answer is 326 + 1,128, which is 1,454. This is, in fact, the 'difference between' (Add up to Subtract) strategy for in subtraction and how all change is calculated in stores.

**Extending to Decimals: Get to One** – Write a decimal on a sheet of paper or on the white board and ask, *How much to get to 1?* **Objective**: Using the same strategies for get to 10 or getting to the next 10 works directly with working with decimals. However, it is absolutely imperative to use the language of values to strengthen place value understanding. "Point 12" has no mathematical meaning; "twelve hundredths" does (.12).

$$.12 \xrightarrow{+.08} .20 \xrightarrow{+.80} 1.00$$

The answer is .88
Read as *"eighty-eight hundredths"*

**Extending to Decimals: Get to the Next One** – Write a mixed number with a decimal and ask, *How much to get to the next one?* Example: *3.46* (Read as, *three and forty-six hundredths*), *how much to get to 4?* **Objective**: Connecting the strategies of how much to get to the next ten to how much to get to the next one. This is the strategy used to calculate change in stores. Reminder: talking in value is an absolute imperative.

**Extending to Decimals: Get from One Number to Another Number** – Write to mixed numbers on the paper or white board. Ask, *How much to get from x to y?* Example: 3.46, how much to get to 5.32?

$$3.46 \xrightarrow{+.04} 3.50 \xrightarrow{+.50} 4.00 \xrightarrow{+1.32} 5.32$$

The answer is *one and eighty-six hundredths,* (1.86).

## *Subtraction*:

**Go Back from Ten** – Draw a card from the deck of playing cards. (Take out the picture cards. Aces are worth one.) Say, *You are at 10, go back x.* **Objective**: The ten facts in reverse.

**Go Back from a Decade** – *Pull* out the decade cards from the 100-card deck (10, 20, 30…) and place them in a deck of its own. Draw a card from the deck of regular playing cards. Say, *Go back #.* Example: Draw 70; draw 4; Say, *Go back 4.* **Objective**: Extending the ten facts in reverse but from any decade marker.

**Go back a Number** – Draw a card from the 100-card deck. Draw a number from the regular card deck. Say, *Go back #.* Example: Draw 82. Draw 6. Say, *Go back 6.* Quickly break 6 apart into 2 + 4, "*Go back 2, back 4.*" 82 – 2 → 80 – 4 → 76. **Objective:** To decompose a number in a form that allows one to *get back to a ten*, then use knowledge of ten facts to move into the decade below.

**Go Back from 100** – Draw a card from the 100-card deck and say, *Go back ##.* Example: Draw 47. Say, *Go back 47.* 100 – 40 → 60 – 7 → 53. It is acceptable for the student to change this into an addition problem. However, it is worthwhile to develop the skill of working backwards in chunks. **Objective:** Use 100 facts and ten facts in reverse.

**Go Back to a Number (100) –** Draw a card from the 100-card deck and say, *How much to get from 100 to ##.* Example: Draw 67. Say, *How much to get from 100 **to** 67?* Example: 100 – 30 → 70 – 3 → 67. The answer is 33. This is different than the activity above. Go Back From 100 follows a "take-away" model of subtraction [100 – 47]. Go back ***to*** a number follows the *"difference between"* structure [100 – $x$ = 67]. **Objective:** To span the difference between the two numbers.

# Descriptions and Objectives of Tasks: *Multiplication & Division*

**Materials needed:** A deck of playing cards with the picture cards removed; the 0 – 100 cards from a hundreds pocket chart or from a cut up hundreds chart; paper and marker as needed to transcribe student thinking and make it visible to all in the group.

**Moving from additive to multiplicative thinking:** As students initially engage with these strategies, they typically begin by adding combinations. As they become more adept, their language shifts from more additive descriptions to multiplicative arrangements. For example "Triple 36" might initially be described by early learners as 30 + 30 + 30 is 90, 6 + 6 + 6 is 18, 90 + 18 is 108. A more multiplicative description may sound like: *Three 30s are 90, three sixes are 18, 90 + 18 is 108.* Transcribing this last statement would be 3 $x$ 30 = 90, 3 $x$ 6 = 18, 90 + 18 = 108. The language the student uses should drive the transcription.

**Language:** It is very important that students describe the values of the numbers with which they are working. These activities are to enhance and deepen place value understanding and sharpen the abstract number strategies. A child who needs to quadruple 80, can use the fact of 4 eights are 32 to figure out $4 \times 80$. However, the scaling up from the smaller fact to the larger is because 32 is then *multiplied by 10* and *not* the result of *"adding a zero to the 32."* This is the mathematical transcription of what really is happening: $4 \times 80 = 4 \times (8 \times 10) = (4 \times 8) \times 10 = 32 \times 10 = 320$. This assists the child to comprehend the *factoring* of 80 into $8 \times 10$ and develop the concept of the *rate of ten.* These two ideas are essential for the mathematics they need in the upper elementary and middle school mathematics programs.

**Strategies:** In multiplication there are only two strategies a student can use. They either decompose the number into addends to multiply the combinations in order to then add the partial products (the distributive property), or they can break the numbers into factors to then commute and associate its various terms (the commutative and associative properties). Example: $24 \times 6$ could be solved as $(20 + 4) \times 6 = (20 \times 6) + (4 \times 6) = 120 + 24 = 148$ (the distributive property), or, $24 \times 6 = (2 \times 12) \times 6 = 2 \times (12 \times 6) = 2 \times 72 = 144$ (factoring to use the associative property). en into either addends or factors there are the only two multiplicative strategies that students can use.

## Multiplication

**Doubling, Tripling, Quadrupling…** – If just beginning, start with doubling a single digit numbers in a deck of playing cards. Build capacity by using the lower multidigit numbers from the 100-card deck and then gradually move to higher multidigit values. Extend this to include three and four digit numbers as student skill expands. Draw a card and say, *Double ##* or *Triple… Quadruple…* Example: Double 46, Triple 23, Quadruple 17. Notice as the student gets good at doubling the numbers get higher. But when tripling, you return to smaller numbers and slowly expand the range. Initially transcribe the strategies as the child talks to allow them to visually reflect upon his or her thinking. **Objective:** To use the distributive property and think in increasingly larger multiples of number combinations:

**Representing students thinking using informal notation**

**Triple 47: By place value**

**47**
**↙ ↘**
**120 + 21**
**141**

**Triple 47: Compensation Strategy**

**47 ← +3**

**50**
**↓**
**150 – 9**
**141**

**Scaling Up -** For this you need paper or a whiteboard. Plan a sequence of related math facts to present to the student(s) one at a time. Example: Write on the paper or board 3 x 8. Say, *What are three groups of 8?* After the students give an answer and how they know, possibly give them 6 x 8 saying, *If you know 3 x 8, can you use that to figure out 6 x 8* [A doubling scale]? It becomes important to transcribe the student thinking in algebraic terms to make the mathematics visible and to allow students to reflect on how the numbers were decomposed and recombined. Example:

$$6 \times 8 = (3 \times 8) + (3 \times 8) = 48$$

This would match a student's description: *I did 3 x 8 and then added another 3 x 8 and got 48.*

It could also be solved as:

**Representing students thinking using formal notation**

**Triple 47**

$$\begin{aligned} 3 \times 47 &= 3 \times (40 + 7) \\ &= 3 \times 40 + 3 \times 7 \\ &= 120 + 21 \\ &= 141 \end{aligned}$$

$$6 \times 8 = 2 \times (3 \times 8) = 2 \times 24$$

This would match the description: *I just double the 3 x 8*.

The sequence of problems can continue and can include the extension of patterns you want students to think about (example: 12 x 8). Scaling based on tripling and quadrupling sequence. Example: Tripling sequence –

3 x 4
9 x 4
27 x 4

**Objective:** Develop relational thinking around scale factor.

## *Division*

**Splitting & Sharing:** Use numbers drawn from either the playing card deck or the 100-card deck. Draw a card and say, *Split ## in two* or *Share ## with two.* Example: Draw 32. *Split 32 in two* or *Share 32 with four.* Remainders can be included which will result in fractional sharing. Sharing remainders may be best discussed initially with single digit numbers. Example: *Share 7 with three.* Initially the sharing of the remainder may need to include drawing of pictures and conversations about how to name the fractional amount along with how to write the fractional amount. This is the "Equal Sharing" model of division. **Objective:** Using the distributive property and known fact combinations to solve efficiently [32 = 20 + 12; 4 x 5 + 4 x 3].

**A Transcription of *"Share 7 with 3"***

7

↙ ↓ ↘

2 2 2 ←→ 6

1/3 1/3 1/3 ←→ 1

2 1/3 2 1/3 2 1/3

Answer: *They each get 2 1/3*

**Scaling Up:** Present one number combination at a time as above but in this instance the question would change to, *How many 4s are in 32?* Use a "T-Chart" to help students organize their thinking and to reflect upon the relationships among the sequence of problems. Example: *If you know how many 4s are in 32, how many 4s are in 64? If you know how many 4s are in each of those, how many 4s are in 96?* **Objective:** recognize relationships among combinations to use scale factor.

**T-Chart for Division Scaling Up**

| | | |
|---|---|---|
| (8) | **4s** | **32** |
| (16) *(2 x 8)* | **4s** | **64** *(2 x 32)* |
| (24) *8 + 16* | **4s** | **96** *(32 + 64)* |

### References


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